6. NIP groups

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Let T be a complete NIP theory, $\mathcal{U}$ a monster model κ-saturated for κ large enough, strongly inaccessible and regular.

1 Definition and examples of NIP groups

Definition 1.1. Let A be a small set. A *type-definable group* (over A) $(G, \cdot)$ is a group where the set G is given by a type (over A) and where the law "$\cdot$" is definable (over A). A relatively-definable subgroup H of G is a subgroup given by the intersection of the group G and a definable set $\phi(\mathcal{U})$: $H = \phi(\mathcal{U}) \cap G$.

example 1.

- The theory T_{DOAG} of non-trivial divisible ordered abelian groups in the language $\mathcal{L} = \{0, +, -, <\}$ is o-minimal, hence NIP.
- The theory of Presburger arithmetic, i.e. the theory of $(\mathbb{Z}, 0, 1, +, -, <)$ is quasi-o-minimal, hence NIP. That means that every definable subset of $\mathcal{U}$ is a boulean combination of intervals and $\emptyset$-definable sets. Here for example, the formula $\phi(x; y_0, \cdots, y_k) \equiv \ \exists z \ nz = mx + \sum_{i=0}^{k} n_i y_i$ is NIP since it doesn't shattered any set $A = \{a_0, \cdots, a_n\}$ of $n+1$ elements. Indeed, there are $i, j \leq n$ such that $a_i \equiv a_j \mod n$, and no instance of $\phi(x; y)$ can separate them.

We have the following general fact:

Fact 1 (Gurevich-Schmitt)**.** *Any pure ordered abelian group* $(\Gamma, 0, +, <)$ *is NIP.*

example 2.

- The theory T of $(\mathbb{Z}, +)$ is stable, in particular NIP. If $G \models T$ is a model, we may define by a type over $\emptyset$ the subgroup of element divisible by every integer n, namely $\bigcap_{n \in \mathcal{N}} nG$.

As well, this example come from a more general fact, say :

Fact 2. *Any pure* $\mathbb{Z}$*-module is stable (this statement is also true for any ring* R*).*

example 3. • Let $T = RCF$ the theory of real closed fields. Then, $(\mathcal{U}, +)$ is in particular a group. There is an $\emptyset$-type defining the subgroup of infinitesimal $\mathrm{Inf} = \bigcap[-1/k, 1/k]$.

• One can also consider the unite circle $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ of $\mathcal{U}^2$ endowed with the complex multiplication : $(x_0, y_0) \cdot (x_1, y_1) = (x_0x_1 - y_0y_1, x_0y_1 + y_0x_1)$. As well, we may define by a $\emptyset$-type the subgroup of infinitesimal: $\mathrm{Inf}_{S_1} = S_1 \cap ((1, 0) + \mathrm{Inf} \times \mathrm{Inf})$.

2 Baldwin-Saxl condition

Let $(G, \cdot)$ be a type-definable group. Let $\psi(x, y)$ a formula, A be a set of parameters from $\mathcal{U}$ and $\{H_a = \psi(\mathcal{U}, a) \cap G\}_{a\in A}$ a *relatively-definable* family of subgroups of G uniformly given by $\psi(x, y)$.

Theorem 2.1 (Baldwin-Saxl). *There is an integer $N = N_\psi$, depending only on $\psi(x; y)$, such that for any finite set of parameters $B \subset A$, there is a subset B_0 of B of size N such that $\bigcap_{a\in B} H_a = \bigcap_{a\in B_0} H_a$. In other words, the intersection of finitely many H_a is equal to a subintersection of at most N of them.*

Proof. Let N be an integer and assume it doesn't satisfy the conclusion of the theorem. There is a set $B = \{a_0, \cdots, a_N\}$ of parameters such that for every $i \leq N$:

$$\bigcap_{a\in B/\{a_i\}} H_a \supsetneq \bigcap_{a\in B} H_a.$$

Concider $c_{\{i\}}$ an element of the first set but not of the second. For $I \subset N$, we pose $c_I = \prod_{i\in I} c_{\{i\}}$, where the product is in the sens of the group G. Then we have

$$\psi(c_I, a_i) \quad \text{if and only if} \quad c_I \in H_{a_i} \quad \text{if and only if} \quad i \notin I.$$

Hence, the formula $\neg\psi^{\mathrm{opp}}(y; x)$ have a VC-dimension greater than N. By NIP, there is a maximal such N.

□

An easy consequence is that the family of finite intersection of H_a is also uniformly definable, given by the formula $\bigwedge_{i=1}^N \psi(x, y_i)$.

Fact 3. *If T is stable and G is definable, then G satisfies the trivial chain condition, namely for any formula $\psi(x, y)$, there is $N' = N'_\psi$ such that if $(H_a)_{a\in A} = (\psi(\mathcal{U}, a) \cap G)_{a\in A}$ is a sequence of relatively-definable subgroups, then any chain $H_{a_1} \subsetneq H_{a_2} \subsetneq \cdots \subsetneq H_{a_m}$ is of length at most N'.*

Indeed, otherwise take $H_{a_1} \subsetneq H_{a_2} \subsetneq \cdots$ a chain of length ω of such subgroups. Then the formula

$$\exists z\ \psi(z,x) \wedge \neg\psi(z,y),$$

will have the order property, witnessed by $(a_i)_{i<\omega}$ and $(a_j)_{j<\omega}$.

Corollary 2.2. Assume T is stable. There is an integer $N = N_\psi$, depending only on $\psi(x;y)$ such that for any set (finite or infinite) of parameters $B \subset A$, there is a subset B_0 of B of size N such that $\bigcap_{a\in B} H_a = \bigcap_{a\in B_0} H_a$.

Proof. Since the family of finite intersection of H_a for $a \in B$ is uniformly definable, it has a minimal element which is necessary the intersection of all H_a, $a \in B$. □

3 Connected component G^0

Recall that T is NIP.

Definition 3.1. We denote by G^0 the subgroup of G defined as the intersection of all relatively-definable subgroups of G of finite index. In stable theory, it is called the connected component of G.

Let $\psi(x,y)$ a formula, k a positive integer, A be a set of parameters and $\{H_a = \psi(\mathcal{U},a)\cap G\}_{a\in A}$ a family of relatively-definable subgroups. By Baldwin-Saxl condition, any finite intersection of subgroups H_a of index at most k is of index at most k^N, where $N = N_\psi$ as in theorem 2.1 (consider $G/\bigcap_{i<N} H_{a_i} \hookrightarrow \prod G/H_{a_i}$). Then it is also true for an infinite intersection : let $\{g_0,\cdots,g_{k^N}\}$ be a subset of representatives of left class of $\bigcap_{i<\omega} H_{a_i}$, i.e. for all $i,j \leq k^N$, $g_i - g_j \notin \bigcap_i H_{a_i}$. Then, its a subset of representatives for a finite intersection, contradiction. As well we show that such an infinite intersection is actually a finite intersection. Then, we define $G^0_{\psi,k}$ as the intersection of all subgroups of index at most k given by an instance of $\psi(x,y)$. It is of index at most k^N and it is relatively definable. Furthermore, it is invariant under any automorphisme, so it is $\emptyset$-relatively definable. Now we define $G^0_\psi = \bigcap_{k<\omega} G^0_{\psi,k}$. It is type-definable over $\emptyset$, of bounded index (at most $2^{\aleph_0}$). Then, $G^0 = \bigcap_\psi G^0_\psi$. We have proved the following:

Proposition 3.2. *The subgroup G_0 is type-definable over $\emptyset$ and of bounded index (at most $2^{|T|}$). It is also a normal subgroup.*

The last point simply come from the fact the set of subgroup of finite index is stable by conjugation.

example 4.
- Let $T = \mathrm{Th}(\mathbb{Z},+)$, and $G \models T$. Then $G^0 = \bigcap_{n\in\mathcal{N}} nG$, since G has exactly one subgroup of index n namely nG.

- Let $T = RCF$, $R \models T$. Then $R^0 = R$ since R doesn't have any non-trivial subgroup of finite index : if H is a subgroup of index n, $nR = R \subset H$. Contradiction.
- Consider $S_1(R)$ as before. As well, it doesn't have any subgroup of finite index. Then $S_1^0 = S_1$. More generally, every divisible abelian group is connected.

4 Strongly connected component G^{00}

Definition 4.1. We say that a type-definable subgroup H of G is *of bounded index* if $[G(\mathcal{U}), H(\mathcal{U})]$ is lower than κ, the cardinal of saturation of $\mathcal{U}$. Equivalently, that means for some model M, the index of $H(M')$ doesn't grow for $M \prec M'$.

Note that if H is definable, the index of H is bounded if and only if it is finite, by saturation.

example 5.

- Consider $R \prec \mathcal{U} \models$ RCF. Then the index of Inf is unbounded, as the type $x > R$ give us in an extension $R' \succ R$ a new left-class for Inf. An other way to see that : we can find in $\mathcal{U}$ a sequence $(a_i)_{i<\kappa}$ such that $a_i + 1 < a_{i+1}$. Hence, each a_i belong to a different left-class of Inf.

- Consider $G \prec \mathcal{U} \models \text{Th}(\mathbb{Z}, +)$, and $H = \bigcap_n nG$. Assume H is non-trivial. Then G/H is isomorphe to the profinite group $\hat{\mathbb{Z}}$, of cardinality 2^ω. In particular, H is of bounded index.
- Now, consider the circle S_1. Every elements in $S_1(\mathcal{U})$ have a standard part (which doesn't hold for R !) and thus $S_1(\mathcal{U}) = \bigcup_{r\in S_1(\mathbb{R})} r\text{Inf}_{S_1}$. Better : $S_1(\mathcal{U})/\text{Inf}_{S_1} \simeq S_1(\mathbb{R})$, and the index of $S_1(\mathcal{U})$ is $2^{\aleph_0}$. In particular it's bounded.

Definition 4.2. We denote by G^{00} the intersection of type-definable subgroups of G of bounded index. If G^{00} is itself a subgroup of bounded index, we say that G^{00} *exists*, and it is *called the strongly connected component* of G.

The subgroup G^{00} is always an $\emptyset$-type-definable normal subgroup, since it is stable by any automorphisme. Clearly, we have $G^{00} \subset G^0$. We give now a classical result from stable group:

Theorem 4.3. *Let A be small, G a A-type-definable group. If T is stable, every A-type-definable subgroup H is the intersection of at most $|T|$-many A-definable subgroups.*

In particular, in stable theory, we have $G^{00} = G^0$ since G^{00} is an intersection of definable subgroup of finite (equivalently bounded) index.

example 6. Let $T = \mathrm{Th}(\mathbb{Z}, +)$, and $G = (\mathcal{U}, +)$. Then we know that $G^{00} = G^0 = \bigcap_{n\in\mathcal{N}} nG$. The quotient G/G^{00} is a profinite group, isomorphe to $\hat{\mathbb{Z}}$. In particular, we have $[G : G^{00}] = 2^{\aleph_0}$.

Lemma 4.4. *Let G be a type-definable group, then every subgroup H is the intersection of subgroups given by a countable types.*

Proof. To simplify, we assume G is definable by a formula $\psi(x)$. One can adapte the proof for type-definible group. Let $p(x)$ be the type defining H. We may assume $p(x)$ is close by conjunction. Take $\phi_0(x) \in p$ a formula in p. We define a sequence $(\phi_n(x))_{n<\omega}$ of formulas of $p(x)$ as follow :

- $\phi_0(x) = \phi(x) \wedge \phi(x^{-1}) \in p(x)$
- if $\phi_n(x) \in p(x)$ is given, by compactness, there is $\phi'_{n+1}(x) \in p(x)$ such that:
$$\forall x, y \quad \phi'_{n+1}(x) \wedge \phi'_{n+1}(y) \to \phi_n(x \cdot y)$$
Pose $\phi_{n+1}(x) = \phi'_{n+1}(x) \wedge \phi'_{n+1}(x^{-1}) \in p(x)$.

Then the countable type $p_\phi(x) = \{\psi(x)\} \cup \{\phi_n(x)\}_{n\omega} \subset p(x)$ define a subgroup G_ϕ containing H. It is clear that $H = \bigcap_{\phi\in p} G_\phi$. □

Lemma 4.5. *Let G an $\emptyset$-definable group. Let $b \equiv a$ two elements of $\mathcal{U}$ with the same type over $\emptyset$. Let $p_0(x, y)$ be a type such that $p_0(x, a)$ define a subgroup H_a of G of bounded index. Then $p_0(x, b)$ also defined a subgroup H_b of G of bounded index.*

Proof. It's clear that $p_0(x, b)$ define a subset of G. Let $\phi(x)$ a formula of $p_0(x)$. By compactness, we find $\phi_0(x, y) \in p(x, y)$ such that:

$$\forall x, y \ \phi_0(x, a) \wedge \phi_0(y, a) \to \phi(x \cdot y).$$

The same formula holds with the parameter b. It follow that $p_0(x, b)$ also define a subgroup of G. For the index, consider an automorphisme σ of $\mathcal{U}$ sending a to b. By $\emptyset$-definability of G, $\sigma(G) = G$. It's clear that any left class of H_a is send to a left class of H_b. Hence, index of H_a and H_b in G must be equal. □

Theorem 4.6 (Shelah)**.** *Assume T is NIP. Let G an $\emptyset$-type-definable group. Then G^{00} exists.*

example 7.
- Let T be the theory of real closed fields, $\mathcal{R}$ the monster model. Then $\mathcal{R}^{00}$ exists, and it is invariant by every automorphism. In particular by every multiplication by a element a, i.e, it is an ideal of $\mathcal{R}$. Since it is of bounded index, $\mathcal{R}^{00}$ can't be $\{0\}$: $\mathcal{R}^{00} = \mathcal{R}$.

- Same assumptions, we consider now Inf, the subgroup of infinitesimals. This is also a strongly connected group: $\text{Inf} = \text{Inf}^{00}$. Consider in $\mathcal{R}$ the ring $O = \bigcup_{k\in\mathbb{N}}[-k,k]$ (the convex hull of the reals). It is a valuation ring (for all element x, either x or x^{-1} is in O), we note v the corresponding valuation. For any a in O, $a\text{Inf}^{00} \subseteq \text{Inf}^{00}$ as $a\text{Inf}^{00}$ is strongly connected. Hence, Inf^{00} is an ideal of O. In particular, it is convex (easily come from the fact R is ordered). If it is proper, take $r \in \text{Inf} \setminus \text{Inf}^{00}$. Then $\text{Inf}^{00} \subseteq rO \subseteq \text{Inf}$. In a extension in the language of valued field, with enough saturation, you find unboundly many such intermediate ideal. This contradict the fact that Inf^{00} is of bounded index.

- Same assumptions, we consider the circle S_1. The extension R_{an} of real closed fields endowed with all analytic fonctions is still o-minimal, hence NIP. We may consider in this theory the definable isomorphisme sending Inf to Inf_{S_1}. By the previous point, Inf_{S_1} is strongly-connected. Since by Example 5 we have $S_1^{00} \subseteq \text{Inf}_{S_1}$, we deduce that $S_1^{00} = \text{Inf}_{S_1}$.

Proof. Assume G^{00} doesn't exist. Then, in particular there is an unbounded sequence $(H_i)_{i<\kappa}$ of type-definable subgroups of bounded index. As in the fact 4.4, each H_i is the intersection of subgroup given by countable type, necessary of bounded index since they contain H_i.

Assume there is boundly many subgroup of G given by a countable type. Then, the number of subgroup H_i as above will be also bounded, contradiction. Then, without lost, we may assume H_i are each given by a countable type. By pigeon-hole principal, there is countable types $p_0(x,y)$, where y stand for parameters (can be of infinite length), such that unboundly-many H_i is given by $p_0(x,a_i)$, for some $a_i \in \mathcal{U}$ with $(\text{tp}(a_i))_{i<\kappa}$ constant.

Then, we may assume for every $a \in \mathcal{U}$, $p_0(x,a)$ defines a subgroup of G. Indeed, for $\phi(x) \in G$, there is $\phi_0(x,y) \in p_0(x,y)$ such that $\phi_0(x,a_0) \to \phi(x)$ and it's harmless to add $\phi_0(x,y) \to \phi(x)$ to $p_0(x;y)$, since $(\text{tp}(a_i))_{i<\kappa}$ is constant. Doing the same for every formula in $p(x)$ and every instance of $p_0(x,y)$ will be a subset of G. Similarly, we add formulas to $p_0(x,y)$ s.t. every instance give a subset stable by $+$, i.e. a subgroup. By Ramsey and compactness, we find an indiscernible sequence $(b_i)_{i<\omega}$, such that $\text{tp}(b_0) = \text{tp}(a_0)$. By fact 4.5, all subgroup $K_i = p(\mathcal{U}, b_i)$ for $i < \omega$ have same bounded index.

We consider now this sequence $(K_i)_{i<\omega}$. □

Claim 1. *For every $i < \omega$, the subgroup K_i doesn't contain* $\bigcap_{j<\omega\ j\neq i} K_j$.

Proof. Assume for a contradiction that for some $i < \omega$, K_i contain $\bigcap_{j<\omega\ j\neq i} K_j$.

We may insert a sequence $(\tilde{b}_j)_{j<\kappa}$ such that $(b_j)_{j<i}(\tilde{b}_j)_{j<\kappa}(b_j)_{i<j<\omega}$ is still indiscernible. Then, by compactness and indiscernibility, every $\tilde{K}_j := p_0(\mathcal{U}, \tilde{b}_j)$ for $j \in \kappa$ contain $\bigcap_{j<\omega \ j\neq i} K_j$. A contradiction since only boundly-many distinct subgroup can contain a subgroup of bounded index. $\square$

Let $c_{\{0\}} \in \bigcap_{j\neq 0} K_j \setminus K_0$. There is a formula $\phi(x, y) \in p_0(x, y)$ such that $\models \neg\phi(c_{\{0\}}, b_0)$. By indiscernability and compactness, we find for all $i < \omega$ an element $c_{\{i\}} \in \bigcap_{j\neq i} K_j \setminus K_i$ satisfying $\models \neg\phi(c_{\{i\}}, b_i)$.

Then, $\models \phi(c_{\{i\}}, b_j)$ if and only if $i \neq j$.

By compactness, there is a formula $\theta(x, y)$ such that

$$\bigwedge_{i=0}^{2} \theta(x_i, y) \rightarrow \phi(x_0 \cdot x_1 \cdot x_2, y).$$

Claim 2. *The formula $\theta(x, y)$ has IP.*

Proof. For $I \subset \mathbb{N}$ a finite subset, let $c_I := \prod_{i\in I} c_{\{i\}}$. Then $\models \neg\theta(c_I, b_i)$ if and only if $i \in I$. Indeed, if $i \notin I$, then $c_I \in K_i$ and in particular $\models \theta(c_I, b_i)$. Reciprocally, if $i \in I$, let's write $c_i = \tilde{c}_0 \cdot k_I \cdot \tilde{c}_1$ where $\tilde{c}_0 = (\prod_{j\in I \ j<i} c_{\{j\}})^{-1} \in K_i$ and $\tilde{c}_1 = (\prod_{j\in I \ j>i} c_{\{j\}})^{-1} \in K_i$. Since $\phi(c_{\{i\}}, b_i)$ doesn't hold, $\theta(c_I, b_i)$ doesn't hold either. $\square$