A $p$-ADIC EXAMPLE FOR THE CHARACTERIZATION OF THE CANONICAL $p$-HENSELIAN VALUATION

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Abstract. The authors have shown recently that the canonical $p$-henselian valuation is uniformly $∅$-definable in the elementary class of fields which have characteristic $p$ or contain a primitive $p$th root of unity $ζ_p$. In order to do this, we proved a classification of the canonical $p$-henselian valuation via case distinction. One of the cases discussed was previously not even known for algebraic extensions of the $p$-adic numbers. The aim of this note is to give a direct proof of this $p$-adic fact.

In [JK14, Theorem 3.1], we give a uniform definition of the canonical $p$-henselian valuation. In this note, we show why one of the main ingredients involved ([JK14, Lemma 4.5]) holds for algebraic extensions of the $p$-adic numbers. The proof is completely independent of the one given in [JK14] and uses specific properties of the $p$-adics. However, it gave us reason to search for a proof of the more general Lemma. For an introduction to $p$-henselian valuations in general and the canonical $p$-henselian valuation in particular, see [EP05, §4.2], [Koe95] and [JK14]. We use the following notation: For a valued field $(F, v)$, we denote the valuation ring by $O_v$ and the maximal ideal by $m_v$. Furthermore, we write $F_v$ for the residue field and $v_F$ for the value group. The canonical $p$-henselian valuation on $F$ is denoted by $v_p^F$.

The fact which we aim to prove in the $p$-adic context is the following:

Lemma (Lemma 4.5 in [JK14]). Let $(F, v)$ be a $p$-henselian valued field with $\text{char}(F) = 0$, $\text{char}(F_v) = p$ and $ζ_p ∈ F$. Assume further that $F_v$ is perfect, $v_F$ has a non-trivial $p$-divisible convex subgroup and that $F_v = F_v(ζ_p)$ holds. Then, we have

$$v = v_p^F ⇐⇒ ∀x ∈ m_v \setminus \{0\} : 1 + x^{-1}(ζ_p - 1)pO_v ⊈ (F^\times)^p.$$  

The aim of this note is to show explicitly that the lemma holds for any algebraic extension $F$ of $Q_p$. For simplicity, we assume $p > 2$. On any such $F$ there is a unique non-trivial $(p)$-henselian valuation, namely the unique extension $v_F$ of the $p$-adic valuation $v_p$ on $Q_p$. Note that $F_vF$ is an algebraic extension of $F_p$ and is thus perfect. Furthermore, the value group $v_FF$ has rank 1, i.e. it contains no non-trivial proper convex subgroups. Thus, in case the value group $v_FF$ contains a non-trivial $p$-divisible subgroup, it is already $p$-divisible. Hence, in the $p$-adic context, the lemma reads as follows:

Lemma ($p$-adic version). Let $p > 2$ and $(F, v_F)$ be an algebraic extension of $(Q_p, v_p)$ with $ζ_p ∈ F$, such that $v_FF$ is $p$-divisible and $v_FF = F_FF(p)$ holds. Then, we have

$$∀x ∈ m_v \setminus \{0\} : 1 + x^{-1}(ζ_p - 1)pO_v ⊈ (F^\times)^p.$$  

The rest of this note is a proof of the above Lemma. Take $(F, v_F) ⊇ (Q_p, v_p)$ an algebraic extension satisfying the conditions of the Lemma. In particular, $F$ is an infinite algebraic extension of $Q_p$. Assume for the sake of a contradiction that the condition in the Lemma does not hold, that is we have

$$(1) \exists x ∈ m_v \setminus \{0\} : 1 + x^{-1}(ζ_p - 1)pO_v ⊆ (F^\times)^p.$$
Note that we have $v_F(p) = 1$ and, as in any algebraic extension of the $p$-adics containing $\zeta_p$, also $v_F(\zeta_p - 1) = \frac{1}{p-1}$. Since we have $0 \notin (F^o)^p$, we get
\[ 1 \notin x^{-1}(\zeta_p - 1)^pO_{v_F} \]
for any $x$ satisfying the condition in 1. In particular, we have $0 < v_F(x) < v_F((\zeta_p - 1)^p)$ for any such $x$.

Consider $\alpha_0 := \inf \left\{ q \in \mathbb{Q} \mid 1 + p^q \cdot O_{v_F} \subseteq (F^o)^p \right\} \in \mathbb{R}$. Then, by the discussion in the last paragraph and since $v_F(p^q) = q$ for any $q \in \mathbb{Q}$ with $p^q \in F$, we have
\[ 0 < \alpha_0 < \frac{p}{p-1}. \]
Note that this implies in particular that
\[ 1 - \alpha_0 \left( 1 - \frac{1}{p} \right) > 0 \]
holds. Take $\alpha \in \mathbb{Q}$, such that we have
\[ \alpha_0 \leq \alpha < \frac{p}{p-1} \]
and
\[ (2) \quad \alpha - \alpha_0 < \frac{1}{2} \left[ 1 - \alpha \left( 1 - \frac{1}{p} \right) \right]. \]
In particular, we get once again
\[ 1 - \alpha \left( 1 - \frac{1}{p} \right) > 0. \]
Note that such $\alpha$ exists since every rational in the real interval
\[ \left[ \alpha_0, \alpha_0 + \frac{1 - \alpha_0 \left( 1 - \frac{1}{p} \right)}{1 + \frac{1}{p}} \right] \]
satisfies condition 2. We want to use $\alpha$ in order to show that there can be no such $\alpha_0$, yielding that there are no algebraic extensions of $\mathbb{Q}_p$ satisfying condition 1.

Take $F \supseteq K \supseteq \mathbb{Q}_p$ such that $K$ is a finite extension of $\mathbb{Q}_p$, say $n = [K : \mathbb{Q}_p] = ef$ with uniformizer $\pi$ (so we have $v(\pi) = \frac{1}{e}$ for the restriction $v$ of $v_F$ to $K$). We assume further that $e$ is ‘large enough’, i.e.
\[ (3) \quad e \alpha \in \mathbb{Z}, \]
\[ (4) \quad p \mid e \alpha, \]
\[ (5) \quad e \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + \frac{1}{p} > 7. \]
Define
\[ r_{\text{max}} := \max \left\{ t \in \mathbb{Z} \mid p \nmid t \text{ and } t \leq e \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + \frac{1}{p} - 1 \right\} \]
so we have $r_{\text{max}} \geq 5$ by assumption 5. Furthermore, calculation yields
\[ (6) \quad r_{\text{max}} \leq e \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + \frac{1}{p} - 1 \leq ep \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right). \]
We want to show first that
\[ 1 + \pi^{ea - r_0} \mathcal{O}_v \subseteq (F^\times)^p \]
holds and then conclude that this contradicts the minimality of \( \alpha_0 \). In order to do this, we consider the element \( 1 + \pi^{ea - r} \) for \( r \leq r_{\text{max}} \) with \( r \in \mathbb{Z} \).

Case 1: We first assume \( p \mid r \) and define \( m := \frac{1}{p}(ea - r) \). Then we have
\[
1 + \pi^{ea - r} = (1 + \pi^m)^p - p\pi^m + \text{terms of higher valuation} \\
\in (1 + \pi^m)^p \subseteq (K^\times)^p (1 + p^s \mathcal{O}_v)
\]
as
\[
v(p\pi^m) = \frac{1}{pe} (ea - r) \geq \frac{1}{pe} (ea - r_{\text{max}}) \geq 1 + \frac{\alpha}{p} - \left(1 - \alpha \left(1 - \frac{1}{p}\right)\right) = \alpha
\]
holds by inequality 6. Thus, as every element \( x \in K \) with
\[
v(x) = v(\pi^{ea - 1}) = \frac{1}{e}(ea - 1)
\]
is of the form \( x = \pi_0^{ea - 1} \) for some uniformizer \( \pi_0 \) of \( K \), we get
\[
(7) \quad 1 + \pi^{ea - r} \mathcal{O}_v \subseteq (K^\times)^p.
\]

Case 2: On the other hand, if \( p \nmid r \), we also consider the element \( 1 + \pi^{ea + 1} \). Then, we have \( \gcd(p, ea + 1) = \gcd(p, ea - r) = 1 \) and thus
\[
1 + \pi^{ea + 1}, \quad 1 + \pi^{ea - r} \not\in (K^\times)^p
\]
and
\[
1 + \pi^{ea + 1} \in p^s \mathcal{O}_v, \quad 1 + \pi^{ea - r} \not\in p^s \mathcal{O}_v
\]
holds. Consider \( L := K \left(\sqrt[p]{1 + \pi^{ea + 1}}\right) \) and let \( v \) be the prolongation of \( v \) to \( L \).

Claim: We have \( 1 + \pi^{ea - r} \in (L^\times)^p \cdot (1 + p^s \mathcal{O}_v) \).

Proof of Claim: We first show that
\[
\rho := 1 - \sqrt[p]{1 + \pi^{ea + 1}} \pi \frac{1}{\pi}
\]
is a uniformizer for \( L \). Indeed, this follows immediately as we have
\[
w(\rho) = \frac{v(\pi)}{p} = \frac{1}{ep}.
\]
So, we get
\[
1 + \pi^{ea + 1} = \left(1 - \rho \pi^\frac{1}{p}\right)^p
\]
\[
= 1 - p\rho \pi^\frac{1}{p} + \cdots + p \left(\rho \pi^\frac{1}{p}\right)^{p-1} - \rho^p \pi^a
\]
Hence, \( \rho \) satisfies the following polynomial equation over \( \mathcal{O}_v \):
\[
\rho^p - p\pi^{\frac{1}{p} - ea} \rho^{p-1} + \cdots + p \pi^{ea \cdot \left(\frac{1}{p} - 1\right)} \rho + \pi = 0
\]
This is in fact an Eisenstein polynomial: Note that
\[
w(\rho \pi^{ea \cdot \left(\frac{1}{p} - 1\right)}) = 1 + \frac{1}{e} \left(\frac{ea}{p}n - ea\right) = 1 + \frac{1}{p}an - \alpha,
\]
so the valuations of the coefficients increase with \( n \). Furthermore,
\[
w(\rho \pi^{ea \cdot \left(\frac{1}{p} - 1\right)}) = 1 - \alpha \left(1 - \frac{1}{p}\right) > 0
\]
holds as \( \alpha < \frac{\rho}{\rho - 1} \). Moreover, we have
\[
\omega(\pi) < \omega\left(\rho^{p\alpha(\frac{1}{p} - 1)}\right) < \cdots \omega\left(\rho^{p\alpha(p - 1) - \rho p^{\rho - 1}}\right)
\]
where the first inequality holds by assumption 5. So the \( \rho \)-adic expansion of \( \pi \) is
\[
\pi = -\rho^p + a_s \rho^s + \text{terms of higher valuation}
\]
with \( s \) such that \( \omega(\rho^s) = \omega\left(\rho^{p\alpha(\frac{1}{p} - 1)}\right) \) and \( a_s \in O_w \), i.e.
\[
s = e \rho \left(1 + \alpha \left(\frac{1}{p} - 1\right)\right) + 1.
\]
Thus, we get
\[
1 + \pi^{e\rho - r} = 1 - \rho^{(e\alpha - r)} + (e\alpha - r) \rho^{(e\alpha - r - 1)} a_s \rho^s + \text{terms of higher valuation}.
\]
Calculation and our assumption on \( r_{\text{max}} \) now yield
\[
\omega\left(\rho^{p\alpha(p - 1) - e \rho^s}\right) \geq \omega\left(\rho^{p\alpha(r_{\text{max}} - 1) - \rho^s}\right) \geq \alpha,
\]
so we get
\[
1 + \pi^{e\rho - r} \in 1 - \rho^{(e\alpha - r)} + p^\alpha O_w.
\]
On the other hand, we also have
\[
(1 - \rho^{e\alpha - r})^p \in 1 - \rho^{(e\alpha - r)p} + p^\alpha + \text{terms of higher valuation}.
\]
Now
\[
\omega(p^\alpha) = 1 + \frac{1}{p^e}(e\alpha - r) \geq 1 + \frac{1}{p^e}(e\alpha - r_{\text{max}}) \geq \alpha
\]
holds by inequality 6. So, we have
\[
(1 - \rho^{e\alpha - r})^p \in 1 - \rho^{(e\alpha - r)p} + p^\alpha O_w
\]
and thus
\[
1 + \pi^{e\rho - r} \in (1 - \rho^{e\alpha - r})^p + p^\alpha O_w \subseteq (L^\times)^p (1 + p^\alpha O_w)
\]
as required. \( \square \)

Hence, for any \( r \leq r_{\text{max}} \), we have shown that \( 1 + \pi^{e\rho - r} \) is a \( p \)th power in \( F \). As in 7, we get
\[
1 + \pi^{e\rho - r_{\text{max}} O_w} \subseteq (F^\times)^p.
\]
Note that we have
\[
r_{\text{max}} > \frac{1}{p} + e \left(1 - \alpha \left(\frac{1}{p} - 1\right)\right) - 3.
\]
Thus, we have
\[ w(\pi^\alpha r_{\text{max}}) \leq \frac{1}{e} \left( e\alpha - \frac{1}{p} - e \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + 3 \right) \]
\[ = \alpha - \frac{1}{ep} - \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + \frac{3}{e} \]
\[ < \alpha - \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + \frac{3}{e} \]
\[ < \alpha - \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) + \frac{1}{2} \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) \quad \text{by assumption 5} \]
\[ = \alpha - \frac{1}{2} \left( 1 - \alpha \left( 1 - \frac{1}{p} \right) \right) \]
\[ < \alpha_0 \quad \text{by inequality 2.} \]

As any element in \( O_p \) is contained in some finite extension \( K \supseteq \mathbb{Q}_p \) with ‘large enough’ inertia degree \( e \) (i.e. such that \( e \) satisfies conditions 3–5), we conclude
\[ 1 + p^{\alpha_0 - \frac{1}{2}(1-\alpha(1-\frac{1}{p}))}O_p \subseteq 1 + p^{\alpha_0}O_p \subseteq (F^p)^p. \]
This contradicts our choice of \( \alpha_0 \).

References


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