Profinite and pseudofinite gps

Plan: Start with profinite gps, basics, Sylow thm
Explain duality profinite - pseudofinite.
Lectures for MMM: stable pseudofin. gps
profinite NIP gps

continue with profinite free gps.

§1 Basics on profinite gps

Def 1.1 (i) A directed p.o. set $(J, \leq)$ is a set J with p.o. $\leq$ and s.th. for $i, j \in J$ there is $k \in J$ s.th. $i, j \leq k$.

(ii) An inverse (or: projective) system of top spaces (or top. gps, rings, ..) over J is given by family

$\{X_i \mid i \in J\}$, X_i top sp.

and contin. maps (or gp homom.) $\varphi_{ij} : X_i \to X_j$, $i \geq j$,

s.th. for $i \geq j \geq k$

$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ik}"] \arrow[dr] & & X_k \\ & X_j \arrow[ur] & \end{tikzcd}$$

comm.

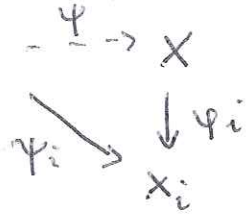
Write $\{X_i, \varphi_{ij}, J\}$ or $\{X_i, \varphi_{ij}\}$

Ex: (i) The const. inverse system $X_i = X$, $\varphi_{ij} = id$.

(ii) Let $X_i \subseteq \mathbb{Z}/p^i\mathbb{Z}$, $\varphi_{ij} : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}$, $x \mapsto x + p^j\mathbb{Z}$

Def 1.2 (i) If Y is a top. space (gp), $\{X_i, \varphi_{ij}, I\}$ an inverse system, $\psi_i : Y \to X_i$ cont., then $\{\psi_i\}$ is compatible with $\{X_i, \varphi_{ij}\}$ if $\varphi_{ij}\psi_i = \psi_j$ for $i \geq j$.

(ii) We call the top. space (gp) X with comp. maps $\{\varphi_i\}$ the inverse limit of $\{X_i, \varphi_{ij}\}$ if X is universal in the following sense: for any comp. system $Y, \{\psi_i\}$



there ex. unique cont. ψ s.th. the diagram commutes for all $i \in I$.

Prop 1.3. If $\{X_i, \varphi_{ij}, I\}$ is an inverse system, then the inverse limit X exists and is unique up to homeom.

In fact we can describe $X \subseteq \prod X_i$ as

$$X = \{x = (x_i)_{i \in I} \mid \varphi_{ij}(x_i) = x_j \ \forall i \geq j\} := \varprojlim X_i.$$

Put $\varphi_i : \prod X_i \to X_i$ the proj.

Pf: Exercise

Lemma 1.4 If $\{X_i, \varphi_{ij}\}$ is an inv. system of Hausd. sp., then $\varprojlim X_i$ is closed (subgp) in $\prod X_i$.

Pf: Let $(x_i) \in (\prod X_i) \setminus \varprojlim X_i$. Then there ex $r \geq s$ s.th $\varphi_{rs}(x_r) \neq x_s$. Let $U, V \subseteq X_s$ be open disj. nbhds of $\varphi_{rs}(x_r)$, x_s, $U' \subseteq X_r$ open nbhd of x_r.

s.t. $\varphi_{rs}(U') \subseteq U$. Then $\omega = \prod V_i$ with

$V_r = U'$, $V_s = V$, $V_i = X_i$, $i \neq r, s$, is open in $\prod X_i$

We have $(x_i) \in \omega$ and $\omega \cap \varprojlim X_i = \varnothing$.

Def A top. space is tot. disc. if every pt is its own conn. comp. A space X is conn. if $X = Y \cup Z$ for Y, Z open implies $Y = \varnothing$ or $Z = \varnothing$.
Ex: Finite Hausd.; discrete spaces → Rem.

Propl. 6 Let (X_i, φ_{ij}) be an inverse system of comp. Hausd. tot. disc. top. spaces (gps). Then $\varprojlim X_i$ is also tot. disc. comp. Hausd.

Pf: $\prod X_i$ is cp., tot. disc.

Propl. 7 The inverse lim. of an inv. system of nonempty comp. Haus. spaces is nonempty.

Pf: Put $Y_i = \{(x_i) \in \prod X_i \mid \varphi_{ik}(x_i) = x_k, \ k \leq i\}$.

Then (AC) Y_i is closed nonempty. Hence $\{Y_i \mid i \in I\}$ has fin. inters. prop. (since I is dir.)

By comp. of $\prod X_i$ then $\bigcap Y_i = \varprojlim X_i \neq \varnothing$.

Def: For inverse systems $\{X_i, \varphi_{ij}, I\}$, $\{X_i', \varphi_{ij}', I\}$ a map of inverse systems is given by cont. $\{\theta_i\}$, $\theta_i : X_i \to X_i'$ s.t.

$$\begin{array}{ccc} X_i & \longrightarrow & X_i' \\ {\scriptstyle \varphi_{ij}}\downarrow & & \downarrow{\scriptstyle \varphi_{ij}'} \\ X_j & \longrightarrow & X_j' \end{array}$$ commutes.

Then $\{\theta_i \ \varphi_i : X \to X_i'\}$ induces a cont. map

$\varprojlim \theta_i : \varprojlim X_i \to \varprojlim X_i'$.

Prop 1.5 X cp. Hausd., $x \in X$. Then the conn. comp C of x is $\bigcap_{x \in U_t} \{U_t \mid U_t \text{ clopen}\}$.

Pf: Let $A = \bigcap U_t$. Then $C \subseteq A$ as any clopen nbhd contains C. Suffices to show A conn.

Ow. $A = U \cup V$, $U \cap V = \varnothing$, U, V clopen

Since X is Hausd., U, V cp. disj., there ex open $U', V' \subseteq X$ s.th. $U \subseteq U'$, $V \subseteq V'$, $U' \cap V' = \varnothing$, so $[X - (U' \cup V')] \cap A = \varnothing$.

Since $X - (U' \cup V')$ is closed, by cp. there is $T' \subset T$ finite, s.th. $[X - (U' \cap V')] \cap \bigcap_{t \in T'} U_t = \varnothing$.

Then $x \in B = \bigcap_{t \in T'} U_t$ is clopen

So $x \in (B \cap U') \cup (B \cap V') = B$. If $x \in B \cap U'$, then $B \cap U'$ is open, and by symmetry closed

Therefore $A \subseteq B \cap U' \subseteq U'$. Hence $A \cap V \subseteq A \cap V' = \varnothing$ so $V' = \varnothing$.

Ex: $I = \mathbb{N}$, $\{\mathbb{Z}, id\}$ the const. inv. system

$\{\mathbb{Z}/p^n\mathbb{Z}, \varphi_{nm}\}$

Then $\theta_n : \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ the canon. proj. yields a map of inverse systems $\theta = \varprojlim \theta_n : \mathbb{Z} \to \varprojlim \mathbb{Z}/p^n\mathbb{Z}$

The image of $\mathbb{Z}$ under $\varprojlim \theta_i$ is $\{(a_n) \mid a_n = t\}$

but (b_n) with $b_n = 1 + \dots + p^{n-1} \in \varprojlim \mathbb{Z}/p^n\mathbb{Z} \setminus \operatorname{im}\theta$.

Thus θ is not onto.

However

Remma 1.9 If $\theta : \{X_i, \varphi_{ij}\} \to \{X_i', \varphi_{ij}'\}$ is a map of inv. systems of comp. Hausd. spaces and each $\theta_i : X_i \to X_i'$ is onto, then $\varprojlim \theta_i : \varprojlim X_i \to \varprojlim X_i'$ is onto

Pf: Let $(x_i') \in \varprojlim X_i'$. Put $\tilde{X}_i = \theta_i^{-1}(x_i')$. Then $\tilde{X}_i \subset X_i$ cl., hence comp. and $\varphi_{ij}(\tilde{X}_i) \subseteq \tilde{X}_j$ for $i > j$. Then $\{\tilde{X}_i, \varphi_{ij}\}$ is an inv. system of cp. spaces. So $\varprojlim \tilde{X}_i \neq \emptyset$. Hence for $(x_i) \in \varprojlim \tilde{X}_i$ is s. th. $\varprojlim \theta(x_i) = (x_i')$.

Rem: This holds in part. for $X = X_i$, $\varphi_{ij} = id$.

We call $X = \varprojlim X_i$ profinite if X_i is finite, discrete

Thm 1.10 Let X be a top. space. Then t.f.a.e.

(i) X is a profinite space

(ii) X is cp. Hausd. and tot. disc.

(iii) X is cp. Hausd. and admits a basis of clopen sets for its topol.

Pf: (i) $\Rightarrow$ (ii) X profinite, $X = \varprojlim X_i$, X_i fin.

Then by Prop. 1.5, X is cp. Hausd. tot. disc.

(ii) $\Rightarrow$ (iii) Let X be cp. Hausd. tot. disc., W open nbhd of $x \in X$. Show W contains clopen nbhd.