UNIFORMLY DEFINING p-HENSELIAN VALUATIONS (JOINT WORK WITH JOCHEN KOENIGSMANN)

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ABSTRACT. This is an extended abstract for a talk given at the Oberwolfach Workshop *Valuation Theory and Its Applications* (26 October – 1 November 2014). The workshop was organized by Zoe Chatzidakis (Paris), Franz-Viktor Kuhlmann (Saskatoon), Jochen Koenigsmann (Oxford) and Florian Pop (Philadelphia).

1. p-HENSELIAN VALUATIONS

Admitting a non-trivial p-henselian valuation is a weaker assumption on a field than admitting a non-trivial henselian valuation. Unlike henselianity, p-henselianity is an elementary property in the language of rings. We are interested in the question when a field admits a non-trivial $\emptyset$-definable p-henselian valuation (in the language of rings). These $\emptyset$-definable p-henselian valuations can then often be used to obtain parameter-free definitions of non-trivial henselian valuations. The aim of this talk is give a classification of elementary classes of fields in which the canonical p-henselian valuation is uniformly $\emptyset$-definable. As our definitions involve Beth's Definability Theorem, they require a careful analysis of when a p-henselian valuation coincides with the canonical p-henselian valuation.

Let K be a field and p a prime.

Definition. *We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree. A valuation v on K is called* p-henselian *if v extends uniquely to $K(p)$. We call K* p-henselian *if K admits a non-trivial p-henselian valuation.*

Clearly, this definition only imposes a condition on v if K admits Galois extensions of p-power degree. Every henselian valuation is p-henselian for any prime p, but there are p-henselian non-henselian valuations. Any p-henselian valuation also satisfies an analogue of Hensel's lemma ([2, Proposition 1.2]) Note that p-henselianity is an elementary property of a valued field (K, v) in the language $\mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a unary predicate interpreted as the valuation ring of v ([2, Theorem 1.5]). Furthermore, being p-henselian is an elementary property of K in $\mathcal{L}_{ring}$, provided that K contains a primitive pth root of unity ζ_p in case $\mathrm{char}(K) \neq p$ and that K is not Euclidean in case $p = 2$ ([2, Corollary 2.2]).

Assume that $K \neq K(p)$. We divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{v \text{ } p\text{-henselian on } K | Kv \neq Kv(p)\}$$

and

$$H_2^p(K) = \{v \text{ } p\text{-henselian on } K | Kv = Kv(p)\}.$$

One can show that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is non-empty, then there exists a unique coarsest valuation v_K^p in $H_2^p(K)$; otherwise there exists a unique finest valuation $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation*. If K is p-henselian then v_K^p is non-trivial.

2. Defining the canonical p-henselian valuation

We want to find a uniform definition of the canonical p-henselian valuation. As p-henselianity is an $\mathcal{L}_{ring}$-elementary property, any sufficiently uniform definition of v_K^p on some field K will also define the canonical p-henselian valuation in any field elementarily equivalent to K. This motivates the following

Definition. *Let K be a field, assume that $K \neq K(p)$ and that $\zeta_p \in K$ in case* $\mathrm{char}(K) \neq p$. *We say that v_K^p is* $\emptyset$-definable as such *if there is a parameter-free $\mathcal{L}_{ring}$-formula $\phi_p(x)$ such that*

$$\phi_p(L) = \mathcal{O}_{v_L^p}$$

holds in any $L \equiv K$.

Recall that a field F is called Euclidean if $[F(2) : F] = 2$. This is an elementary property in $\mathcal{L}_{\mathrm{ring}}$: Every Euclidean field is uniquely ordered, the positive elements being exactly the squares. Note that Euclidean fields are the only fields for which $F(p)$ can be a proper finite extension of F.

We are now in a position to state our main theorem:

Theorem 2.1 (Main Theorem in [1])**.** *Fix a prime p. There exists a parameter-free $\mathcal{L}_{ring}$-formula $\phi_p(x)$ such that for any field K with either* $\mathrm{char}(K) = p$ *or $\zeta_p \in K$ the following are equivalent:*

(1) *ϕ_p defines v_K^p as such.*
(2) *v_K^p is $\emptyset$-definable as such.*
(3) *$p \neq 2$ or Kv_K^p is not Euclidean.*

Note that it may well happen that v_K^p is definable, but not definable as such (see the example given in [1, p. 3]). The proof of the theorem uses Beth's Definability Theorem and a complete 1st-order characterization of v_K^p as described in [1] (see Lemmas 4.1, 4.3 and 4.5 as well as Corollary 4.2).

References

[1] Franziska Jahnke and Jochen Koenigsmann, *Uniformly defining p-henselian valuations*, Manuscript, arXiv:1407.8156, 2014.
[2] Jochen Koenigsmann, *p-Henselian Fields*, Manuscripta Mathematica 87(1):89–99, 1995.

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