In order to show that $Aut(G)$ has fin. rk if G does, we consider embeddings into lin. gps: $GL_n(\mathbb{Z}_{p^l})$.
For this we go back to unif. powerful pro-p gps and their relation to p-adic analytic gps.
We will show (later)

Prop: Let G be a unif. pro-p gp of rk d. Then $Aut(G)$ embeds into $GL_d(\mathbb{Z}_p)$.

With this we get

Thm 4.8 Let G be profin. If G is virt. pro-p of fin rk, then so is $Aut(G)$.

Pf: G is fin. gen. Now let H (as in the pf of 4.7) be a char. open uniformly powerful pro-p of dim d.
With the same notation as in 4.7 we now have:
Δ is top. isom. to a closed subgp of $Z(H)^m$, which is pro-p of fin rk. Thus Δ has fin. rk.
By the prec. prop. (to be shown) there is a contin. embed. $\mu: Aut(H) \to GL_d(\mathbb{Z}_p)$. Then $\pi\mu: \mathbb{A} \to GL_d(\mathbb{Z}_p)$ is contin. and hence $\Xi^{\pi\mu} \leq_c GL_d(\mathbb{Z}_p)$. By Thm 4.2 $\Xi^{\pi\mu}$ has fin. rk and Ξ/Δ is top. isom. to $\Xi^{\pi\mu}$.
Thus Ξ has fin. rk, is pro-p and an open normal subgp of $Aut(G)$. (unif. powerf.)

More on unif. powerful pro-p gps

Lemma 4.9 Let G be a unif. powerf. pro-p gp with $d(G) = d$. Write $G_n = P_n(G) (= \{g^{p^n} \mid g \in G\})$.
Let $G = \langle a_1, \dots a_d \rangle$. Then

every $g \in G$ has a unique expn.

$g = a_1^{\lambda_1} \cdots a_d^{\lambda_d} = \bar{a}(\bar{\lambda})$, $\lambda_i \in \mathbb{Z}_p$.

Thus there is a bij. $\Theta : G \to \mathbb{Z}_p^d$, which is a homeom.

In part., also the inverse $\psi : (\lambda_1, \ldots \lambda_d) \mapsto a_1^{\lambda_1} \cdots a_d^{\lambda_d}$ is a homeom.

Pf: We already know that $G = \langle \bar{a}_1 \rangle \cdots \langle \bar{a}_d \rangle$; thus every $g \in G$ has an expn. of the form $g = a_1^{\lambda_1} \cdots a_d^{\lambda_d}$, $\lambda_i \in \mathbb{Z}_p$. For fixed k, the gp G/G_{k+1} has order p^{kd} and is a prod. of its cycl. subgps $\langle a_1 G_{k+1} \rangle, \ldots \langle a_d G_{k+1} \rangle$. Each of this cycl. subgps has order $\leq p^k$ and hence $|\langle a_i G_{k+1} \rangle| = p^k$. Thus every $g \in G/G_{k+1}$ has a unique expn. $g = a_1^{e_1} \cdots a_d^{e_d} G_{k+1}$, $e_i \in \mathbb{Z}/p^k\mathbb{Z}$. Thus uniqueness of λ_i follows inductively, $\lambda_i = \varprojlim e_i$. We have a bij. $\Theta : G \to \mathbb{Z}_p^d$ with inverse $\psi : \mathbb{Z}_p^d \to G$, $(\lambda_1, \ldots \lambda_d)^\psi = a_1^{\lambda_1} \cdots a_d^{\lambda_d}$.

By Prop. 2.3 ψ is continuous (every abstr. homom. from fin. gen. pro-p to profin gp is contin.) and because $\mathbb{Z}_p^d$, G are cp Hausd., ψ is homeom.

We next consider an add. str. on G. First recall

Lemma 4.10 If G is unif. powerful pro-p, then for any $n \in \mathbb{N}$, $f_n : G \to G_{n+1}$, $x \mapsto x^{p^n}$ is a homeom inducing bij. $G_k \to G_{k+n}$ and $G_k/G_{k+m} \to G_{k+n}/G_{k+m+n}$.

Pf: We have $f_n(G_k) = G_{k+n}$ by Prop. 2.12, f_n is contin. and if $x \equiv y \bmod G_{k+m}$, then $f_{n+1}(x) \equiv f_n(y) \bmod G_{k+m+n}$.

Rem: By the prev. lemma, $x \mapsto x^{p^{-n}}$ is well-def.

Lemma 4.11: Let G be unif. powerf. pro-p and ~~define~~

$$x +_n y = (x^{p^n} y^{p^n})^{p^{-n}}, \quad x, y \in G, \ n \geq 1.$$

Then for $u, v \in G_n$

$$xu +_n yv \equiv x +_n v \equiv x +_{n-1} y \pmod{G_n}$$

and hence for $m > n$

$$x +_m y \equiv x +_n y \pmod{G_{n+1}}.$$

Pf: For any pro-p gp G we have

$$[P_i(G), P_j(G)] \leq P_{i+j}(G), \quad i, j \in \mathbb{N}. \qquad \text{(Exercise)}$$

Hence we have $[G_n, G_n] \leq G_{2n}$. Thus

$$x^{p^n} y^{p^n} \equiv \left(x^{p^{n-1}} y^{p^{n-1}}\right)^p \pmod{G_{2n}}$$

$$= (x +_{n-1} y)^{p^n}.$$

By extracting p^n-th roots and Lemma 4.10 it follows

$$x +_n y = (x^{p^n} y^{p^n})^{p^{-n}} \equiv x +_{n-1} y \pmod{G_n}$$

Since $(xu)^{p^n} \equiv x^{p^n} \pmod{G_{2n}}$ and $(yv)^{p^n} \equiv y^{p^n} \pmod{G_n}$

we also get $xu +_n yv \equiv x +_n y \pmod{G_n}$.

This implies that for fixed (x, y) the sequence $(x +_n y)$ is Cauchy.

Def 4.12 For $x, y \in G$ put $x + y = \lim_{n \to \infty} x +_n y$

Rem By Lemma 4.11 we have

$$x + y \equiv x +_n y \pmod{G_{n+1}}$$

and for $u, v \in G_n$ also $xu + yv \equiv x + y \pmod{G_n}$.

Prop 4.13 The set G with $+$ is an ab. gp with identity 1_G and inversion given by $x \mapsto x^{-1}$.