Exactly in the same way

Thm 1.28 (Schur-Zassenhaus for prof. gps)

Let K be a normal Hall subgp of a prof. gp G. Then $G = K \rtimes H$ for some $H \leq_c G$, and any two complements H are conjugate.

Pf: Use Schur-Zassenhaus for fin. quotients and argue as before: let $\mathcal{U} = \{U \trianglelefteq_o G\}$, $G = \varprojlim G/U$.

$S_U = \{H_U \leq G/U \mid \text{complement for } KU/U\} \neq \emptyset$.

is inverse system of fin. sets, $\varprojlim S_U \neq \emptyset$ etc.

Similarly for conjugacy of complements.

Def 1.29 If G is a topol. gp, then $X \subseteq G$ generates G if $\langle \overline{X} \rangle = G$, i.e. $\langle X \rangle$ is dense in G.

A topol. gp G is fin. gen., if $\langle \overline{X} \rangle = G$ for X finite.

Prop 1.30 Let G be profin., $H \leq_c G$.

(i) $X \subseteq H$ gener. H iff XN/N gener. HN/N for all $N \trianglelefteq_o G$.

(ii) If there is $d \in \mathbb{N}$ s.th. HN/N is d-gener. for all $N \trianglelefteq_o G$, then H is top. d-generated.

Pf: (i) Note that for $X \subseteq H$ we have $\overline{X} = \bigcap_{N \trianglelefteq_o H} XN$. (see 1.18)

(ii) As before: let $\langle y_N \rangle = HN/N$, $|y_N| = d$.

This is div. system of fin. sets. A limit sequence $(x_N) \in \varprojlim y_N$ is s.th. $x_1, \dots x_d$ gener. H top. by (i).

Prop. 1.31. If G is a fin. gen. profin gp, then for any $m \in \mathbb{N}$ G has only fin. many (open) subgps of index m, and any open subgp contains an open top. char. subgp.

Pf: If G is top. d-gen., there are at most $(n!)^d$ contin. homom. of G to S_n. If $H \leq_o G$, $[G:H] = n$, then the right reg. action of G on G/H is continuous and $H = \rho_G^{-1}(A)$ for $A \leq S_n$ a 1-pt. stab. in S_n. Hence there are at most $n \cdot (n!)^d$ poss. for H.

For each cont. autom α of G, $[G:H^\alpha] = [G:H] = n$. Hence $\bigcap H^\alpha$ is a fin. intess., topol. char. and of fin. index.

Prop 1.32 If G is a fin. gen. profin. gp, then every open subgp is fin. gen.

Pf: Let $G = \langle X \rangle$, $|X| = d$, $X = X^{-1}$. Let $H \leq_o G$, $T \subseteq H \backslash G$, $1 \in T$, a transversal, so T is finite. For $x \in X$, $t \in T$ there is $s = s(t,x) \in T$ s.th. $Htx = Hs$. Put

$$y = \{ tx\, s(t,x)^{-1} \mid t \in T, x \in X \}.$$

Then $\overline{\langle y \rangle} = H$: clearly $\overline{\langle y \rangle} tx = \overline{\langle y \rangle} tx\, s(t,x)^{-1} s(t,x) \in \overline{\langle y \rangle} TX = \overline{\langle y \rangle} T$

Now $\overline{\langle y \rangle} T \supseteq \langle X \rangle$ as $1 \in \overline{\langle y \rangle}, T$.

$\overline{\langle y \rangle} T = G$ since T is finite, $\overline{\langle y \rangle} \leq H$, so

$$H = H \cap \overline{\langle y \rangle} T = \overline{\langle y \rangle} (H \cap T) = \overline{\langle y \rangle}.$$

Def 1.33 For a (pro)fin. gp G, the Frattini subgp

$$\Phi(G) = \bigcap \{ M \mid M \leq G \text{ max proper (open)} \}.$$

Prop 1.34 If G is profinite, then

(i) $\Phi(G) \trianglelefteq_c G$

(ii) If $K \trianglelefteq_c G$, $K \leq \Phi(G)$, then $\Phi(G/K) = \Phi(G)/K$

(iii) For $X \subseteq G$ TFAE

(a) $\langle \overline{X} \rangle = G$

(b) $\langle \overline{X \cup \Phi(G)} \rangle = G$

(c) $X\,\Phi(G)/\Phi(G)$ generates $G/\Phi(G)$ topol.

Pf: (i), (ii) ✓

(iii) (a) $\Rightarrow$ (b) $\Rightarrow$ (c). For (c) $\Rightarrow$ (a), let $X \subseteq K \leq_o G$. If $K \neq G$, then $K \leq M$ for some max $M \leq_o G$ and $\langle \overline{X} \rangle \Phi(G)/\Phi(G) \leq M/\Phi(G) \neq G/\Phi(G)$ ↯ Hence $K = G$ and $\langle \overline{X} \rangle = G$ by Prop. 1.18.

§2 Pro-p gps

Lemma 1.33 Let G be a finite p-gp. Then

(i) $\Phi(G) = [G,G]\,G^p$

(ii) If $X \subset G$ and $\langle X\,\Phi(G) \rangle = G/\Phi(G)$, then $\langle X \rangle = G$.

(iii) $G/\Phi(G) \simeq \mathbb{F}_p^d$ where d is min. s.th. G is d-gener.

Pf: (i) $G/[G,G]\,G^p$ is an elem. ab. p-gp, so its max proper subgps index p. Trivially. Hence $\Phi(G) \leq [G,G]\,G^p$. Any max subgp of G has index p, so equality holds.

(ii) follows as in 1.34

(iii) follows from (i) and (ii).

Recall that in a finite p-gp max subgps are normal.

Prop 2.28 If G is a pro-p-gp, then $\overline{\Phi(G)} = G^p[G,G]$.

Pf: If $M \leq_o G$ maxi, there is $N \trianglelefteq_o G$ with $N \leq M$. Then M/N is a max. subgp of G/N and hence $M \trianglelefteq G$ with $[G:M] = p$. Hence $G^p[G,G] \leq M$. So $\Phi(G) = \bigcap M \geq G^p[G,G]$.

Since $\Phi(G)$ is closed, also $\overline{\Phi(G)} \geq \overline{G^p[G,G]}$.

Now $G/\overline{G^p[G,G]} =: Q$ is a pro-p gp, ~~therefore~~ so

open normal subgps intersect trivially. If $N \trianglelefteq_o Q$ then Q/N is fin. elem. ab. p-gp, so $\Phi(Q/N) = 1$. Thus $\Phi(Q) \leq \bigcap N = 1$ and by Prop. 1.34(iii) and first part we have $\Phi(G)/\overline{G^p[G,G]} = \bar{\Phi}(Q) = 1$.

Prop 2.3 For fin. gen. pro-p gps a subgp is open iff it has finite index.

Pf: see [D, dS, M, S] Thm 1.17

Here generally this holds for fin. gen. profin. gps by a deep result of Nikolov-Segal.

Prop 2.4 A pro-p gp G is fin. gen. iff $\Phi(G) \leq_o G$.

Pf: "$\Rightarrow$" If $\Phi(G) <_o G$, then $G/\Phi(G)$ is fin. Thus there is fin. $X \subseteq G$ s.th. $G = X\Phi(G)$, so $\langle \overline{X} \rangle = G$ by 1.34

"$\leq$" Suppose $G = \langle \overline{X} \rangle$, $|X| = d$. For $\Phi(G) \leq N \trianglelefteq_o G$, G/N is elem. ab. p-gp by 1.36 generated by d elts, so $[G:N] \leq p^d$. Let $N_0 \trianglelefteq_o G$ be s.th. $[G:N_0]$ is max, so $N_0 \leq N$ for all $N \trianglelefteq_o G$. Since $\Phi(G) \trianglelefteq_c G$ we have $\Phi(G) = \bigcap \{ N \trianglelefteq_o G \mid \Phi(G) \leq N \} = N_0 <_o G$.

The lower p-series of a pro-p gp

Def. 2.5 Let G be a pro-p gp. Then $P_1(G) = G$ and for $i \geq 1$ $P_{i+1}(G) = P_i(G)^p [P_i(G), G]$.

Rem: $P_2(G) = \Phi(G)$ and $P_{i+1}(G) \geq \Phi(P_i(G))$ $\forall i$.

We will be interested in "$=$".