Let $\{U_t \mid t \in T\}$ be family of clopen nbhds of x.

Then $\{x\} = \bigcap U_t$ by Rem 1.5

Since $X \setminus W$ is closed and disj. from $\{x\} = \bigcap U_t$, by comp. there is finite $T' \subseteq T$ s.t.

$(X \setminus W) \cap (\bigcap_{T'} U_t) = \emptyset$, and $\bigcap_{t \in T'} U_t$ is clopen nbhd of X in W.

(iii) $\Rightarrow$ (i) Let X be cp Hausd. with basis of clopen nbhds.

Let $\mathcal{R}$ = all equiv. rel. R on X with ^{xR}clopen. Dann ist X/R diskrete and finite. Def. part. ord $\geq$ on $\mathcal{R}$ by $R \geq R'$ if R is refines R'. Clearly this is directed by $R_1 \cap R_2 \geq R_1, R_2$. For $R \geq R'$ there is a natural proj $\varphi_{RR'} : X/R \to X/R'$, $xR \longmapsto xR'$ which makes $\{X/R, \varphi_{RR'}\}$ an inv. system.

Claim $\quad X \cong \varprojlim X/R \quad$ (homeom.)

Pf: Let $\psi : X \to \varprojlim X/R$ induced by $\varphi_R : X \to X/R$

By Lemma 1.9. ψ is onto. Since ψ is cont., X cp, it suffices to show that ψ is inj. Let $x, y \in X$.

There ex clopen $U \subseteq X$, $x \in U$, $y \notin U$. The part. $U, X \setminus U$ def. an equiv. rel. on X in $\mathcal{R}$ separating x and y.

Def 1.11 For a nonempty class $\mathcal{C}$ of fin. gps a pro-$\mathcal{C}$-gp G is an inv. limit $G = \varprojlim G_i$, $G_i \in \mathcal{C}$

Eg: $\mathcal{C}$ could be

(i) class of fin. gps $\to$ profinite

(ii) cyclic $\to$ procyclic

(iii) soluble $\to$ prosoluble

(iv) p-gps $\to$ pro-p gps

(v) nilp $\to$ pro-nilp.

Sometimes we require $\mathcal{C}$ to be closed under one or more of the following

(Sub) subgp closed

(Quot) closed under quotients

(Dir) closed under fin. dir. prod.

(Subdir) $G/N_1, G/N_2 \in \mathcal{C} \Rightarrow G/(N_1 \cap N_2) \in \mathcal{C}$

(Ext) $1 \to K \to G \to H \to 1$, $K, H \in \mathcal{C} \Rightarrow G \in \mathcal{C}$.

<u>Lemma</u> 1.12 Let $G = \varprojlim G_i$ be prof., $\varphi_i : G \to G_i$ the corresp. proj. Then $\{S_i \mid S_i = \operatorname{Ker} \varphi_i\}$ is basis of open nbhds of 1.

<u>Pf</u>: Consider basis of $1 \in \prod G_i$ given by

$$\{1\}_{i_1} \times \dots \times \{1_{G_{i_k}}\} \times \prod_{i \neq i_1, \dots, i_k} G_i$$

Note that each G_i is discrete.

For $i_0 \in J$ with $i_0 > i_1, \dots, i_k$ we have

$$\underbrace{G \cap \left[\left(\prod_{i \neq i_0} G_i \right) \times \{1_{i_0}\} \right]}_{\ker(\varphi_{i_0})} = G \cap \left[\left(\prod_{i \neq i_1, \dots, i_k} G_i \right) \times \{1\}_{i_1} \times \dots \{1_{i_k}\} \right]$$

is basis in G.

<u>Rem</u> 1.13 (i) If G is a cp gp, then $U \leq G$ is open iff U is closed of fin. index

(ii) If $H \leq_{fin} G$, then the <u>core</u> $H_G = \bigcap_{g \in G} H^g \trianglelefteq_{fin} G$

as $H_G = \bigcap_{g \in H \backslash G} H^g$.

Thm 1.14 If $\mathcal{C}$ is closed under (Quot), (Subdir), then the foll. are equiv. for top. gp G.

(i) G is pro-$\mathcal{C}$.

(ii) G is cp. Hausd. tot. disc. and $G/U \in \mathcal{C}$ for any $U \trianglelefteq_o G$

(iii) G is cp, 1 has nbhd basis $\mathcal{N}$ s.th. $\bigcap_{U \in \mathcal{N}} U = 1$ where $U \trianglelefteq_o G$, $G/U \in \mathcal{C}$.

(iv) $G = \varprojlim G/U$, $\{U \trianglelefteq_o G \mid \text{nbhd basis for } 1\}$.

Pf: (i) $\Rightarrow$ (iii) If $G = \varprojlim G_i$, $\varphi_i : G \to G_i$, $U \trianglelefteq_o G$, then by Lemma 1.13 there is $i \in I$ s.th. $\ker \varphi_i \leq U$. Hence G/U is quotient of $G/\ker \varphi_i \in \mathcal{C}$. By (Quot), $G/U \in \mathcal{C}$.

(iii) $\Rightarrow$ (iii) By Thm 1.10, the clopen nbhds of 1 form a nbhd basis for 1 and $\bigcap_{V \text{ clopen}} V = 1$

Hence shows: any clopen V contains open normal subgp.

Let $X \subseteq G$, let $X^{(n)} = \{x_1 \cdots x_n \mid x_i \in X\}$, $X^{-1} = \{x^{-1} \mid x \in X\}$

Let $F = (G \setminus V) \cap V^2$. Since V is cp, so is V^2. Hence F is closed and cp. Let $x \in V \subseteq G \setminus F$. Since multipl. is contin., there is are open nbhds V_x, S_x of $x, 1$, resp s.th. $V_x, S_x \subseteq V$ and $V_x S_x \subseteq G \setminus F$. By comp. of V, there are $x_1, \ldots x_n$ s.th. $V \subseteq \bigcup V_{x_i}$. Put $S = \bigcap S_{x_i}$, $W = S \cap S^{-1}$. Then $W \subseteq V$, $VW \subseteq G \setminus F$ and W is symm.

Hence $VW \cap F = \varnothing$ and $VW \subseteq V^2$. Then $VW \cap (G \setminus V) = \varnothing$, i.e. $VW \subseteq V$ and hence $VW^{(n)} \subseteq V$. Since W is symm.

$R = \bigcup W^{(n)} \subseteq V$ is an open subgp of G.

Then $R_G = \bigcap_{g \in G} R^g$ is open normal and $R_G \leq V$.

(iii) $\Rightarrow$ (iv) Let $\mathcal{N}$ be as in (iii). Make $\mathcal{N}$ into dir. poset by $U \geq V$ if $U \leq V$. Then the natural inv. system $\{G/U, \varphi_{UV}\}$ with can. epim. $\varphi_U : G \to G/U$ induces a cont. homom. $\varphi : G \to \varprojlim G/U$.

Then φ is epim. by Lemma 1.14. Since G is cp., it suffices to show φ is inj., but this is clear since for $x \in G$ s.th $\varphi(x) = 1$ we have

$$x \in \bigcap_{U \in \mathcal{N}} U = 1.$$

(iv) $\Rightarrow$ (i) is clear.

Prop 1.16 Let $\{H_i \mid i \in I\}$ be a family of closed subgps of a profin. gp G, $\bigcap_I H_i \leq U \leq_o G$. Then there is a finite set $J \subseteq I$ s.th. $\bigcap_J H_i \leq U$.

Pf: The $\{G \setminus H_i\}_{i \in I}$ form an open covering of $G \setminus U$.

Cor 1.17 If $\{U_i \mid i \in I\}$ is any coll. of open subgps of profinite G s.th. $\bigcap_I U = 1$. Then

$$\mathcal{V} = \{\bigcap_{i \in J} U_i \mid J \subseteq I \text{ fin.}\}$$ is a nbhd basis of 1 in G.

Pf: Follows from 1.15.

Examples and def. 1.17 Let $\mathcal{C}$ be a class of fin. gps closed und (Quot) and (Subdir), G any gp. Let $\mathcal{N} = \{N \trianglelefteq_{fin} G \mid G/N \in \mathcal{C}\}$. Then $G \in \mathcal{N} = \emptyset$. Then $\mathcal{N}$ is a dir. p.o. set by reverse incl. and

$G_{\mathcal{C}} = \varprojlim G/N$ is called the pro-$\mathcal{C}$-completion

of G. (In pwb. G_ℓ is a pro-ℓ-gp.)

Question: When is $G \leq G_\ell$?

Ex: (i) Let $G = \mathbb{Z}$, the profinite compl. $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$

the pro-p-completion is $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Then $\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$ are disc. topol. ring with 1. profinite rings, i.e. cp, tot.

(ii) In particular for a profinite ring R, the gp of units R^* is a profin. gp, cp subset of R

(iii) for a prof. ring R, $GL_n(R)$ is a prof. ring.

(iv) $$UT_n(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & * & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$$

is a pro-p-gp.

Rem: Pro abelian $\Rightarrow$ abelian

pronilp. $\not\Rightarrow$ nilpot.

Next we collect some basic properties of pro-ℓ-gps.

Prop 1.18 Let $\mathcal{L}$ be class of fin. gps closed under (Quot) and (Subobj).

(i) If G is pro-$\mathcal{L}$-gp, $K \trianglelefteq_c G$, then G/K is pro-$\mathcal{L}$.
If $\mathcal{L}$ is closed under taking subgps., then K is pro-$\mathcal{L}$.

(iii) A direct prod. of pro-$\mathcal{L}$-gps is pro-$\mathcal{L}$.

Pf: not hard.

For (ii) show that for $U \trianglelefteq_o G$, we have $G/U \in \mathcal{L}$

There is fin. set s.th. $U \geq \prod X_i$, $X_i = U_i$, $i \in I$, basic open. Then G/U is quot. of $G/\prod X_i = \prod G_i/U_i$

Prop 1.20 An inf. profin. gp G is not the cbl. union of closed subsets with empty interior.

In part. G is fin. or uncbl.

Pf: Suppose $G = \bigcup_{i \in \mathbb{N}} C_i$, $\emptyset \neq C_i \subsetneq G$ closed with empty interior. Then $D_i = G \setminus C_i$ is dense open in G.

If $U_0 \subseteq G$ is open, then $U_0 \cap D_i \neq \emptyset$ open. By Thm 1.10 there is $\emptyset \neq U_1 \subseteq U_0 \cap D_1$ clopen. Now $U_1 \cap D_2 \neq \emptyset$ is open, and we find clopen $U_2 \subseteq U_1 \cap D_2$.

By cpn. we have $\emptyset \neq \bigcap U_i \subseteq \bigcap D_i = G \setminus (\bigcup C_i) = \emptyset$ ↯

Hence the cardin. of a profin. gp does not carry much inform. (in contrast to fin. gps).

Def: 1.21 A supernatural number is a formal product $n = \prod_{p \in \mathbb{P}} p^{n(p)}$, $\mathbb{P}$ = all primes.

Put $n < \infty$, $\infty + \infty = \infty = \infty + n = n + \infty$.