Motivic Integration

Immi Halupczok

Contents

0 Notation and Language

0.1 Notation: (K, v) is a henselian valued field with value group Γ, valuation ring $\mathcal{O}_K$, maximal ideal M_K, and residue field $k = O_K/M_K$. The residue map is denoted $\mathrm{res} : \mathcal{O}_K \to k$. We write p for the characteristic of k, $p \in \mathbb{P} \cup \{0\}$.

0.2 Main Examples:

- $K = \mathbb{Q}_p = \{\sum_{i=N}^{\infty} a_i p^i : N \in \mathbb{Z}, a_i \in \{0, \dots, p-1\}\}$, the p-adic numbers, with $v(a)$ equal to the minimal i such that $a_i \neq 0$. In this example, the value group is $\mathbb{Z}$, the valuation ring is $\mathbb{Z}_p$, the maximal ideal is $p\mathbb{Z}_p$, and the residue field is $\mathbb{F}_p$. We have an angular component map $\mathrm{ac}(a) = a_{v(a)}$.

- $K = k((t)) = \{\sum_{i=N}^{\infty} a_i t^i : N \in \mathbb{Z}, a_i \in k\}$, with $v(a)$ equal to the minimal i such that $a_i \neq 0$. Here, the value group is again $\mathbb{Z}$, the valuation ring is $k[[t]]$, the maximal ideal is $t \cdot k[[t]]$, the residue field is k, and there is an angular component map $\mathrm{ac}(a) = a_{v(a)}$.

0.3 Language: We work in the Denef-Pas language L_{DP}, which is three sorted:

- K, with $L_{\mathrm{ring}} = \{0, 1, +, -, \cdot\}$
- k with L_{ring}
- Γ with $L_{\mathrm{oag}} = \{0, +, -, <\}$

We have two maps between sorts: a valuation $v : K \to \Gamma \cup \{\infty\}$ and an angular component $\mathrm{ac} : K \to k$. Strictly speaking, ∞ should probably be part of the Γ-sort, but because we're not worrying about quantifier elimination in this course, we can be a bit hand-wavy about things.

0.4 Theory: We will mostly work in the L_{DP} theory $\mathrm{HVFZ}_{0,0}$ of henselian valued fields with value group $\mathbb{Z}$ and $(\mathrm{char}(K), \mathrm{char}(k)) = (0, 0)$. We also assume that v is a valuation and ac is an angular component map for models of $\mathrm{HVFZ}_{0,0}$ (in particular, we assume that models of the theory have a well-defined angular component map).

0.5 Example: If k is a field of characteristic 0, then $k((t)) \models \mathrm{HVFZ}_{0,0}$. Note that if $\theta \in \mathrm{HVFZ}_{0,0}$ then θ depends on only finitely many axioms, and hence $\mathbb{Q}_p \models \theta$ for sufficiently large $p \in \mathbb{P}$.

1 What is motivic integration

1.1 Overview

1.1 Example:
- (p-adic measure): On $\mathbb{Q}_p$, one has a Lebesgue measure, and definable sets are measurable. For example, $\mu_p(\mathbb{Z}_p) = 1$ and $\mu_p(p\mathbb{Z}_p) = \frac{1}{p}$. There is even an algorithm to determine what the measure of a definable set is, and definable sets have rational measure (ie $\mu(\phi) \in \mathbb{Q}$ for all formulas ϕ). If $(X_s)_{s \in S}$ is a definable family of sets, we can describe the map $s \mapsto \mu_p(X_s)$.
- (Uniform p-adic measure): Given an L_{DP}-formula ϕ, we can try to understand how $\mu_p(\phi(\mathbb{Q}_p))$ depends on p. To do this, we can consider the uniform measure $\mu_u(\phi) = (\mu_p(\phi(\mathbb{Q}_p)))_{p \in \mathbb{P}}$. For example,
  - $\mu_u(v(x) \geq 0) = (1)_p = (1, 1, 1, \dots)$
  - $\mu_u(v(x) > 0) = (p^{-1})_p = (2^{-1}, 3^{-1}, 5^{-1}, \dots)$
  - $\mu_u(v(x) \geq 0 \wedge \exists y (y^2 = x)) = (\frac{p}{2p+1})_p$ for $p \geq 3$. Because we almost exclusively care about the eventual behaviour of p-adic formulas, we will say that two sequences are equal if they are eventually equal.
  - For any L_{ring}-formula ψ, we have $\mu_u(v(x) \geq 0 \wedge \psi(\mathrm{res}(x))) = (\frac{1}{p} \cdot \#\psi(\mathbb{F}_p))_p$. The number of points in $\mathbb{F}_p$ satisfying ψ can be very complicated; for example, if ψ describes an elliptic curve, we don't know how $\psi(\mathbb{F}_p)$ grows with respect to p.
- Similar measures can be defined $\mathbb{F}_p((t))$.

1.2 Recall: (Ax-Kochen/Ershov Transfer Principle) If θ is an L_{PD}-sentence then for $p \gg 1$, $\mathbb{Q}_p \models \theta$ iff $F_p((t)) \models \theta$. This has a measure version: if $\phi(x)$ is an L_{PD}-formula then for $p \gg 1$, $\mu(\phi(\mathbb{Q}_p)) = \mu(\phi(\mathbb{F}_p((t))))$.

For example, if θ is $\exists y (y^2 = 5)$, then by Hensel's lemma, for $p \geq 7$, we have $\mathbb{Q}_p \models \theta$ iff $\mathbb{F}_p \models \theta$ iff $\mathbb{F}_p((t)) \models \theta$.

1.3 Definition: To every L_{DP}-formula ϕ, we associate an abstract "measure" $\mu_{\mathrm{mot}}(\gamma) \in \mathcal{C}^0_{\mathrm{mot}}$ such that for each $p \in \mathbb{P}$ there exists a map $\mathrm{sp}_p : \mathcal{C}^0_{\mathrm{mot}} \to \mathbb{Q}$ (the specialization at p) with $\mathrm{sp}_p \circ \mu_{\mathrm{mot}} = \mu_p$. Moreover, we should obtain a "measure" of $\phi(K)$ for any $K \models \mathrm{HVFZ}_{0,0}$. We call μ_{mot} the *motivic measure*.

What is $\mathcal{C}^0_{\text{mot}}$? It is a ring which contains, for every ring-formula $\psi(x)$, an element $[\psi]$ such that $\text{sp}_p[\psi] = \#\psi(\mathbb{F}_p)$. One element which we will use quite frequently is $\mathbb{L} = [x = x]$. We will define things more precisely later in the course, but $\mathcal{C}^0_{\text{mot}}$ is generated by the $[\psi]$ and certain $\frac{1}{f}$ where $f \in \mathbb{Z}[\mathbb{L}]$.

1.4 Example: $\mu_{\text{mot}}(v(x) > 0) = \frac{1}{\mathbb{L}}$. Observe that $\text{sp}_p(\mathbb{L}) = p$, and so

$$\text{sp}_p(\mu_{\text{mot}}(v(x) > 0)) = \text{sp}_p(\frac{1}{\mathbb{L}}) = \frac{1}{p} = \mu_p(v(x) > 0)$$

as we claimed.

1.5 Example: Write $\phi(x) = v(x) \geq \wedge\psi(\text{res}(x))$ for some L_{ring}-formula ψ. Then we have $\mu_{\text{mot}}(\phi(x)) = \frac{1}{\mathbb{L}}[\psi]$.

For example, take $\psi(x) = \exists y(y^2 = x)$. Then by some basic facts about finite fields, for $p \geq 3$ we have $\mu(\phi(\mathbb{Q}_p)) = \frac{1}{p} \cdot \frac{p+1}{2}$. So it is tempting to say that $\mu_{\text{mot}}(\phi) = \frac{\mathbb{L}+1}{2\mathbb{L}}$. But consider $\mathbb{C}((t))$. This has residue field $\mathbb{C}$, in which every element is a square, so here ϕ is equivalent to the formula $v(x) \geq 0$, and we clearly have $\mu_{\text{mot}}(v(x) \geq 0) = 1$. So we really need to write $\mu_{\text{mot}}(\phi) = \frac{1}{\mathbb{L}}[\psi]$ and not try to simplify it further.

1.6 Remark: Using our definition of the motivic measure, it is very difficult to check that we actually have a measure. We could instead define $\mathcal{C}^0_{\text{mot}}$ using some universal properties in order to ensure that we have a measure, but in that case we'd have even less of an idea of the structure of $\mathcal{C}^0_{\text{mot}}$. It is currently a work in progress that both definitions show the same μ_{mot}.

1.2 Applications

1.7 Point Counting: Suppose $f \in \mathbb{Z}[x]$, $x = (x_1, \ldots, x_n)$, and for a ring R, set $V(R) = \{a \in R^n : f(a) = 0\}$. (Note: for the following, we will assume V is the zero set of a single polynomial, but everything still works with easy adjustments for zero sets of several polynomials.) Determining whether $V(\mathbb{Z}) = \emptyset$ is undeciable. One way to approach this problem is to ask whether $V(\mathbb{Z}/m\mathbb{Z})$ has solutions; for each m, it is a finite problem, and so is much easier. We can then extend this to ask what $N_m = \#V(\mathbb{Z}/m\mathbb{Z})$ is for a given m, and even how does N_m depend on m?

Using the Chinese Remainder Theorem, it suffices to know N_{p^s} for $p \in \mathbb{P}$, $s \in \mathbb{N}$. For m_1, m_2 coprime, CRT gives $N_{m_1 m_2} = N_{m_1} N_{m_2}$. Fix $p \in \mathbb{P}$; the question now becomes how does N_{p^s} depend on s? Define the Poincaré series by

$$P_{V,p}(T) = \sum_{s=0}^{\infty} N_{p^s} T^s \in \mathbb{Z}[[T]].$$

Denef, Igusa, Mevser (1980s) proved that $P_{V,p}(T) \in \mathbb{Q}(T)$ (thinking of $\mathbb{Z}[[T]]$ and $\mathbb{Q}(T)$ as subrings of the formal series field $\mathbb{Q}((T))$). For example, if $n = 1$ and f is the zero polynomial then $N_{p^s} = p^s$ for all s, and hence

$$P(T) = \sum_{s=0}^{\infty} p^s T^s = \sum_{s=0}^{\infty} (pT)^s = \frac{1}{1 - pT} \in \mathbb{Q}(T).$$

This is a very strong statement in the following sense: $\mathbb{Z}[[T]]$ is uncountable, but $\mathbb{Q}(T)$ is countable. So this shows that there are in fact very few options for $P_{V,p}(T)$, and hence for the function $s \mapsto N_{p^s}$. One proof of this result uses p-adic integration.

Now we have that $P_{V,p}(T) = \frac{g_p(T)}{h_p(T)}$ for $g_p, h_p \in \mathbb{Q}[T]$, and we can ask how do g_p, h_p depend on p? We can obtain a dependence on p using uniform p-adic integration: $h_p(T) = h(p,T) \in \mathbb{Z}(p,T)$, $\deg(g_p)$ is bounded, and the coefficients of g_p are things like $\#\psi(\mathbb{F}_p)$ for some L_{ring}-formula ψ.

1.8 Transfer Principles: Like the AKE transfer principle mentioned earlier.

1.9 Invariants of Varieties over $\mathbb{C}$: Suppose V and W are Calabi-Yav-manifolds (meaning there is something like a measure on them) and are birational (ie there are dense open subsets $V_0 \subseteq V$ and $W_0 \subseteq W$ with an algebraic bijection between V_0 and W_0). Then V and W have the same Betti numbers (some sort of cohomology dimension; the zeroth gives the number of connected components, the first gives the number of holes, etc.). It follows somehow that $[V] = \mu(V(\mathbb{C}[[T]])) = \mu(W(\mathbb{C}[[T]]) = [W]$. Then by some general nonsense, $[V]$ determines the Betti numbers of V.

2 Uniform p-adic measure

2.1 Measure on $\mathbb{Q}_p$

2.1 Definition: Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a unique (up to scalar multiple) measure which is translation invariant, the *Haar measure*. We set $\mu_p(p^r\mathbb{Z}_p) = p^{-r}$ for $r \in \mathbb{Z}$. Since every definable set $X \subseteq \mathbb{Q}_p$ is a countable disjoint union of balls and points, we can compute $\mu_p(X)$ by summing over the balls (which are translates of $p^r\mathbb{Z}_p$). We can extend this to ${\mathbb{Q}_p}^n$ by using the product measure, ie $\mu_p(X \times Y) = \mu_p(X) \cdot \mu_p(Y)$.

2.2 Remark: Note that in $\mathbb{Q}_p$ (and in any model of $\mathrm{HVFZ}_{0,0}$) $\{x : v(x) > 0\} = \{x : v(x) \geq 1\}$, so this set has measure $1/p$, whereas $\mathbb{Z}_p = \{x : v(x) \geq 0\}$ has measure 1. Thus, unlike the real case, the open and closed balls of a given radius do not have the same measure.

2.3 Application: (point counting) If $f \in \mathbb{Z}[x]$, $x = (x_1, \ldots, x_n)$, we set $V(R) = \{a \in R^n : f(a) = 0\}$ for any ring R. Then we define

$$N_{p^s} = \#V(\mathbb{Z}/p^s\mathbb{Z}) = |\{a \in \mathbb{Z}^n : f(a) \in p^s\mathbb{Z}\}/\sim|$$

where $\sim$ is the equivalence relation $a \sim a'$ iff $a - a' \in p^r\mathbb{Z}$. By some straightforward ring theory, $\mathbb{Z}/p^s\mathbb{Z} \cong \mathbb{Z}_p/p^s\mathbb{Z}_p$. If we write $X_s = \{a \in \mathbb{Z}_p^n : f(a) \in p^s\mathbb{Z}_p\}$, then we have

$$N_{p^s} = \#V(\mathbb{Z}_p/p^s\mathbb{Z}_p) = |X_s/\sim|.$$

An easy exercise shows that X_s is a disjoint union of translates of $(p^s\mathbb{Z}_p)^n$, and N_{p^s} is the number of translates appearing in this union. Moreover,

$$\mu_p(X_s) = N_{p^s}\mu_p(p^s\mathbb{Z}_p)^n = N_{p^s} \cdot p^{-sn}.$$

Yesterday, we claimed that $P_{V,p}(T) = \sum_{s\geq 0} N_{p^s}T^s = \sum_{s \geq 0} \mu_p(X_s)p^{sn}T^s$ was equal to a function in $\mathbb{Q}(T)$. Note that $(X_s)_{s\in\mathbb{N}}$ is defined by

$$v(x_1) \geq 0 \wedge \ldots \wedge v(x_n) \geq 0 \wedge v(f(x_1, \ldots, x_n)) \geq s.$$

Then the fact that $P_{V,p}$ is rational follows by substituting p^nT for T in the following theorem:

2.4 Theorem: Suppose $(X_s)_{s\in\mathbb{N}}$ is a definable family with $X_s \subseteq \mathbb{Q}_p^n$ and $\mu(X_s) < \infty$ for all s. Then we have

$$P_{(X_s)}(T) = \sum_{s\geq 0} \mu_p(X_s)T^s \in \mathbb{Q}(T).$$

(Here we are considering the index set $\mathbb{N}$ as a subset of Γ.)

2.2 Uniform p-adic theorems

2.5 Definition: Let $\mathcal{C}_u^0$ be the subring of $\{(a_p)_{p\in\mathbb{P}} : a_p \in \mathbb{Q}\}$ generated by

1. $\left(\frac{1}{p}\right)_p$ and $\left(\frac{1}{1-p^{-l}}\right)_p$ for every $l \in \mathbb{N}$
2. $(\#\psi(\mathbb{F}_p))_p$ where $\psi(z)$ is an L_{ring}-formula

2.6 Example:

1. $\mu_p(v(x) \geq 1) = \frac{1}{p}$
2. If $\phi(x)$ is the formula $\text{ac}(x) = 1 \wedge l \mid v(x) \wedge v(x) \geq 0$ then

$$\mu_p(\phi) = p^{-1} + p^{-1-l} + p^{-1-2l} + \ldots = p^{-1}\sum_{r\geq 0}(p^{-l})^r = p^{-1}\left(\frac{1}{1-p^{-1}}\right)$$

3. Recall from yesterday that

$$\mu_p(\text{squares in } \mathbb{Z}_p) = \frac{p}{2(p+1)} = \frac{p(p+1)(p-1)}{2(p+1)^2(p-1)} = \frac{p+1}{2}\cdot\frac{p(p-1)}{(1+p^{-1})(1-p^{-2})p^3}$$

where $\frac{p+1}{2} = \#\{x \in \mathbb{F}_p : \exists y(y^2 = x)\}$ is allowed by item (2).

2.7 Theorem: For every L_{DP}-formula $\phi(x)$ (each x_i a K-variable), there exists a sequence $a = (a_p)_p \in \mathcal{C}_u^0$ such that for all $p \gg 1$, $\mu_p(\phi(\mathbb{Q}_p)) = a_p$.

2.8 Definition: A *definable set* is a tuple $X = (X_K)_K$ (with K ranging over all valued fields with an angular component map) given by an L_{DP}-formula $\phi(x)$ (without parameters), in the sense that $\phi(K) = X_K$. A *deifnable function* is a tuple $f = (f_K)_K$ such that the tuple $(\text{graph}(f_K))_K$ is a definable set.

2.9 Definition: Let S be a definable set (in any sorts) and $\mathcal{V} = \{\mathbb{Q}_p : p \in \mathbb{P}\} \cup \{\mathbb{F}_p((t)) : p \in \mathbb{P}\}$. We write $\mathcal{C}_u(S)$ for the subring of $\{(f_K)_{K\in\mathcal{V}} : f_K : S_K \to \mathbb{Q}\}$ generated by:

1. $(s \mapsto p^{-1})_{K\in\mathcal{V}}$ and $\left(s \mapsto \frac{1}{1-p^{-1}}\right)_{K\in\mathcal{V}}$, where p is the residue characteristic of K
2. $(s \mapsto \#Z_{s,K})_{K\in\mathcal{V}}$ for any definable family $(Z_s)_{s\in S}$ with $Z_{s,K} \subseteq k^n$ (the residue field)
3. $(s \mapsto \alpha_K(s))_{K\in\mathcal{V}}$ for any definable function $\alpha : S \to \Gamma$ (considering $\Gamma \cong \mathbb{Z}$ as a subset of $\mathbb{Q}$)
4. $(s \mapsto p^{\alpha_K(s)})_{K\in\mathcal{V}}$ for any definable function $\alpha : S \to \Gamma$

(Note: these f_K will not be definable functions, in part because their range in $\mathbb{Q}$, not K.)

2.10 Example: Consider any definable $\alpha : S \to \Gamma$ and the formula $\phi(x, s)$ given by $v(x) \geq -\alpha(s)$. Then $X_s = \{x : v(x) \geq -\alpha(s)\}$ and for each $s \in S_{\mathbb{Q}_p}$ we have

$$X_{s,\mathbb{Q}_p} = \phi(\mathbb{Q}_p, s) = \{x \in \mathbb{Q}_p : v(x) \geq -\alpha_{\mathbb{Q}_p}(s)\} = p^{-\alpha_{\mathbb{Q}_p}(s)}\mathbb{Z}_p,$$

so $\mu(X_{s,\mathbb{Q}_p}) = p^{\alpha_{\mathbb{Q}_p}(s)}$ for each s. Thus, we need functions of type (4) to be allowed.

2.11 Theorem: For every definable family $(X_s)_{s\in S}$ (where each $X_{s,K} \subseteq K^n$), there exists an $f = (f_K)_{K\in\mathcal{V}} \in \mathbb{C}_v(S)$ such that, for $p \gg 1$, if $K = \mathbb{Q}_p$ or $K = \mathbb{F}_p((t))$ and $s \in S_K$, $\mu_p(X_{s,K}) = f_K(s)$.

2.12 Corollary: $\sum_s \mu_p(X_{s,\mathbb{Q}_p})T^s \in \mathbb{Q}_(T)$.

Proof: (assuming the theorem holds) Assume $S = \mathbb{N}$. Then it is enough to prove $\sum_{s\geq 0} f_p(s)T^s \in \mathbb{Q}(T)$ for $f_p \in \mathcal{C}_u(S)$. We only have $f_p(s) = \mu_p(X_{s,\mathbb{Q}_p})$ for sufficiently large p, but changing the first finitely many terms won't change the rationality of the sequence.

1. Constant, so irrelevant
2. Γ and k are orthogonal, ie any definable $Z \subseteq \Gamma^m \times k^n$ is a finite disjoint union of sets of the form $A \times B$ where A is a definable subset of Γ^m and B is a definable subset of k^n. So after a finite definable partition of $\mathbb{N}$ (as a subset of the value group $(\mathbb{Z}, +, <)$), we may assume that the contribution of (2) is constant.

We are left with $A \subseteq \mathbb{N}$ definable (one of the partitions from (2)), $\alpha_i, \beta_i : A \to \mathbb{Z}$ definable, and $f(s)$ is a product of things as in (3) and (4), ie

$$f(s) = \prod_i \alpha_i(s) \prod_j p^{\beta_j(s)} = \left(\prod_i \alpha_i(s)\right) p^{\sum_j \beta_j(s)} = \left(\prod_i \alpha_i(s)\right) p^{\beta(s)},$$

where $\beta = \sum \beta_j$ is definable since a sum of definable functions is again definable. By partitioning A if necessary, we may assume that α_i, β are (affine) linear, ie of the form $s \mapsto bs + c$. Now

$$f(s) = \left(\prod_i \alpha_i(s)\right) p^{\beta(s)} = h(s)p^{bs}p^c$$

where $h \in \mathbb{Z}[c]$. By moving the p^c into $h(s)$, we may in fact assume $f(s) = h(s)p^{bs}$. We now have

$$\sum_s \mu_p(X_{s,\mathbb{Q}_p})T^s = \sum_A \sum_{s\in A} h(s)p^{bs}T^s$$

which (exercise) is rational.

2.13 Notation: We use VF, RF, VG to denote the valued field, residue field, and value group sorts. Ie, given a valued field K, we write $VF_K = K$, $RF_K = k$, and $VG_K = \Gamma$. If we write $X \subseteq VF^n$, for example, we mean that $X_K \subseteq K^n$ for any valued field K.

2.14 Recall: This notation allows us to say that $\mathcal{C}_u(S)$ is the ring generated by:

1. $s \mapsto 1/p$, $s \mapsto 1/(1 - p^{-l})$
2. $s \mapsto \#Z_S$ for $Z \subseteq S \times RF^m$ definable
3. $s \mapsto \alpha(s)$ for $\alpha : S \to VG$ definable
4. $s \mapsto p^{\alpha(s)}$ for $\alpha : S \to VG$ definable

3 Uniform p-adic integration

3.1 Example: Suppose we want to compute the measure of a definable set $X \subseteq \mathbb{Q}_p^n$. This is fairly straightforward if $n = 1$, but quickly becomes complicated if $n > 1$. So one reason we might want to be able to integrate is to be able to compute

$$\mu(X) = \int_{\mathbb{Q}_p^n} 1_X(x)\ \mathrm{d}x = \int_{\mathbb{Q}_p^{n-1}} \left(\int_{\mathbb{Q}_p} 1_X(x)\ \mathrm{d}x_0 \right)\ \mathrm{d}\hat{x}$$

where $x = (x_0, \hat{x})$. Of course, to do this, we need to have $\int_{\mathbb{Q}_p} 1_X(x)\ \mathrm{d}x_0$ to be an integrable function. This is where $\mathcal{C}_u(S)$ comes in.

3.2 Remark: We define integrals using Lebesgue integration, which works for any measure space X and functions $f : X \to \mathbb{R}$. In particular, we can take X to be $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

3.3 Theorem: Suppose $f \in \mathcal{C}_u(S \times VF)$ and suppose that for $p \gg 1$ and $K = \mathbb{Q}_p$ or $\mathbb{F}_p((t))$, for every $s \in S_K$, we have $f_K(s, \cdot) : K \to \mathbb{Q}$ is integrable. Then there exists $g \in \mathcal{C}_u(S)$ such that for $p \gg 1$ (possibly larger than the earlier bound on p) and $g_K(s) = \int_K f_K(s,x)\ \mathrm{d}x$ is an element of $\mathcal{C}_u(S)$.

3.4 Notation: We write $\int_{VF} f(s,x)\ \mathrm{d}x$ for the element $g \in \mathcal{C}_u(S)$ in the above theorem. Note: this definition is not completely unique, but any two functinos satisfying the condition will agree for p sufficiently large.

3.5 Remark: Today's Theorem implies Yesterday's Theorem:

Claim: If $(X_s)_{s\in S}$, $X_s \subseteq VF^n$, then $(s \mapsto \mu(X_s)) \in \mathcal{C}_u(S)$ for sufficiently large p.

Proof: From the definition of Lebesgue integration we have

$$\mu(X_s) = \int_{VF} \cdots \int_{VF} 1_{X_s}(x_1, \ldots, x_n)\ \mathrm{d}x_n \ldots\ \mathrm{d}x_1,$$

which is an element of $\mathcal{C}_u(S)$, provided 1_{X_s} is integrable. But 1_{X_s} is even in $\mathcal{C}_u(S)$: i

3.6 Fact: $\mathcal{C}_u(S)$ is the smallest family of rings which are closed under $\int$ and contain $1_{S'}$ for definable $S' \subseteq S$. (Closure under $\int$ is the theorem; to see $1_{S'} \in \mathcal{C}_u(S)$, consider S' as a subset of $S \times RF^0$, and then $1_{S'}$ is the function $s \mapsto \#S'_s$.)

3.7 Example: A key ingredient of the proof of the theorem is cell decomposition. Cell decomposition for valued fields is not nearly as pretty as for o-minimal fields, so we begin with an example. Consider

$$X = \{x \in VF : v(x) \geq \wedge \exists y (y^2 = x)\}.$$

Then $x \in X$ if and only if $v(x) \geq 0$, $2 \mid v(x)$, and $\mathrm{ac}(x)$ is a square (in the residue field); the forward direction is clear, and the reverse holds by Hensel's lemma for $p \geq 3$. Thus

$$\mu_p(X(\mathbb{Q}_p)) = \left(\#\text{squares in } \mathbb{F}_p^\times\right)\left(p^{-1} + p^{-3} + p^{-5} + \ldots\right) = \left(\frac{(p-1)p^2}{2}\right)\left(\frac{p^{-1}}{1-p^{-2}}\right) = \frac{p}{2(p+1)}$$

3.8 Definition: Let K be a valued field. A *cell* is a (definable) set C of one of the following forms:

0. $C = \{c\}$
1. $C = \{c + x : v(x) \equiv \lambda \mod m \wedge \alpha \leq v(x) \leq \beta \wedge \alpha \leq v(x) \leq \beta \wedge \mathrm{ac}(x) \in Z\}$

where $c \in K$, $\lambda, m \in \mathbb{N}$, $\alpha, \beta \in \Gamma \cup \{\pm\infty\}$, and $Z \subseteq RF$ definable.

A *cell family* is a family $(C_s)_{s \in S}$ of sets such that $C_s \neq \emptyset$ iff $s \in S'$ for some definable $S' \subseteq S$ and, for $s \in S'$, C_s has one of the following forms:

- $C_s = \{c(s)\}$
- $C_s = \{c(s) + x : v(x) \equiv \lambda \mod m \wedge \alpha(s) \leq v(x) \leq \beta(s) \wedge \mathrm{ac}(x) \in Z\}$

where $c : S' \to K$, $\lambda, m \in \mathbb{N}$, $\alpha, \beta : S' \to \Gamma \cup \{\pm\infty\}$ definable, and $Z_s \subseteq RF$ definable.

3.9 Remark: We would like the function $c : S' \to K$ to be definable, but this is not possible, because if a set contains $\pm\sqrt{2}$ (for example), there isn't a way to definably distinguish between the postive and negative root. The function c is not completely arbitrary, but because we only care about measurable sets, and measures are invariant under translation, we will be able to save ourselves considerable detail work and not worry about what the restrictions on c are.

A similar issue occurs with Z_s; each set is definable, but they are not definable as a family. This causes more issues for our purposes, but to make the rest of the course more manageable, we will pretend that $(Z_s)_{s \in S}$ is a definable family.

3.10 Theorem: (Cell Decomposition) Suppose $X \subseteq VF$ is definable and K a henselian valued field with $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p \gg 1$. Then X_K is a finite union of cells.

Proof: The characteristic zero case follows without too much work from Denef-Pas quantifier elimination. The positive characteristic case follows using usual algebraic tricks that show that anything true in characteristic zero is true in characteristic p for sufficiently large p.

3.11 Theorem: (Uniform Cell Decomposition) Suppose $(X_s)_{s \in S}$ is a definable family. Then there exist finitely many cell families $(C_s)_{s \in S}$ such that for K as before and $s \in S_K$, $X_{s,K}$ is a disjoint union of the $C_{s,K}$.

Proof: (of Theorem 3.3) We want $s \mapsto \mu(X_s)$ to be an element of $\mathcal{C}_u(S)$. From the uniform cell decomposition theorem, we may assume X_s is a cell family. Moreover, we may assume $S = S'$ in the definition of a cell family. If we are in case (0), $\mu(X_s) = 0$. If we are in case (1),

$$\mu(C_s) = (\#Z_s) \sum_{\substack{\alpha(s) < i < \beta(s) \\ i \equiv \lambda \mod m}} p^{-1-i} = \ldots = (\#Z_s)(p^{\alpha'(s)} - p^{\beta'(s)}) \frac{1}{1 - p^{-m}}.$$

Now given $f \in \mathcal{C}_u(S \times VF)$, we want $s \mapsto \int f(s, x)\, \mathrm{d}x$ to be an element of $\mathcal{C}_u(S)$. Without loss of generality, we may assume f is a product of functions of the form (2), (3), and (4). In particular, $f(s)$ only depends on $\alpha_i(s)$ for certain definable $\alpha_i : S \times VF \to VG$ and certain definable $Z_i \subseteq S \times VF \times RF^m$.

It is a theorem that given finitely many α_i, Z_i as above, we can find a cell family decomposition of $S \times VF$ such that for each cell family $(C_s)_{s \in S}$ of type (1),

- $\alpha_i(s, c(s) + x)$ only depends on $(s, v(x))$
- $(Z_i)_{s,c(s)+x}$ only depends on $(s, v(x))$

The rest of the proof follows using similar tricks to the theorem yesterday to simplify the combinations of the α_is, but we don't have time to finish today.

4 Motivic Integration

4.1 Idea: We want a map $\mu_{\text{mot}} : \bigcup_n \{\text{definable sets in } VF^n\} \to \mathcal{C}^0_{\text{mot}}$, where $\mathcal{C}^0_{\text{mot}}$ is some (commutative) ring (with unity), with μ_{mot} behaving like a measure, ie

- $\mu_{\text{mot}}(X \cup Y) = \mu_{\text{mot}}(X) + \mu_{\text{mot}}(Y)$
- $\mu_{\text{mot}}(X \times Y) = \mu_{\text{mot}}(X)\mu_{\text{mot}}(Y)$
- If $X \subseteq VF^n$ and $\dim X \leq n$ then $\mu_{\text{mot}}(X) = 0$
- "Compatibility with measure-preserving bijections:" In $\mathbb{Q}_p$¡ if $X, Y \subseteq \mathbb{Q}_p^n$, $f : X \to Y$ an n-to-1 map, and $v((Jac(f))(x)) = 0$ for all x, then $\mu_{\text{mot}}(X) = \mu_{\text{mot}}(Y)$

4.2 Cluckers-Loeser Approach: In this approach, we define $\mathcal{C}^0_{\text{mot}}$, $\mathcal{C}_{\text{mot}}(S)$ in analogy to the uniform p-adic setting. But we need to define these rings abstractly, using generators and relations. In particular, $\mathcal{C}^0_{\text{mot}}$ is the ring generated by

1. $[X]$, for $X \subseteq RF^m$ definable.
2. $\frac{1}{\mathbb{L}}$, $\frac{1}{1-\mathbb{L}^{-l}}$, where $\mathbb{L} = [RF]$

and with relations

- If $f : X \to Y$ is a definable bijection then $[X] = [Y]$
- $[X \cup Y] = [X] + [Y]$
- $[X \cdot Y] = [X][Y]$

The ring generated by $[X]$ module the above relations is called the *Grothendieck* ring of definable sets over RF, and is denoted by $K_0(RF)$. So $\mathcal{C}^0_{\text{mot}} = K_0(RF)[\mathbb{L}^{-1}, (1 - \mathbb{L}^{-1})^{-1}]$.

Define $\mathcal{C}_{\text{mot}}(S)$ similarly. Given $f \in \mathcal{C}_{\text{mot}}(S \times VF)$, we define $s \mapsto \int_{VF} f(s, x)\, \mathrm{d}x$ in $\mathcal{C}_{\text{mot}}(S)$ using cell decomposition. We can prove that this map is well-defined, and that this behaves "like integration" in essentially the same way that μ_{mot} behaves "like a measure."

4.3 Remark: This approach is good enough for many applications, but the definitions seem a little bit arbitrary.

4.4 Hrushovski-Kazhdan Approach: Define $\mu_{\text{mot}} : \bigcup_n \{\text{definable sets in } VF^n\} \to \mathcal{C}^0_{\text{mot}}$ to be the universal map satisfying $(*)$ ("behaving like a measure" from before), ie $\mathcal{C}^0_{\text{mot}}$ is generated by $[[X]]$ for definable $X \subseteq VF^n$ ($\mu_{\text{mot}}(X) = [[X]]$) and has relations $(*)$. It follows from $(*)$ that, for example $[[X \cup Y]] = [[X]] + [[Y]]$.

Universality shows that there is a map from the H-K version of $\mathcal{C}^0_{\text{mot}}$ to the C-L version of $\mathcal{C}^0_{\text{mot}}$. This version also has the advantage that it works equally well in ACVF, although we won't use this fact.

4.5 Theorem: (work in progress, Cluckers-H) The map between the H-K and C-L versions of $\mathcal{C}^0_{\text{mot}}$ is actually a ring isomorphism, after modding the H-K version out by the additional relation (∗∗): if $Z, Z' \subseteq RF^n$ are definable and there is a definable bijection $f : Z \to Z'$, then $[[\text{res}^{-1}(Z)]] = [[\text{res}^{-1}(Z')]]$.

5 Uniform p-adic integration: further developments and applications

5.1 Big Topic: Admissible representations over $\mathbb{C}$ of reductive groups G over non-archimidean local fields K. What do these words mean?

- A non-archimidean local field is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$; in fact, everything we've done in the course so far can be extended to finite extensions without too much difficulty. We will stick to the assumption that K is equal to $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for today.
- A reductive group is (something like) GL_n or SL_n.
- An admissible representation is an action of $G(K)$ on $\mathbb{C}$-vector spaces, which is continuous with respect to the discrete topology on $\mathbb{C}$.

Some thoughts on the topic:

- One can classify the irreducible representations of $G(K)$.
- The theory uses p-adic integration (eg integrate functions $G(K) \to \mathbb{C}$)
- Idea of Hales: Make this uniform in K using motivic integration
  - Far goal: obtain "$Irr(G)$", where there is a bijection $Irr(G)_K \to$ {irreducible representations of $G(K)$}.
  - Application: transfer between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$.
  - Fundamental Lemma of the Langlands Program (this is not a lemma, but rather a series of conjectures that have been open for a long time). One version that has been proven is the $\mathbb{F}_p((t))$ version by Ngo; this version only works in $\mathbb{F}_p((t))$, and won him a Fields medal, so this is a hard problem in general.

5.2 Theorem: Suppose $g \in \mathcal{C}_u(X)$. Then for $p \gg 1$, we have: $\forall x \in X_{\mathbb{Q}_p}$, $g_{\mathbb{Q}_p}(x) = 0$ iff $\forall x \in X_{\mathbb{F}_p((t))}$, $g_{\mathbb{F}_p((t))}(x) = 0$.

5.3 Corollary: The fundamental lemma holds in $\mathbb{Q}_p$ for $p \gg 1$. (Hales: the fundamental lemma holds in $\mathbb{Q}_p$ for all p).

5.4 Application: (Harish-Chandra Characters) Given an irreducible representation of $G(K)$, obtain a fucntion $f : G(K) \to \mathbb{C}$ (the Harish-Chandra character).

- In $\mathbb{Q}_p$, this f will be locally integrable (proven by H-Ch in the 70s)
- Local integrability in the $\mathbb{F}_p((t))$ case is still open has been shown for $p \gg 1$ by Cluckers-Gordon-H in 2014, but is still open for small values of p.

5.5 Fact: There is a ring $\mathcal{C}_u^{\exp}(S)$, which contains $\mathcal{C}_u(S)$ and is closed under integration. Theorem 5.2 actually holds with $g \in \mathcal{C}_u^{\exp}(X)$ and the Harish-Chadra characters are in $\mathcal{C}_u^{\exp}(G)$. A similar ring $\mathcal{C}_{\mathrm{mot}}^{\exp}(S)$ can be defined, and many of the results earlier in the week can be shown to hold (with much more difficulty) in $\mathcal{C}_u^{\exp}(S)$ and $\mathcal{C}_{\mathrm{mot}}^{\exp}(S)$. $\mathcal{C}_u^{\exp}(S)$ contains "additive characters:" continuous group homomorphisms $(K,+) \to (\mathbb{C}^\times, \cdot)$, with the discrete topology on $\mathbb{C}$.

5.6 Remark: When we say $f \in \mathcal{C}_u^{\exp}(S)$, we really mean that f is a term for a function from K to $\mathbb{C}$. For example, we might have $f(s) = \alpha(s) + p^{\beta(s)}$ with $\alpha, \beta : S \to VG$. We call such f *motivic terms.*

This allows us to get new formulas, called Locus formulas, for example, $\{x, \in VF : v(x) - p^{v(y)} = 0\}$. Locus sets are sets definable by Locus formulas (generics of definable sets). If $X \subseteq VF^n$ is definable then $1_X \in \mathcal{C}_u(VF^n)$ implies $X = \{x : 1_X(x) - 1 = 0\}$ is a Locus set.

5.7 Theorem:

- If ϕ and ψ are Locus formulas then so are $\phi \wedge \psi$ and $\phi \vee \psi$.
- If $\phi(x,y)$ is a Locus formula then $\forall y \phi(x,y)$ is a Locus formula.
- If ϕ is a Locus sentence then for all $p \gg 1$, $\mathbb{Q}_p \models \phi$ iff $\mathbb{F}_p((t)) \models \phi$.
- If $f(x,y) \in \mathcal{C}_u^{\exp}(X,Y)$ then $\{x : f(x,\cdot)$ is integrable over $Y\}$ is a Locus set.

5.8 Corollary: Local integration is a Locus set.