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The Theorem of Kaplan-Scanlon-Wagner

K char p. $\wp : (K,+) \to (K,+)$ $x \mapsto x^p - x$
"Artin-Schreier map"

$\wp$ is a homom. with $\ker(\wp) = \mathbb{F}_p$

Fact: $K/\wp(K) \cong \mathrm{Hom}(\mathrm{Gal}_K, \mathbb{Z}/p\mathbb{Z})$ corresponds to the Galois extensions of K of degree p.

Fact (Kummer): $\ell \neq p$ prime, $\mu_\ell \subseteq K$ ℓ^{th} roots of unity then $K^\times/(K^\times)^\ell \cong \mathrm{Hom}(\mathrm{Gal}_K, \mathbb{Z}/\ell\mathbb{Z})$

Macintyre: K ω-stable field, then $\wp(K) = K$, $(K^\times)^\ell = K^\times$, $\mu_\ell \subseteq K^\times$. Same holds for finite ext. of K, so K is alg. closed. (since K is also perfect).

In NIP: $(\mathbb{Q}_p, +, \cdot)$ is NIP but $(\mathbb{Q}_p^\times)^\ell \neq \mathbb{Q}_p^\times$ for any prime ℓ.

Theorem [KSW]: Suppose $(K, +, \cdot)$ is an infinite NIP of char. $p > 0$. Then $\wp(K) = K$.

Corollary 1: K NIP, $\mathrm{char}(K) = p > 0$. Then K has no Galois ext. of degree p.

Corollary 2: K NIP, $\mathrm{char}(K) = p > 0$. Then K has no Galois ext. of degree div. by p.
(eq: no sep. ext. of degree p).

Pf: $L \supseteq K$ sep., $p \mid [L:K]$. L' = normal closure of L
$\exists$ $K \leq K' \leq L'$ s.th. $K' \leq L'$ is Gal. of degree p.
But $(K', +, \cdot)$ is interpreted in K, so K' is NIP
$\lightning$ Cor. 1 $\boxtimes$

Corollary 3: K inf. NIP field of char $p>0$

$\Rightarrow \mathbb{F}_p^{alg} \cap K = \mathbb{F}_p^{alg}$.

Proof: $k := \mathbb{F}_p^{alg} \cap K$

K is AS-closed $\Rightarrow$ k is AS-closed

$\Rightarrow$ k is infinite

$\overset{\text{Lang-Weil}}{\Rightarrow}$ k is PAC

$\overset{\text{Duret}}{\Rightarrow}$ k has IP or is sep. closed

$\overset{\text{Duret}}{\Rightarrow}$ k is sep. closed, so $k = \mathbb{F}_p^{alg}$ $\boxtimes$

Proof of Thm: Wlog, K is $\aleph_0$-saturated.

By Baldwin-Saxl applied to $(a\wp(K))_{a\in K}$

[note that $a\wp(K) \leq K$ additive subgrp. f.a. $a \in K$].

ex. some N s.th. f.a. $(a_0,..,a_m) \in K^{m+1}$ ex. some

subtuple $(a_{i_0},..,a_{i_{N-1}})$ with $\bigcap_{j=0}^{m} a_i \wp(K) = \bigcap_{k=0}^{N-1} a_k \wp(K)$ $(*)$

Define $G_{\bar a} = \{\bar x \mid a_0 \wp(x_0) = \ldots = a_m \wp(x_m)\} \leq K^{m+1}$

By $(*)$, $pr: G_{\bar a} \to G_{\bar a'}$ is surjective

$(x_0,..,x_m) \mapsto (x_{i_0},..,x_{i_{N-1}})$

whenever $m > N$.

Let $k = K^{p^\infty} = \bigcap_{n\geq 0} K^{p^n}$ perfect subfield of K. Choose

$\bar b = (b_0,..,b_N) \in k^{N+1}$ alg. indep. (so trdeg$(\bar b) = N+1$,

ex. by $\aleph_0$-saturation of K).

Then $pr: G_{\bar b} \twoheadrightarrow G_{\bar b'}$ is surjective where wma

$\bar b' = (b_1,..,b_N)$

<u>Claim 1</u>: If $\bar a \in k^{<\omega}$ alg independent, then $G_{\bar a}$ is

isomorphic $/k$ to the additive group of the

field as algebraic group. (Get $G_{\bar a}(\bar K) \cong (\bar K, +)$)

[proof at the end if there is time]

Then pr induces $G_{\bar b}(K) \overset{pr}{\twoheadrightarrow} G_{\bar b'}(K)$ a surj. hom.

$$\begin{array}{ccc} G_{\bar b}(K) & \overset{pr}{\twoheadrightarrow} & G_{\bar b'}(K) \\ \wr\wr & & \wr\wr \\ (K,+) & \twoheadrightarrow & (K,+) \end{array}$$

$\theta : (k,+) \twoheadrightarrow (k,+)$ with $|\ker(\theta)| = |\ker \wp| = p$.

<u>Claim 2</u>: $\theta(t) = a\wp(ct)^{p^m}$ for some $a, c \in k^\times$, $m \geq 0$

Hence, if $x \in k$, there ex. $y \in k$ s.th. $a x^{p^m} = a\wp(y)^{p^m}$

hence $x = \wp(y)$

[so done if we prove Claim 1 & 2]

Proof of Claim 2.

<u>Fact</u>: $\theta(t) = \sum_{m=0}^{s} d_m t^{p^m}$ for some $d_m \in k$.

for any θ an alg. homom. $(k,+) \xrightarrow{\theta} (k,+)$

in ~~char~~ $\mathrm{char}(k) = p$

Now $|\ker \theta| = p$ for $\theta : (\bar{k},+) \to (\bar{k},+)$.

say $\theta(t) = \theta_0(ct)^{p^{m_0}}$ where $\theta_0(x) = \sum_m e_m x^{p^m}$

with $e_0 \neq 0$ and $\theta_0(1) = 0$

Then $(\theta_0', \theta_0) = 1$, so θ_0 is sep. so $\deg(\theta_0) = p$ hence

$\ker(\theta_0) = \mathbb{F}_p$

so $\theta_0(x) = e_0 x + e_1 x^p$ and $\theta_0(1) = 0 \Rightarrow e_0 = -e_1$

so $\theta_0(x) = e_1(x^p - x) = e_1 \wp(x)$.

so $\theta(t) = e_1 \wp(ct)^{p^{m_0}}$ ☑

Proof of Claim 1:

$G_{\bar{a}} = \{a_0 \wp(x) = \ldots = a_n \wp(x_n)\}$ alg. group.

Let $\alpha_0, \ldots, \alpha_n \in k$ s.th.

$$\sum_i a_i^{-p^{-m}} \alpha_i = \begin{cases} 1 & m = 0 \\ 0 & m = 1, \ldots, n \end{cases}$$

(exists since $(a_i^{-p^{-m}})_{i,m}$ is non-sing. since

trdeg$(\bar{a}) = n+1$ and $a_i^{-p^{-m}} \in k$ since k is perfect)

Set $t := \sum \alpha_i x_i$. Then $t^{p^k} = \sum \alpha_i^{p^k} x_i$

[proof by induction on k: $t^{p^{k+1}} = (t^{p^k})^p = \sum_i \alpha_i^{p^{k+1}} x_i^p$

$= \sum_i \alpha_i^{p^{k+1}} (\wp(x_i) + x_i)$

$$= \underbrace{\sum \frac{\alpha_i p^{k+1}}{\partial_i} \partial_i p(x_i)}_{=0} + \sum \alpha_i^{p^{k+1}} x_i$$

Claim: $(\alpha_i^{p^m})_{i,m}$ is non-singular (i.e. invertible)

Then $x_i = \sum_m \beta_{i,m} t^{p^m}$

Homework!