$\mathcal{C}$ formation of finite groups (closed under Quot. and finite subdirect prod.)

✓ Existence of free pro-$\mathcal{C}$ groups

Problem. How to define a rank.? (One needs to control open normal subgps
The rank should be as ~~finite~~ in profinite case!

Def 5.3 G profinite, X set. A map $\varphi: X \to G$ is convergent to 1 if $X \setminus \varphi^{-1}(H)$ is finite for each H open normal subgp of G.

Def 5.4 A free pro-$\mathcal{C}$ gp on X converging to 1 is a pro-$\mathcal{C}$ gp $\hat{F}$ with a map $i: X \to \hat{F}$, s.t.:

(i) i converges to 1 and $\overline{\langle i(X) \rangle} = F$

(ii) $\forall \varphi: X \to G$, G pro-$\mathcal{C}$ which conv. to 1 and $\overline{\langle \varphi(X) \rangle} = G$
$\exists!$ epimorphism $\hat{\varphi}: \hat{F} \xrightarrow{\text{cont.}} G$ with $\hat{\varphi} \circ i = \varphi$

$$\begin{array}{ccc} \hat{F} & \overset{\hat{\varphi}}{\dashrightarrow} & G \\ {\scriptstyle i}\uparrow & \mathcal{C} & \nearrow {\scriptstyle \varphi} \\ X & & \end{array}$$

Prop 5.5 $\mathcal{C}$ formation, X set. There exists a free pro-$\mathcal{C}$ group $\hat{F}$ on X. converging to 1.

More precisely: Let F free gp on X. $\mathcal{N} = \{N \trianglelefteq F : F/N \in \mathcal{C},\ X \setminus N$ finite$\}$
Let $\hat{F} := \varprojlim F/N$ be the profinite completion of F w.r.t $\mathcal{N}$, $\theta: F \to \hat{F}$ canonical homomorph., $i: X \to \hat{F}$ restriction of θ to X. Then $\hat{F}$ is is a free pro-$\mathcal{C}$ gp on X conv. to 1.

Pf. Similar to Satz 5.2; $\hat{F}$ is pro-$\mathcal{C}$ gp since $\mathcal{N}$ directed family ✓.
$\hat{F} = \overline{\langle i(X) \rangle}$ [$\hat{x}\hat{N}$ intersects F for any $\hat{x} \in \hat{F}$, $N \in \mathcal{N}$: take $x \in \pi_N(\hat{x}) \in F/N$; $\pi_N^{-1}(\pi_N(x)) = \hat{x}\hat{N} \Rightarrow x \in \hat{x}\hat{N}$].
$\hat{H} \trianglelefteq \hat{F}$ open then $\theta^{-1}(\hat{H}) \in \mathcal{N}$ so $X \setminus \theta^{-1}(\hat{H})$ finite $\Rightarrow i$ conv. to 1.

Consider $\varphi: X \to G$, G pro-$\mathcal{C}$ conv. 1. and $\overline{\langle \varphi(X) \rangle} = G$.

Extend φ in a unique way to $\varphi: F \to G$ (as abstract gps).

epimorphism. Let $H \leq G$ op., $\varphi^{-1}(H) \leq F$ and $\varphi(F) \cdot H = G$.

$\Rightarrow$ $F/\varphi^{-1}(H) \cong G/H \in \mathcal{C}$. Also $X - \varphi^{-1}(H)$ finite $\Rightarrow$ $\varphi^{-1}(H) \in \mathcal{N}$.

$\Rightarrow$ $\exists\ \hat{\varphi}: \hat{F} \xrightarrow{\text{cont epi}} G$ s.t $\hat{\varphi} \circ \Theta = \varphi$. □.

Remark. The universal property of a free pro-$\mathcal{C}$ gp on a set X implies its uniqueness, up to isomorphisms. (Easy).

We denote the "unique" free pro-$\mathcal{C}$ gp on X conv. to 1 by $\hat{F}_X(\mathcal{C})$ (If X is empty, $\hat{F}_X(\mathcal{C}) = 1$).

Lemma 5.7. $\mathcal{C}$ formation of fin gps, X set, $i: X \to \hat{F}_X(\mathcal{C})$ conv. to 1.

Suppose $\exists\ C \in \mathcal{C}$ non-trivial with $\text{rank}(C) \leq |X|$. Then i is injective

Moreover, $\forall x \in X \quad i(x) \neq 1$.

Pf. WMA $|X| \geq 2$.

$r = \text{rank}(C)$, $x \neq y$, $x, y \in X$. Let $c_1, \dots, c_r$ gen. of C, $c_i \neq 1$.

Case 1. $r = 1$, define $\varphi(x) = c_1$ and $\varphi(z) = 1$ $\forall z \in X - \{x, y\}$.

Case 2. $r \geq 2$, choose $x_3, \dots, x_r \in X - \{x, y\}$, and set

$\varphi(x) = c_1$, $\varphi(y) = c_2$, $\varphi(x_i) = c_i$ for $i \geq 3$, $\varphi(z) = 1$ $\forall z \in X - \{x, y, x_3, \dots, x_r\}$.

In both cases, $\langle \varphi(X) \rangle = C$. and $\varphi: X \to C$ conv. to 1. Hence

$\exists\ \hat{\varphi}: \hat{F}_X(\mathcal{C}) \to C$ epimorph. with $\hat{\varphi} \circ i = \varphi$. By construction, $\varphi(x) \neq \varphi(y)$

$\Rightarrow$ $i(x) \neq i(y)$.

Moreover, as $\varphi(x) \neq 1$, then $i(x) \neq 1$. As x was arbitrary, we obtain the second part of the statement.

Let $\mathcal{C}$, X, i and C as above, we may identify X with a subset of $\hat{F}_X(\mathcal{C})$ which does not contain 1 and $i: X \to \hat{F}_X(\mathcal{C})$ with the inclusion.

Then: $\hat{F}_X(\mathcal{C})$ becomes a free pro-$\mathcal{C}$ group with basis X.

Def 5.8 $\mathcal{C}$ formation of fin gps., $\hat{F}$ pro-$\mathcal{C}$ gp., $X \subseteq \hat{F}$ which $1 \notin X$.
$\hat{F}$ is a free pro-$\mathcal{C}$ gp with basis X (equiv. to 1) if:

(i) $\overline{\langle X \rangle} = \hat{F}$ and X converges to 1. (i.e $i: X \hookrightarrow \hat{F}$ conv. to 1).

(ii) $\forall\, \varphi: X \to G$ conv. to 1, G pro-$\mathcal{C}$ gp., $G = \overline{\langle \varphi(X) \rangle}$ extends uniquely to an epimorphism $\hat{\varphi}: \hat{F} \to G$.

Remark 5.9 (Relation with Definition 5.1, page 49).
The following are equivalent:

1) $\hat{F}$ is a free pro-$\mathcal{C}$ gp with basis X conv. to 1.

2) $\hat{F}$ is a free pro-$\mathcal{C}$ gp in the sense of Def. 5.1 and $i: X \hookrightarrow \hat{F}$ conv. to 1. and i inclusion.

Lemma 5.10 $\mathcal{C}$ form fin gp., $\hat{F}$ free pro-$\mathcal{C}$ gp with basis X.
$\mathcal{C}$ contains a non-trivial gp of rank $\leq |X|$. Then:

(a) $\operatorname{rank}(\hat{F}) = |X|$.

(b) Suppose $e = |X| < \infty$. Then every set of generators of $\hat{F}$ of size e is a basis of $\hat{F}$.

(c) F free gp on X, $\mathcal{N}(X) = \{ N \trianglelefteq F : F/N \in \mathcal{C},\ X \setminus N$ finite $\}$
$\hat{F}$ is ~~the canonical~~ profinite completion of F wrt $\mathcal{N}(X)$ and
$\theta: F \to \hat{F}$ canonical map ~~maps~~ is s.t $\theta(x) = x \ \ \forall x \in F$.

Pf.

a) X infinite. By def X conv to 1. $\Rightarrow |X| = rk(\hat{F})$

By def. $rk(\hat{F})$ is the size of a system of generators that conv to 1
It is well-defined by ex. 2.

a), b) X finite, $X = \{x_1, \dots, x_e\}$. Let $s_1, \dots, s_d$ a syst of gen of $\hat{F}$, with $d \leq e$. Set $s_{d+1} = \dots = s_e = 1$.

$$\begin{array}{ccc} X & \longrightarrow & \hat{F} \\ x_i & \mapsto & s_i \end{array}$$

Consider the map , extends to an epimorphism $\hat{F} \xrightarrow{\hat{\varphi}} \hat{F}$

ex. 4 $\Rightarrow \hat{\varphi}$ is an auto. Since $x_i \neq 1$, $d = e$ so $\{s_1, \dots, s_e\}$ basis of $\hat{F}$

b) follows from the above.

c) By 5.5 and 5.7, $F_X(\mathcal{C}) = \varprojlim_{N} F/N$, $N \in \mathcal{N}(X)$ is By Ex. 4.

$F_X(\mathcal{C}) \underset{X}{\cong} \hat{F}$. So $\theta : F \to \hat{F}$.

$x \mapsto \dot{x}$

Remark. $\mathcal{C}$ form. fin. gps. $m \geq 1$. By the above, if $\mathcal{C}$ has a gp of $rk \leq m$, then $\hat{F}_X(\mathcal{C})$ is uniquely determined up to iso by m and $\mathcal{C}$. We denote it by $\hat{F}_m(\mathcal{C})$ and call it the free pro-$\mathcal{C}$ gp of rank m.

Ex. $\mathcal{C}$ all finite gps. $\hat{F}_1(\mathcal{C}) = \hat{\mathbb{Z}}$.

$\mathcal{C}$ all finite p-gps, $\hat{F}_1(\mathcal{C}) = \mathbb{Z}_p$.

Remark Sup. $rk(G) > m$ for any $G \in \mathcal{C}$ non-trivial. ($\mathcal{C}$ finite).
$F = F(X)$, with $|X| = m$. Then F has no proper normal subgp N, s.t. $F/N \in \mathcal{C}$
$\Rightarrow \hat{F}_X(\mathcal{C}) = 1$. and $\hat{F}_m(\mathcal{C})$ not def. So, by convention $\hat{F}_m(\mathcal{C})$ assume that $\mathcal{C}$ has non-trivial subgp of $rk \leq m$.

Lemma. 5.11 $\mathcal{C}$ form of gps. $e \in \mathbb{Z}^{>0}$. G pro-$\mathcal{C}$ gp. of $rk \leq e$.
Each epimorphism $\beta : G \to \hat{F}_e(\mathcal{C})$ is an isom.

Pf. By assump $\exists\, \alpha : \hat{F}_e(\mathcal{C}) \xrightarrow{epi} G$. Now $\beta \circ \alpha$ is epi from $\hat{F}_e(\mathcal{C})$ onto itself, so it is an automorphism. (Exercise 2). Hence β iso. (α too!). □

~~Every free~~

~~Prop. 5.12. Every pro-$\mathcal{C}$ gp G is a quotient of a free pro-$\mathcal{C}$ gp. If $rk(G) = m \Rightarrow G$ quotient of $\hat{F}_m(\mathcal{C})$.~~

~~Pf. Every profinite gp has a generating set X that conv. to 1. (Douady) So $\hat{F}_X(\mathcal{C})/\mathrm{Ker}(\hat{\varphi}) \cong G$ where $\hat{\varphi}: \hat{F}_X(\mathcal{C}) \twoheadrightarrow G$.~~ □

Lemma 5.13. $\mathcal{C}$ form. of fin. gps, F free pro-$\mathcal{C}$ gp with basis X, $C \in \mathcal{C}$, $C \neq 1$.

a) $A \subseteq X$, $|A| \geq rk(C)$. Then $\overline{\langle A \rangle}$ free pro-$\mathcal{C}$ gp with basis A. Moreover, if N ~~max.~~ min. closed normal subgp of F, with $X \setminus A \subseteq N$, then $F = \overline{\langle A \rangle} \ltimes N$. Thus $F/N \cong \overline{\langle A \rangle}$.

b) $F = \varprojlim \overline{\langle A \rangle}$, $A \subseteq X$, $|A| \geq rk(C)$.

Pf. a) $X \setminus A \subseteq N$, $N \cdot \overline{\langle A \rangle} = F$. Let $\varphi_0 : X \to \hat{F}_A(\mathcal{C})$

$$x \mapsto \begin{cases} x & \text{if } x \in A \\ 1 & \text{if } x \in X \setminus A. \end{cases}$$

$\exists$ epimorph. $\varphi_A : F \to \hat{F}_A(\mathcal{C})$ identity on A.

$X \setminus A \subseteq \mathrm{Ker}\, \varphi_A \Rightarrow N \leq \mathrm{Ker}\, \varphi_A$ so $N \cap \overline{\langle A \rangle} = 1$, i.e. $F = N \rtimes \overline{\langle A \rangle}$.

(In fact, $\mathrm{Ker}\, \varphi_A = N$.) $\Rightarrow F/N \cong \overline{\langle A \rangle}$.

b) $A \subseteq B$ fin. $rk(C) \leq |A|$. By (a) $\varphi_{B,A} : \hat{F}_B(\mathcal{C}) \to F_A(\mathcal{C})$ with

$$\varphi_{B,A}(x) = \begin{cases} x & x \in A \\ 1 & x \in B \setminus A \end{cases}$$

Hence $\varphi_{B,A} \circ \varphi_B = \varphi_A$.

$F' = \varprojlim \hat{F}_A(\mathcal{C})$. $\pi_A : F' \to \hat{F}_A(\mathcal{C})$

$\{\varphi_A\}_A$ ~~define~~ induce an epimorph. $\varphi : F \to F'$. For A, $\varphi_A = \pi_A \circ \varphi$

$\Rightarrow \varphi_A$ inj on $\langle A \rangle$. $\Rightarrow \varphi$ isomorphism since the gps $\langle A \rangle$ generate F. □