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Dp-Minimality

Def (1) An ind. pattern is a pair of formulas $\phi^0(x,y)$, $\phi^1(x,y)$ together with $k_0, k_1 \in \omega$ s.th. there is an (infinite) array $(a_i^j)_{i<\omega}$ with $j \in \{0,1\}$ s.th.

(a) $\{\phi^0(x,a_i^0) \mid i \in \omega\}$ is k_0-~~consistent~~ inconsistent

~~(b)~~ $\{\phi^1(x,a_i^1) \mid i \in \omega\}$ is k_1-~~consistent~~ -"-

(b) for any $f: \{0,1\} \to \omega$ $\{\phi^0(x,a^0_{f(0)}), \phi^1(x,a^1_{f(1)})\}$ is consistent.

(2.) A theory T is dp-minimal if there is no ind. pattern and T is NIP

(3.) $A = (a_i)_{i<\omega}$, $B = (b_i)_{i<\omega}$ are mutually ind. if A is ind./B and B is ind./A

Lemma: If T is NIP, TFAE (~~[illegible]~~ In general, we have (6) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) and (6) $\Rightarrow$ (1), but (1) $\not\Rightarrow$ (6))

(1.) There is no ind pattern

(2.) For A, B mutually ind. and $c \in M \models T$, one of $(tp(a_i/c))_{i<\omega}$ or $(tp(b_i/c))_{i<\omega}$ is constant.

(3.) For A, B mutually ind. and $c \in M \models T$, one of A or B is ind. over c.

(4.) For any $A = (a_i)_{i \in I}$ ind., I dense lin. order and any c there is some $i_0 \in \overline{I}$ ← completion of I s.th. both $(tp(a_i/c))_{i<i_0}$ and $(tp(a_i/c))_{i>i_0}$ are constant.

(5.) -"- both are ind. over c.

(6.) T is dp-minimal.

<u>Proof</u>: (3.) $\Rightarrow$ (2.) ✓ (5.) $\Rightarrow$ (4.) ✓ (2.) $\Rightarrow$ (1.) :

Assume not (1.) $\Rightarrow$ in the negation of (1.), we can assume that $(a_i^1)_{i<\omega}$ and $(a_i^0)_{i<\omega}$ are mutually ind.

By the definition of ~~ind~~ ind. pattern, have

$c \in \phi_0(M, a_i^0) \cap \phi_1(M, a_0^1)$.

$\Rightarrow tp(a_i^0/c)$ is not constant, $tp(a_i^1/c)$ not constant.

(2.) $\Rightarrow$ (3.):

Assume A, B, C witness $\neg$(3.). As A, B are mutually ind. and $\exists$ $i_1 < i_2$, $j_1 < j_2$ and a fmla $\varphi(x, y_1, y_2)$ s.th. $\models \varphi(c, a_{i_1}, a_{i_2}) \wedge \neg\varphi(c, a_{j_1}, a_{j_2})$

(really need to do this argument with N-tuples! Argument works just the same).

Take $n > i_1, j_2$

Claim $\Rightarrow$ $\models \varphi(c, a_{i_1}, a_{i_2}) \wedge \neg\varphi(c, a_{2n}, a_{2n+1})$

(if not true change i_1, i_2 to j_1, j_2! so wlog.

Either $\models \varphi(c, a_{i_1}, a_{i_2}) \wedge \neg\varphi(c, a_{2n}, a_{2n+1})$

or $\models \varphi(c, a_{j_1}, a_{j_2}) \wedge \neg\varphi(c, a_{2n}, a_{2n+1})$ holds ...)

Then, define $A' := \{(a_{i_1}, a_{i_2})\} \cup \{(a_{2k}, a_{2k+1}) : k \geq n\}$

$B' := \{(b_{j_1}, b_{j_2})\} \cup \{(b_{2k}, b_{2k+1}) : k \geq n\}$

Using 4, we get that neither $tp(A'/c)$ nor $tp(B'/c)$ is constant.

(3.) $\Rightarrow$ (5.): Let $(a_i)_{i \in I}$ be ind., I dense LO, $c \in M$, $i_0 \in \bar{I}$.

Then, by (3.), $(a_i)_{i < i_0} =: A_{<i_0}$

or $(a_i)_{i > i_0} =: A_{>i_0}$ are ind. /c.

$\leftarrow$ assume $A_{>i_0}$ is ind.

(Note: wlog $A_{<i_0}$ and $A_{>i_0}$ are ~~ind~~ mutually ind. - standard lemma on ind. sequences!)

Take $j_0 := \inf\{i \leq i_0 : A_{>i} \text{ ind. } /c\}$, $A_{>j_0}$ is ind./c.

<u>Case 1</u>: Assume there is no element in I smaller than j_0.

$\rightsquigarrow$ done

Case 2: There are some $i < j_0$.

If $A_{<j_0}$ is not ind. (i.e. $\neg$(5.) for j_0)

$\Rightarrow$ find $j_1 < j_0$, $A_{<j_1}$ is not ind. /c,

but this contradicts the definition of j_0!

(4.) $\Rightarrow$ (2.) Assume A, B, C witness $\neg$(2.). Then

(*) $\exists i_0 \in I \; \forall i_1 < i_0 < i_2 \; \exists i', i \;\; i_1 < i < i_0 < i' \leq i_2$

with $tp(a_i/c) \neq tp(a_{i'}/c)$.

(dense order, not constant type, ind.)

and same for B ($\exists j_0 \ldots tp(b_j/c) \neq tp(b_{j'}/c)$)

$\rightarrow$ moreover, wma $j_0 \neq i_0$.

Claim: $(a_i b_i)_{i \in I}$ is ind., and ~~[illegible]~~ one of $tp(a_i b_i/c)_{i > i_0}$ and $tp(a_i b_i/c)_{i < i_0}$ is not constant.

(ind. because of mut. ind.; non constant by (*)). $\boxtimes$ claim.

(6.) $\Rightarrow$ (2.): Assume A, B, C witness $\neg$(2.), say we have $\models \phi(c, a_0) \wedge \neg \phi(c, a_1)$.

Define $A' := (a_{2k}, a_{2k+1})_{k \in \omega}$

$$\phi'(x, y_1, y_2) := \phi(x, y_1) \wedge \neg \phi(x, y_2)$$

Claim (needs NIP!): $\{\phi'(x, y_1, y_2) : (y_1, y_2) \in A'\}$ is k_i inconsistent for some k_i.

This implies the claim. $\longrightarrow$

Alternatively: ind. sequences can only alternate finitely many times.

(Fact: If f.a. $n < \omega$, there are a_{2j}, a_{2j+1} with $j < n$ s.th. f.a. $I \subseteq n$ $\{\phi'(x, a_{2j}, a_{2j+1}) : j \in I\} \cup \{\neg \phi'(x, a_{2j}, a_{2j+1}) : j \notin I\}$ is consistent, then ϕ' has IP.)

Do the same for $B' := (b_{2k}, b_{2k+1})$,

$$\psi'(x, y_1, y_2) := \psi(x, y_1) \wedge \neg \psi(x, y_2)$$

Then choosing $\phi^0 \equiv \phi'$ and $\phi^1 \equiv \psi'$ gives an ind. pattern with $A', B'/c$

- rows are k_1- (resp. k_2)- inconsistent by the claims for A' resp. B'
- $c \in \Phi'(x, a_{2j}, a_{2j+1}) \wedge \psi'(x, b_{2j}, b_{2j+1})$.

Now (5.) $\Rightarrow$ (6.)

By the previous parts of the proof, (5.) $\Rightarrow$ (1.).
So, why does (5.) imply NIP?
Because of finite alternation! $\boxtimes$

<u>Def</u>: - $\varphi(x, a)$ k-divides $/A$ if there ex. an (ind$_A$) sequence $(a_i)_{i\in\omega}$, s.th. $\{\varphi(x, a_i)\}_{i<\omega}$ is k-incons.
- with $a_0 = a$ [s.th. $tp(a_i/A) = tp(a_j/A)$ $\forall i, j$]
- $\varphi(x, a)$ forks $/A$ if $\varphi(x, a) \vdash \bigvee_{l<k} \psi_l(x)$ s.th. each ψ_l divides $/A$

<u>Rem</u>: Obviously, dividing $\Rightarrow$ forking.

Assume now $(M, <, \ldots)$ is an ordered structure w/o ind. pattern.

Lemma: $X_{\bar{a}}$ cofinal in M, then $X_{\bar{a}}$ does not fork $/A$.

Pf: Later.

Let $X \subseteq M$. $\mathrm{conv}(X) = \{x \in M : \exists x_1, x_2 \in X \text{ with } x_1 < x < x_2\}$

Now $M = G$ ordered group.

Lemma: If G is inp-minimal (i.e. no indep. pattern), $H \leq G$ def'ble, $C = \mathrm{conv}(H)$ $\Rightarrow$ $[C : H] < \infty$.

Proof: Wlog $G = C$.
Suppose $[C : H] = \infty$. Then $\exists (g_i)_{i<\omega}$ $g_i H \neq g_j H$ for $i \neq j$.
(an assume $(g_i)_{i<\omega}$ is ind.
$\Rightarrow$ then $g_0 H$ 2-divides$/\emptyset$ $\Rightarrow$ forks$/\emptyset$ $\lightning$ to Lemma

[since H is cofinal in C, gH is cofinal in C f.a. $g \in C$] $\boxtimes$

<u>Prop</u>: G inp-minimal $\Rightarrow$ G is abelian.

<u>Pf</u>: Claim 1: $a, b \in G$, $a^n = b^n \Rightarrow a = b$

(o/w, say $a < b$, so $a^n < b^* a^{n-1} < \dots < b^n$).

Claim 2: $C(x)$ centralizer, then

$$D(x) := \mathrm{Conv}(C(x)) = C(x).$$

Pf: $g \in D(x) \overset{?}{\Rightarrow} xg \in D(x)$

We have $z_1 < g < z_2$, $z_i \in C(x)$

$\Rightarrow \underset{\in C(x)}{z_1 x} < xg < \underset{\in C(x)}{z_2 x}$

Now $C(x) \subseteq D(x)$ fin. index

$\Rightarrow$ ex. $n \in \mathbb{N}$ $(xg)^n \in C(x)$.

$\Rightarrow (gx)^n = x^{-1}(xg)^n x = (xg)^n$

$\overset{\text{Claim 1}}{\Rightarrow} D(x) = C(x)$. $\boxtimes$ Claim 2

To finish the proof: suppose $x < y$.

$\Rightarrow C(y)$ is a convex subgrp. containg x

$\Rightarrow x \in C(y)$. $\boxtimes$