5. Introduction to NIP theories

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Let T be a complete theory in a language $\mathcal{L}$ and $\mathcal{U} \models T$ a monster model. For a set I, we will use the notation $J \Subset I$ for "J is a finite subset of I". We will make no difference between elements (resp. single variables) and tuples (resp. tuples of variables). For exemple, if $b = (b_1, \cdots, b_n)$ is a tuples of elements of $\mathcal{U}$, we will simply write $b \in \mathcal{U}$.

1 Definition of NIP

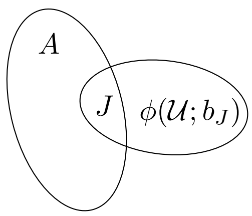
Let x and y be two tuples of variables and $\phi(x; y)$ a partitioned formula. That means we make a distinction between variables before and variables after the semicolon (x are object variables, y are parameters variables). When we write $\phi(x_0, \cdots, x_n; y_0, \cdots, y_m)$, we will automatically assume that variables x_i and y_j are of size one.

Definition 1.1. Let $A = \{a_i\}_{i \in \lambda}$ be a set of $|x|$-tuples of size λ. We say A is *shattered* (*zerschmettert*) by $\phi(x; y)$ if there is a set $\{b_I\}_{I \subseteq \lambda}$ of $|y|$-tuples such that:

$$\mathcal{U} \models \phi(a_i; b_I) \quad \text{if and only if} \quad i \in I.$$

It's the exact same thing to say that there is a set $\{b_J\}_{J \subseteq I}$ of $|y|$-tuples such that for every $a \in A$:

$$A \cap \phi(\mathcal{U}; b_J) = J$$



Remark 1.

- *It's a property of the equivalence class of $\phi(x; y)$: if $T \models \phi(x; y) \simeq \psi(x; y)$, A is shattered by $\phi(x; y)$ if and only if A is shattered by $\psi(x; y)$.*

- *Futhermore, it's also clear that A is shattered by $\phi(x;y)$ if and only if A is shattered by $\neg\phi(x;y)$.*

The following partial type over A:

$$\pi\left((y_I)_{I\subseteq\lambda}\right) = \{\phi(a_i;y_I) \mid i\in I\} \cup \{\neg\phi(a_i;y_I) \mid i\notin I\}$$

says "$\{y_I\}_{I\subseteq\lambda}$ witness that A is shattered by $\phi(x;y)$".

By compactness, it's clear that A is shattered by $\phi(x;y)$ if and only if every finite part X of A is shattered by $\phi(x;y)$. Note that if X is finite, "X is shattered by $\phi(x;y)$" can be described by one formula. Thus, "A is shattered by $\phi(x;y)$" can be written as a partial type:

$$\{\text{"}X \text{ is shattered by } \phi(x;y)\text{"} \mid X \Subset A\}.$$

By compacteness again, if an infinite set is shattered by $\phi(x;y)$, then for any cardinal λ, a set of cardinality λ is shattered by $\phi(x;y)$. If no infinite set is shattered by $\phi(x;y)$, then there is maximal integer n such that a set of size n is shattered by $\phi(x;y)$. Now we can give the following definition:

Definition 1.2. We define the VC-*dimension* of $\phi(x;y)$ as follow:

$$\mathrm{VC}(\phi(x;y)) = \begin{cases} \infty & \text{if } \phi(x;y) \text{ shatter an infinite set,} \\ \text{n} & \text{where } n \text{ maximal such that } \phi(x;y) \text{ shatter a set of size } n. \end{cases}$$

A formula is say NIP (*Non Independence Property*) if its VC-dimension is finite (i.e. $\phi(x;y)$ does not shatter a infinite set). Otherwise, we say that $\phi(x;y)$ has IP.

A theory is say NIP if every formula $\phi(x;y)\in\mathcal{L}$ is NIP. Note that this is equivalente to say every formula $\phi(x;y\,\hat{}\,b)\in\mathcal{L}(\mathcal{U})$ is NIP, as $\phi(x;y\,\hat{}\,b)$ is NIP if and only if $\phi(x;y\,\hat{}\,z)\in\mathcal{L}$ is NIP.

Remark 2. *Recall that a formula $\phi(x;y)$ is stable if it doesn't have the order property, which says that there are two sets $\{a_i\}_{i\in\omega}$ and $\{b_j\}_{j\in\omega}$ such that*

$$\mathcal{U}\models\phi(a_i;b_j) \quad \textit{if and only if} \;\; i\le j.$$

It's clear that an IP formula has the order property : if $\{a_i\}_{i\in\omega}$ and $\{b_I\}_{I\subseteq\omega}$ witness $\phi(x;y)$ is IP, take $b_j = b_{I_j}$ where $I_j = \{a_i\}_{i\le j}$. Thus, a stable theory is in particular NIP.
An another proof is obtained by coumpting types: assume a formula $\phi(x;y)$ has IP. For any cardinal λ, there is a set $A=\{a_i\}_{i\in\lambda}$ shattered by $\phi(x;y)$, witnessed by some $\{b_I\}_{I\subseteq\omega}$. It's clear that $\mathrm{tp}(b_I/A)$ for $I\subseteq\omega$ are pairwise distinct. Hence for any λ, there is a set A of cardinality λ such that $S(A)\ge 2^\lambda$, this contradicts stability.

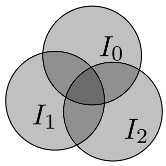
example. • Let $T = DLO$ and $\phi(x; y) = x \leq y$. Then $\phi(x; y)$ doesn't shatter any set of size 2. Indeed, take $A = \{a_0, a_1\}$ with $a_0 \leq a_1$, there is no $b_{\{1\}} \in \mathcal{U}$ such that $\phi(\mathcal{U}; b_{\{1\}}) \cap A = \{a_1\}$. Thus, $\mathrm{VC}(\phi(x; y)) = 1$. As well, if $\psi(x; y_0, y_1) = y_0 \leq x \leq y_1$, we can easily prove that $\mathrm{VC}(\psi(x; y_0, y_1)) = 2$.

- Let $T = \mathrm{Th}(\{\mathbb{Z}, 0, 1, +, .\})$. Then $\phi(x; y) =$ "x divides y" has IP: consider $A = \{p_i\}_{i<\omega}$ the set of prime, and for $I \Subset \omega$, $b_I = \prod_{i \in I} p_i$. Hence any finite part of A is shattered by $\phi(x; y)$ and we are done.

- Let T the theory of random graphs in the language $\mathcal{L} = \{R\}$, and let $\phi(x; y) = xRy$. Then, every small set is shattered by $\phi(x; y)$!

Notation. Let $\phi(x; y)$ a partitioned formula. We pose $\phi^{\mathrm{opp}}(y; x) = \phi(x; y)$. The formula is the same, but the role of y and x are inverted.

example. • Take again $T = DLO$. Then $\psi^{\mathrm{opp}}(y_0, y_1; x) = y_0 \leq x \leq y_1$ has also VC-dimension 2. That's mean we can't find three intervals I_0, I_1, I_2 such that $I_0^{\epsilon_0} \cap I_1^{\epsilon_1} \cap I_2^{\epsilon_2}$ is not empty for every $\epsilon_0, \epsilon_1, \epsilon_2 \in \{0, 1\}$.



- Take T the theory of equality. Then $\phi(x; y_0, y_1, y_2) = (x = y_0 \vee x = y_1 \vee x = y_2)$ has VC-dimension 3, but $\phi^{\mathrm{opp}}(y_0, y_1, y_2; x)$ only has VC-dimension 2.

Lemma 1.3. *A formula $\phi(x; y)$ has IP if and only if $\phi^{\mathrm{opp}}(y; x)$ has IP.*

Proof. Assume $\phi(x; y)$ has IP. By the discussion above, $\phi(x; y)$ shatter in particular a set $A = \{a_i\}_{i \in \mathfrak{P}(\omega)}$ of size $2^\omega = |\mathfrak{P}(\omega)|$. Let $\{b_I\}_{I \subseteq \mathfrak{P}(\omega)}$ witness that. For $j < \omega$, we define $I_j = \{X \in \mathfrak{P}(\omega) \mid j \in X\}$. Let $J \subseteq \omega$. Then

$$\phi(a_J; b_{I_j}) \Leftrightarrow J \in I_j \Leftrightarrow j \in J.$$

This show that the set $B = \{b_{I_j}\}_{j<\omega}$ is shattered by $\phi^{\mathrm{opp}}(y; x)$. Therefore $\phi^{\mathrm{opp}}(y; x)$ has IP. □

2 Characterizations

2.1 Recall on indiscernible sequences

Definition 2.1. Let $\mathcal{I}$ be a ordened set of index and $(a_i)_{i \in \mathcal{I}}$ be a sequence of tuples of same lenght. Let Δ be a set of formulas. We say $(a_i)_{i \in \mathcal{I}}$ is

Δ-*indiscernible* if for every ordered-tuples $i_0 < \cdots < i_n \in \mathcal{I}$ and $j_0 < \cdots < j_n \in \mathcal{I}$ and every $\phi(x_1, \cdots, x_n) \in \Delta$, we have:

$$\phi(a_{i_0}, \cdots, a_{i_n}) \leftrightarrow \phi(a_{j_0}, \cdots, a_{j_n}).$$

Let A a subset of $\mathcal{U}$. A sequence is say *indiscernible* (resp. *indiscernible over* A) if it is $\mathcal{L}$-indiscernible (resp. $\mathcal{L}(A)$-*indiscernible*).

We define the *Ehrenfeucht-Mostowski type* of $I = (a_i)_{i\in\mathcal{I}}$, noted $\mathrm{EM}(I)$, as the maximal set of $\mathcal{L}(\mathcal{U})$-formula such that I is $\mathrm{EM}(I)$-indiscernible and $I \models \mathrm{EM}(I)$. An explicite definition is given by

$$\{\phi(x_1, \cdots, x_n) \in \mathcal{L}(\mathcal{U}) \mid \forall n \in \mathbb{N},\ \forall\ i_0 < \cdots < i_n \in \mathcal{I},\ \phi(a_{i_0}, \cdots, a_{i_n})\}.$$

It's a partial type, and it is global if and only if I is $\mathcal{U}$-indiscernible.

Lemma 2.2. *Let A be any small set and $J = (\alpha_i)_{i\in\mathcal{I}}$ a sequence. There is an A-indiscernible sequence $I = (a_i)_{i\in\mathcal{I}}$ such that $\mathrm{EM}(J) \subseteq \mathrm{EM}(I)$.*

Proof. By Ramsey's theorem and compactness. □

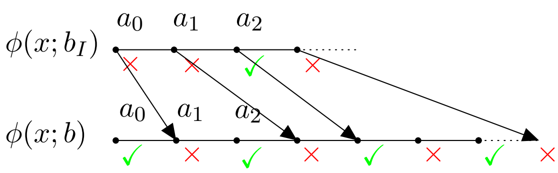
2.2 Characterizations

Lemma 2.3. *A formula $\phi(x; y)$ has IP if and only if there is an indiscernible sequence $(a_i)_{i<\omega}$ of $|x|$-tuple and a $|y|$-tuple b such that*

$$\models \phi(a_i; b) \quad \Leftrightarrow \quad i \text{ is even.}$$

Proof. ($\Leftarrow$): Assume there is $(a_i)_{i<\omega}$ and b as above. Let $I \subseteq \omega$. For $i < \omega$, we note

$$j_i = \begin{cases} 2i & \text{if } i \in I, \\ 2i+1 & \text{otherwise.} \end{cases}$$



By indiscernability of the sequence, there is an automorphisme σ such that $\sigma(a_i) = \sigma(a_{j_i})$. Let $b_I = \sigma^{-1}(b)$. Then:

$$\phi(a_i, b_I) \Leftrightarrow \phi(a_{j_i}, b) \Leftrightarrow j_i \text{ is even} \Leftrightarrow i \in I.$$

($\Rightarrow$) Assume a set $A = \{\alpha_i\}_{i<\omega}$ is shattered by $\phi(x; y)$. By Ramsey and compactness, there is a sequence $(a_i)_{i<\omega}$ such that its EM-type contain the EM-type of $(\alpha_i)_{i<\omega}$. In particular, $(a_i)_{i<\omega}$ is shattered by $\phi(x; y)$, witnessed by some $(b_I)_{I\subseteq\omega}$: indeed for any integer n and any index $i_0 < \cdots < i_n$, "$\{a_{i_0}, \cdots, a_{i_n}\}$ is shattered by $\phi(x; y)$" is true, and the corresponding formula belong to the EM-type. Take now $b = b_{2\mathbb{N}}$. □

Let $\phi(x;y)$ be a NIP formula, $I = (a_i)_{i<\lambda}$ an indiscernible sequence and $b \in \mathcal{U}$. By compactness, there is a maximal integer n, called the alternation rank and denoted $\mathrm{alt}(\phi(x;b), I)$, such that we can find $i_0 < \cdots < i_n < \lambda$ with $\neg\phi(a_{i_k};b) \leftrightarrow \phi(a_{i_{k+1}};b)$ for all $k < n$.

Futhermore, by compactness again, you can find an integer n_ϕ, depending only on $\phi(x;y)$, such that there is no indiscernible sequence $(a_i)_{i\leq n_\phi}$ and b such that

$$\neg\phi(a_k;b) \leftrightarrow \phi(a_{k+1};b) \quad \text{for every } k < n_\phi.$$

In particular, for every indiscernible sequence $I = (a_i)_{i<\lambda}$,

$$\mathrm{alt}(\phi(x;b), I) < n_\phi.$$

Remark 3. *In an NIP theory, every indiscernible sequence $I = (a_i)_{i<\lambda}$ has a "limit behavior" in the sense that the truth value of $\phi(a_i, b)$ is constant for i big enough. We may define the limit type of I by:*

$$\lim(I) = \{\phi(x;b) \in \mathcal{L}(\mathcal{U}) \mid \models \phi(a_i;b) \textit{ for cofinally many } i\}.$$

It's clearly finitely consistent.

Corollary 2.4. A boolean combination of NIP formulas is NIP.

Proof. We already know that a formula $\phi(x;y)$ is NIP if and only if $\neg\phi(x;y)$ is NIP. Let $\phi_0(x;y)$ and $\phi_1(x;y)$ two formulas. It's enough to show that $\phi_0(x;y) \wedge \phi_1(x;y)$ is NIP. By the previous lemma, it's equivalent to show that for every indiscernible sequence $(a_i)_{i\in\omega}$ and $b \in \mathcal{U}$, $\phi_0(a_i;b) \wedge \phi_1(a_i;b)$ is constant for i big enough. But there is i_0 (resp. i_1) such that $\phi_0(a_i;b)$ (resp. $\phi_1(a_i;b)$) is constant for $i > i_0$ (resp. $i > i_1$). Take $i = \max(i_0, i_1)$. □

We need to improve this argument a little bit:

Proposition 2.5. *Fix λ a infinite cardinal and let $\phi(x;y)$ be a formula. The following are equivalent :*

1. *$\phi(x;y)$ is NIP,*
2. *for every indiscernible sequence $(a_i)_{i<\lambda}$ and every b, there is $i_0 < \lambda$ such that the truth value of $\phi(a_i, b)$ is constant for $i > i_0$.*

Proof. $\neg(1) \Rightarrow \neg(2)$. Assume $\phi(x;y)$ is IP. By compactness, the partial type

$$\pi((x_i)_{i<\lambda}, y) = \{\phi(x_i, y) \leftrightarrow \neg\phi(x_{i+1};y)\}_{i\in\lambda} \cup \{\text{"}(x_i)_{i<\lambda} \text{ is indiscernible"}\}$$

is consistent. Thus, we have $\neg(2)$
$\neg(2) \Rightarrow \neg(1)$. Assume not (2). We construct by induction an increasing sequence of index $(i_n)_{n<\omega}$ such that

$$\mathcal{U} \models \phi(a_{i_n}) \leftrightarrow \neg\phi(a_{i_{n+1}}) \quad \text{for every } n < \omega.$$

Hence, $\phi(x;y)$ has IP by the previous caracterization. □

Theorem 2.6. *The following are equivalent :*

1. *the theory T is NIP*
2. *every formula $\phi(x;y)$ with $|x| = 1$ are NIP*
3. *every formula $\phi(x;y)$ with $|y| = 1$ are NIP*

By lemma 1.3, (2) $\Leftrightarrow$ (3) is clear. We will prove (3) $\Rightarrow$ (1).

Claim 1. *Let $(a_i)_{i<|T|^+}$ be an indiscernible sequence and $b \in \mathcal{U}$ with $|b| = 1$. Then there is $\alpha < |T|^+$ such that $(a_i)_{\alpha<i<|T|^+}$ is indiscernible over b.*

Proof. Assume not. For any $\alpha < |T|^+$, there is a formula $\phi_\alpha(x_1, \cdots, x_{k(\alpha)}; y)$ with $|y| = 1$ such that we can find $\alpha < i_1 < \cdots < i_{k(\alpha)} < |T|^+$ and $\alpha < j_1 < \cdots < j_{k(\alpha)} < |T|^+$ with:

$$\mathcal{U} \models \phi_\alpha(a_{i_1}, \cdots, a_{i_{k(\alpha)}}; b) \wedge \neg\phi_\alpha(a_{j_1}, \cdots, a_{j_{k(\alpha)}}; b)$$

By pigeon hole principal, there is a formula $\phi(x_1, \cdots, x_k; y)$ and cofinally many value of α such that $\phi_\alpha(x_1, \cdots, x_{k(\alpha)}; y) = \phi(x_1, \cdots, x_k; y)$. By induction, we construct a sequence of index $(i_1^n, \cdots, i_k^n)_{n<\omega}$ such that

- $i_k^n < i_1^{n+1} < \cdots < i_k^{n+1}$ for all $n \geq 0$,
- $\mathcal{U} \models \phi(a_{i_1^n}, \cdots, a_{i_k^n}; b)$ if and only if n is even.

As the sequence $(a_{i_1^n}, \cdots, a_{i_k^n})_{n<\omega}$ is indiscernible, this contradicts the assomption that $\phi(x;y)$ is NIP. □

Let $\phi(x; y_1, \cdots, y_n)$ a formula, $(a_i)_{i<|T|^+}$ a indiscernible sequence of $|x|$-tuples and $b = (b_1, \cdots, b_n) \in \mathcal{U}$ a n-tuple. By induction, there is α_n such that for all $i > \alpha_n$, $(a_i \,\hat{}\, (b_1, \cdots, b_{n-1}))$ is indiscernible over b_n. The truth value of $\phi(a_i; b)$ is constant for $i > \alpha_n$. By the proposition 2.5, $\phi(x;y)$ is NIP.

example.

- Take $T = DLO$. By quantifier elimination, every formula $\phi(x; y_1, \cdots, y_n)$ with $|x| = 1$ is equivalent to a boolean combination of $x < y_i$, which is NIP. By the theorem 2.6, DLO is NIP.
- Assume more generaly that T is o-minimal. Let $\phi(x;y)$ be a formula with $|x| = 1$. By definition of o-minimality, each instance $\phi(x;b)$ (for $b \in \mathbb{U}$) is a union of intervals. However, we need to be more precise to conclude: by compactness there is an integer n such that each instance is an union of at most n intervals. Then, $\mathrm{VC}(\phi(x;y)) \leq 2n$. By 2.6, any o-minimal theory is NIP.

3 Baldwin-Saxl condition

Assume T is NIP, let $(G, \cdot)$ be a definable group. That means that G as a set is defined by a formula $\phi(x, b)$ with parameter $b \in \mathcal{U}$, and the partial fonction "$\cdot$" is also defined by a formula $\theta(x, y, c)$ with some parameter $c \in \mathcal{U}$. Let A be a set of parameters from $\mathcal{U}$ and $\{H_a\}_{a \in A}$ a uniformly definable set of subgroupz of G, say $H_a = \psi(x; a)$.

Theorem 3.1 (Baldwin-Saxl)**.** *There is an integer $N = N_\psi$, depending only on $\psi(x; y)$ such that for any $B \Subset A$, there is a subset B_0 of B of size N such that $\bigcap_{a \in B} H_a = \bigcap_{a \in B_0} H_a$. In other words, the intersection of finitely many H_a is equal to a subintersection of at most N of them.*

Proof. Let N be an integer and assume it doesn't satisfy the conclusion of the theorem. There is a set $B = \{a_0, \cdots, a_N\}$ of parameters such that for every $i \leq N$:

$$\bigcap_{a \in B/\{a_i\}} H_a \supsetneq \bigcap_{a \in B} H_a.$$

Concider $c_{\{i\}}$ an element of the first set but not of the second. For $I \subset N$, we pose $c_I = \prod_{i \in I} c_{\{i\}}$, where the product is in the sense of the group G. Then we have

$$c_I \in H_{a_i} \quad \text{if and only if} \quad i \in I.$$

The formula $\neg\psi^{\text{opp}}(y; x)$ have a VC-dimension greater than N. By NIP, there is a maximal such N.

□

We may assume any instance of $\psi(x; y)$ is a subgroup of G. Indeed, it's harmless to concider the formula $\psi(x; y) \wedge (\forall x_1, x_2\ (\psi(x_1; y) \wedge \psi(x_2; y)) \rightarrow (\phi(x_1, b) \wedge \phi(x_1 \cdot x_2, b)))$.

Corollary 3.2.