of G. (In pwb. G_ℓ is a pro-ℓ-gp.)

Question: When is $G \leq G_\ell$?

Ex: (i) Let $G = \mathbb{Z}$, the profinite compl. $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$

the pro-p-completion is $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Then $\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$ are disc. topol. ring with 1. profinite rings, i.e. cp, tot. disc.

(ii) In particular for a profinite ring R, the gp of units R^* is a profin. gp, cp subset of R

(iii) for a prof. ring R, $GL_n(R)$ is a prof. ring.

(iv) $$UT_n(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$$

is a pro-p-gp.

Rem: Pro abelian $\Rightarrow$ abelian

pro nilp. $\not\Rightarrow$ nilpot.

< p. 9b

Next we collect some basic properties of pro-ℓ-gps.

Prop 1.18 Let $\mathcal{L}$ be class of fin. gps closed under (Quot) and (Subobj).

(i) If G is pro-$\mathcal{L}$-gp, $K \trianglelefteq_c G$, then G/K is pro-$\mathcal{L}$.

If $\mathcal{L}$ is closed under taking subgps., then K is pro-$\mathcal{L}$.

(ii) A direct prod. of pro-$\mathcal{L}$-gps is pro-$\mathcal{L}$.

Pf: not hard.

For (ii) show that for $U \trianglelefteq_o G$, we have $G/U \in \mathcal{L}$

There is fin. set s.th. $U \geq \prod X_i$, $X_i = U_i$, $i \in J$, basic open. Then G/U is quot. of $G/\prod X_i = \prod G_i/U_i$

Prop 1.18 Let G be a prof. gp, $H \leq_c G$.

(i) If $H \leq V \leq_o G$, then V contains an open subgp of the form $HU \leq V$, $U \trianglelefteq_o G$.

(ii) H is the intersection of all open subgps cont. H. If $H \trianglelefteq G$, then $H = \bigcap_{H \leq V \trianglelefteq_o G} V$.

Pf: (i) If $H \leq V$, then $HV_G \leq V$ where $V_G = \bigcap V^g \trianglelefteq_o G$.

(ii) Clearly, $H \leq \bigcap_{V \leq_o G} HV$. Now that V is closed.

Claim: $\bigcap_{V \leq_o G} HV = H \bigcap_{V \leq_o G} V \quad (= H)$ (holds for any family filtered from below)

This is clear if inters. is fin., and "$\geq$" holds in general. To see "$\leq$" use comp.ness.

Let $x \in \bigcap HV$, so $x \in \bigcap_{i \leq n} HV_i = HV_0$, so $Hx \cap \bigcap_{i \leq n} V_i \neq \emptyset$ for all fin. subfamilies $V_{i_1} .. V_{i_n}$. By cpness $Hx \cap \bigcap V \neq \emptyset$.

Prop 1.20 An inf. profin. gp G is not the cbl. union of closed subsets with empty interior.

In part. G is fin. or uncbl.

Pf: Suppose $G = \bigcup_{i\in\mathbb{N}} C_i$, $\emptyset \neq C_i \subseteq G$ closed with empty interior. Then $D_i = G \setminus C_i$ is dense open in G.

If $U_0 \subseteq G$ is open, then $U_0 \cap D_i \neq \emptyset$ open. By Thm 1.10 there is $\emptyset \neq U_1 \subseteq U_0 \cap D_1$ clopen. Now $U_1 \cap D_2 \neq \emptyset$ is open, and we find clopen $U_2 \subseteq U_1 \cap D_2$.

By cpn. we have $\emptyset \neq \bigcap U_i \subseteq \bigcap D_i = G \setminus (\bigcup C_i) = \emptyset$ ↯

Hence the cardin. of a profin. gp does not carry much inform. (in contrast to fin. gps).

Def: 1.21 A supernatural number is a formal product $n = \prod_{p\in\mathbb{P}} p^{n(p)}$, $\mathbb{P}$ = all primes.

Put $n < \infty$, $\infty + \infty = \infty = \infty + n = n + \infty$.

We say $m = \prod p^{m(p)}$ divides n if $m(p) \leq n(p)$ for all $p \in \mathbb{P}$.

For a family $\{n_i = \prod p^{n(p,i)} \mid i \in I\}$

put $\prod n_i = \prod p^{n(p)}$, where $n(p) = \sum_i n(p,i)$

$\gcd\{n_i\} = \prod p^{n(p)}$ $\quad n(p) = \min_i \{n(p,i)\}$

$\operatorname{lcm}\{n_i\} =$ $\quad \max_i$

For a profin. gp G, U = open subgps of G, and closed subgp $H \leq G$ put

$$[G:H] = \operatorname{lcm}\{[G/U : HU/U] \mid U \in \mathcal{U}\}$$

so $|G| = [G:1] = \operatorname{lcm}\{[G/U] \mid U \in \mathcal{U}\}$.

Prop 1.22 Let G be a prof. gp.

(i) If $H \leq_c G$, then $[G:H] \in \mathbb{N} \Leftrightarrow H \leq_o G$.

(ii) If $H \leq_c G$, then $[G:H] = \operatorname{lcm}\{[G:U] \mid H \leq U \leq_o G\}$.

(iii) If $K \leq_c \leq H \leq_c G$, then $[G:K] = [G:H][H:K]$.

(iv) If $G = \varprojlim G_i$, then $|G| = \operatorname{lcm}\{|G_i| \mid i \in I\}$.

Pf: (i), (ii) by Prop 1.18, (iii) is easy.

π-Hall subgps

1.23 Def Let $\pi \subseteq \mathbb{P}$, $\pi' = \mathbb{P} \setminus \pi$. Then $n = \prod_p p^{n(p)}$ is a π-number if $n(p) = 0$ for $p \notin \pi$.

A profin. gp G is a pro-π-gp if $|G|$ is a π-number.

For $\pi = \{p\}$ we write pro-p-gp.

If G is a profin. gp, $H \leq_c G$ is called π-Hall-subgp if $|H|$ is a π- and $[G:H]$ a π'-number.

For $\{p\} = \pi$, we say p-Sylow-subgp.

Lemma 1.24 (i) Profinite gp G is a π-gp iff $G = \varprojlim G_i$, each G_i π-gp.

(ii) $H \leq_c G$ is a π-Hall subgp of $G = \varprojlim G_i$ iff HU/U is π-Hall subgp of G/U, $\forall U \trianglelefteq_o G$.

Pf: (i) is clear

(ii) We have $H = \varprojlim \varphi_i(H)$. So H is π-gp iff each $\varphi_i(H)$ is π-gp, and $[G:H] = \operatorname{lcm}\{[G/S_i : HS_i/S_i] \mid S_i = \ker \varphi_i\}$.

Since each φ_i is epim, we have $[G/S_i : HS_i/S_i] = [G_i : \varphi_i(H)]$. Hence $[G:H]$ is a π'-number iff each $[G_i : \varphi_i(H)]$ is π'-number.

Thm 1.25 Let $\pi \subset \mathbb{P}$, $G = \varprojlim G_i$. Assume all G_i satisf.

(i) G_i contains a π-Hall subgp

(ii) Any π-subgp of G_i is contained in π-Hall subgp

(iii) Any two π-Hall subgps of G_i are conj.

Then (i') G contains a π-Hall subgp

(ii') any closed π-gp H is contained in π-Hall subgp

(iii') any two π-Hall subgps of G are conj.

Pf (i') Let $\mathcal{H}_i$ = family of π-subgps of G_i, $\mathcal{H}_i \neq \emptyset$ by (i).

Then $\{\mathcal{H}_i, \varphi_{ij}\}$ is inverse system of fin. sets,

hence $\varprojlim \mathcal{H}_i \neq \emptyset$ by Prop. 1.7,

Let $(H_i) \in \varprojlim \mathcal{H}_i$. Then $H = \varprojlim H_i$ is π-Hall subgp

(ii') $\mathcal{S}_i = \{\varphi_i(H) \leq S_i \leq G_i \mid$ for some π-Hall subgp $S_i\} \neq \emptyset$, etc.

system $(S_i) \in \varprojlim \mathcal{S}_i$. Then

$H = \varprojlim \varphi_i(H) \leq \varprojlim S_i = S$ is π-Hall.

(iii') H, K π-Hall in G. Then $\varphi_i(H), \varphi_i(K)$ are

π-Hall in G_i and hence conj., so

$$Q_i = \{q_i \in G_i \mid \varphi_i(H)^{q_i} = \varphi_i(K)\} \neq \emptyset \quad \text{etc.}$$

Cor 1.26 Let G be a profin. gp., $p \in \mathbb{P}$. Then

(i) G contains p-Sylow subgps

(ii) any p-subgp is contained in p-Sylow

(iii) All p-Sylow subgps are conj.

Similarly

Prop 1.27 A profin. gp G is pronilp. iff it contains a unique p-Sylow subgp G_p for each prime p.

So $G = \prod G_p$.

Pf Exercise