A p-ADIC EXAMPLE FOR THE CHARACTERIZATION OF THE CANONICAL p-HENSELIAN VALUATION

FRANZISKA JAHNKE AND JOCHEN KOENIGSMANN

Abstract. The authors have shown recently that the canonical p-henselian valuation is uniformly $\emptyset$-definable in the elementary class of fields which have characteristic p or contain a primitive pth root of unity ζ_p. In order to do this, we proved a classification of the canonical p-henselian valuation via case distinction. One of the cases discussed was previously not even known for algebraic extensions of the p-adic numbers. The aim of this note is to give a direct proof of this p-adic fact.

In [JK14, Theorem 3.1], we give a uniform definition of the canonical p-henselian valuation. In this note, we show why one of the main ingredients involved ([JK14, Lemma 4.5]) holds for algebraic extensions of the p-adic numbers. The proof is completely independent of the one given in [JK14] and uses specific properties of the p-adics. However, it gave us reason to search for a proof of the more general Lemma. For an introduction to p-henselian valuations in general and the canonical p-henselian valuation in particular, see [EP05, §4.2], [Koe95] and [JK14]. We use the following notation: For a valued field (F, v), we denote the valuation ring by $\mathcal{O}_v$ and the maximal ideal by $\mathfrak{m}_v$. Furthermore, we write Fv for the residue field and vF for the value group. The canonical p-henselian valuation on F is denoted by v_F^p.

The fact which we aim to prove in the p-adic context is the following:

Lemma (Lemma 4.5 in [JK14]). *Let (F, v) be a p-henselian valued field with* $\mathrm{char}(F) = 0$, $\mathrm{char}(Fv) = p$ *and $\zeta_p \in F$. Assume further that Fv is perfect, vF has a non-trivial p-divisible convex subgroup and that $Fv = Fv(p)$ holds. Then, we have*

$$v = v_F^p \iff \forall x \in \mathfrak{m}_v \setminus \{0\} : \ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \not\subseteq (F^\times)^p.$$

The aim of this note is to show explicitely that the lemma holds for any algebraic extension F of $\mathbb{Q}_p$. For simplicity, we assume $p > 2$. On any such F there is a unique non-trivial (p-)henselian valuation, namely the unique extension v_F of the p-adic valuation v_p on $\mathbb{Q}_p$. Note that Fv_F is an algebraic extension of $\mathbb{F}_p$ and is thus perfect. Furthermore, the value group $v_F F$ has rank 1, i.e. it contains no non-trivial proper convex subgroups. Thus, in case the value group $v_F F$ contains a non-trivial p-divisible subgroup, it is already p-divisible. Hence, in the p-adic context, the lemma reads as follows:

Lemma (p-adic version). *Let $p > 2$ and (F, v_F) be an algebraic extension of $(\mathbb{Q}_p, v_p)$ with $\zeta_p \in F$, such that $v_F F$ is p-divisible and $Fv_F = Fv_F(p)$ holds. Then, we have*

$$\forall x \in \mathfrak{m}_{v_F} \setminus \{0\} : \ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_{v_F} \not\subseteq (F^\times)^p.$$

The rest of this note is a proof of the above Lemma. Take $(F, v_F) \supseteq (\mathbb{Q}_p, v_p)$ an algebraic extension satisfying the conditions of the Lemma. In particular, F is an infinite algebraic extension of $\mathbb{Q}_p$. Assume for the sake of a contradiction that the condition in the Lemma does not hold, that is we have

$$\exists x \in \mathfrak{m}_{v_F} \setminus \{0\} : \ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_{v_F} \subseteq (F^\times)^p. \tag{1}$$

Note that we have $v_F(p) = 1$ and, as in any algebraic extension of the p-adics containing ζ_p, also $v_F(\zeta_p - 1) = \frac{1}{p-1}$. Since we have $0 \notin (F^\times)^p$, we get

$$1 \notin x^{-1}(\zeta_p - 1)^p \mathcal{O}_{v_F}$$

for any x satisfying the condition in 1. In particular, we have $0 < v_F(x) < v_F((\zeta_p - 1)^p)$ for any such x.

Consider

$$\alpha_0 := \inf \left\{ q \in \mathbb{Q} \;\middle|\; 1 + p^q \cdot \mathcal{O}_{v_F} \subseteq (F^\times)^p \right\} \in \mathbb{R}.$$

Then, by the discussion in the last paragraph and since $v_F(p^q) = q$ for any $q \in \mathbb{Q}$ with $p^q \in F$, we have

$$0 < \alpha_0 < \frac{p}{p-1}.$$

Note that this implies in particular that

$$1 - \alpha_0 \left(1 - \frac{1}{p}\right) > 0$$

holds. Take $\alpha \in \mathbb{Q}$, such that we have

$$\alpha_0 \leq \alpha < \frac{p}{p-1}$$

and

$$\alpha - \alpha_0 < \frac{1}{2}\left[1 - \alpha\left(1 - \frac{1}{p}\right)\right]. \tag{2}$$

In particular, we get once again

$$1 - \alpha\left(1 - \frac{1}{p}\right) > 0.$$

Note that such α exists since every rational in the real interval

$$\left[\alpha_0, \alpha_0 + \frac{1 - \alpha_0\left(1 - \frac{1}{p}\right)}{1 + \frac{1}{p}}\right)$$

satisfies condition 2. We want to use α in order to show that there can be no such α_0, yielding that there are no algebraic extensions of $\mathbb{Q}_p$ satisfying condition 1.

Take $F \supseteq K \supseteq \mathbb{Q}_p$ such that K is a finite extension of $\mathbb{Q}_p$, say $n = [K : \mathbb{Q}_p] = ef$ with uniformizer π (so we have $v(\pi) = \frac{1}{e}$ for the restriction v of v_F to K). We assume further that e is 'large enough', i.e.

$$e\alpha \in \mathbb{Z}, \tag{3}$$

$$p \mid e\alpha, \tag{4}$$

$$e\left(1 - \alpha\left(1 - \frac{1}{p}\right)\right) + \frac{1}{p} > 7. \tag{5}$$

Define

$$r_{\max} := \max \left\{ t \in \mathbb{Z} \;\middle|\; p \nmid t \text{ and } t \leq e\left(1 - \alpha\left(1 - \frac{1}{p}\right)\right) + \frac{1}{p} - 1 \right\}$$

so we have $r_{\max} \geq 5$ by assumption 5. Furthermore, calculation yields

$$r_{\max} \leq e\left(1 - \alpha\left(1 - \frac{1}{p}\right)\right) + \frac{1}{p} - 1 \leq ep\left(1 - \alpha\left(1 - \frac{1}{p}\right)\right). \tag{6}$$

We want to show first that

$$1 + \pi^{e\alpha - r_{\max}} \mathcal{O}_v \subseteq (F^\times)^p$$

holds and then conclude that this contradicts the minimality of α_0. In order to do this, we consider the element $1 + \pi^{e\alpha - r}$ for $r \leq r_{\max}$ with $r \in \mathbb{Z}$.

Case 1: We first assume $p \mid r$ and define $m := \frac{1}{p}(e\alpha - r)$. Then we have

$$\begin{aligned} 1 + \pi^{e\alpha - r} &= (1 + \pi^m)^p - p\pi^m + \text{ terms of higher valuation} \\ &\in (1 + \pi^m)^p \subseteq (K^\times)^p \left(1 + p^\alpha \mathcal{O}_v\right) \end{aligned}$$

as

$$v\left(p\pi^m\right) = \frac{1}{pe}\left(e\alpha - r\right) \geq \frac{1}{pe}\left(e\alpha - r_{\max}\right) \geq 1 + \frac{\alpha}{p} - \left(1 - \alpha\left(1 - \frac{1}{p}\right)\right) = \alpha$$

holds by inequality 6. Thus, as every element $x \in K$ with

$$v(x) = v(\pi^{e\alpha - 1}) = \frac{1}{e}(e\alpha - 1)$$

is of the form $x = \pi_0^{e\alpha - 1}$ for *some* uniformizer π_0 of K, we get

$$1 + \pi^{e\alpha - r} \mathcal{O}_v^\times \subseteq (K^\times)^p. \tag{7}$$

Case 2: On the other hand, if $p \nmid r$, we also consider the element $1 + \pi^{e\alpha + 1}$. Then, we have $\gcd(p, e\alpha + 1) = \gcd(p, e\alpha - r) = 1$ and thus

$$\begin{aligned} &1 + \pi^{e\alpha + 1},\ 1 + \pi^{e\alpha - r} \notin (K^\times)^p \text{ and} \\ &1 + \pi^{e\alpha + 1} \in p^\alpha \mathcal{O}_v,\ 1 + \pi^{e\alpha - r} \notin p^\alpha \mathcal{O}_v \end{aligned}$$

holds. Consider $L := K\left(\sqrt[p]{1 + \pi^{e\alpha + 1}}\right)$ and let w be the prolongation of v to L.

Claim: We have $1 + \pi^{e\alpha - r} \in (L^\times)^p \cdot (1 + p^\alpha \mathcal{O}_w)$.

Proof of Claim: We first show that

$$\rho := \frac{1 - \sqrt[p]{1 + \pi^{e\alpha + 1}}}{\pi^{\frac{e\alpha}{p}}}$$

is a uniformizer for L. Indeed, this follows immediately as we have

$$w(\rho) = \frac{v(\pi)}{p} = \frac{1}{ep}.$$

So, we get

$$\begin{aligned} 1 + \pi^{e\alpha + 1} &= \left(1 - \rho\pi^{\frac{e\alpha}{p}}\right)^p \\ &= 1 - p\rho\pi^{\frac{e\alpha}{p}} + \cdots + p\left(\rho\pi^{\frac{e\alpha}{p}}\right)^{p-1} - \rho^p\pi^{e\alpha} \end{aligned}$$

Hence, ρ satisfies the following polynomial equation over $\mathcal{O}_w$:

$$\rho^p - p\pi^{\frac{e\alpha}{p}(p-1) - e\alpha}\rho^{p-1} + \cdots + p\pi^{e\alpha(\frac{1}{p} - 1)}\rho + \pi = 0$$

This is in fact an Eisenstein polynomial: Note that

$$w\left(p\pi^{\frac{e\alpha}{p}n - e\alpha}\right) = 1 + \frac{1}{e}\left(\frac{e\alpha}{p}n - e\alpha\right) = 1 + \frac{1}{p}\alpha n - \alpha,$$

so the valuations of the coefficients increase with n. Furthermore,

$$w\left(p\pi^{e\alpha(\frac{1}{p} - 1)}\right) = 1 - \alpha\left(1 - \frac{1}{p}\right) > 0$$

holds as $\alpha < \frac{p}{p-1}$. Moreover, we have

$$w(\pi) < w\left(p\pi^{e\alpha(\frac{1}{p}-1)}\right) < \cdots < w\left(p\pi^{\frac{e\alpha}{p}(p-1)-e\alpha}\rho^{p-1}\right)$$

where the first inequality holds by assumption 5. So the ρ-adic expansion of π is

$$\pi = -\rho^p + a_s\rho^s + \text{ terms of higher valuation}$$

with s such that $w(\rho^s) = w\left(p\pi^{e\alpha(\frac{1}{p}-1)}\right)$ and $a_s \in \mathcal{O}_w^\times$, i.e.

$$s = ep\left(1 + \alpha\left(\frac{1}{p} - 1\right)\right) + 1.$$

Thus, we get

$$1 + \pi^{e\alpha-r} = 1 - \rho^{p(e\alpha-r)} \pm (e\alpha - r)\rho^{p(e\alpha-r-1)}a_s\rho^s + \text{ terms of higher valuation.}$$

Calculation and our assumption on $r_{\max}$ now yield

$$w\left(\rho^{p(e\alpha-1)}a_s\rho^s\right) \geq w\left(\rho^{p(e\alpha-r_{\max}-1)}a_s\rho^s\right) \geq \alpha,$$

so we get

$$1 + \pi^{e\alpha-r} \in 1 - \rho^{p(e\alpha-r)} + p^\alpha\mathcal{O}_w.$$

On the other hand, we also have

$$(1 - \rho^{e\alpha-r})^p = 1 - \rho^{(e\alpha-r)p} + p\rho^{e\alpha-r} + \text{ terms of higher valuation.}$$

Now

$$w(p\rho^{e\alpha-r}) = 1 + \frac{1}{pe}(e\alpha - r) \geq 1 + \frac{1}{pe}(e\alpha - r_{\max}) \geq \alpha$$

holds by inequality 6. So, we have

$$(1 - \rho^{e\alpha-r})^p \in 1 - \rho^{(e\alpha-r)p} + p^\alpha\mathcal{O}_w$$

and thus

$$1 + \pi^{e\alpha-r} \in (1 - \rho^{e\alpha-r})^p + p^\alpha\mathcal{O}_w \subseteq (L^\times)^p\,(1 + p^\alpha\mathcal{O}_w)$$

as required. $\square_{\text{Claim}}$

Hence, for any $r \leq r_{\max}$, we have shown that $1 + \pi^{e\alpha-r}$ is a pth power in F. As in 7, we get

$$1 + \pi^{e\alpha-r_{\max}}\mathcal{O}_v \subseteq (F^\times)^p.$$

Note that we have

$$r_{\max} > \frac{1}{p} + e\left(1 - \alpha\left(1 - \frac{1}{p}\right)\right) - 3.$$

Thus, we have

$$\begin{aligned} w\left(\pi^{e\alpha - r_{\max}}\right) &\leq \frac{1}{e}\left(e\alpha - \frac{1}{p} - e\left(1-\alpha\left(1-\frac{1}{p}\right)\right) + 3\right) \\ &= \alpha - \frac{1}{ep} - \left(1-\alpha\left(1-\frac{1}{p}\right)\right) + \frac{3}{e} \\ &< \alpha - \left(1-\alpha\left(1-\frac{1}{p}\right)\right) + \frac{3}{e} \\ &< \alpha - \left(1-\alpha\left(1-\frac{1}{p}\right)\right) + \frac{1}{2}\left(1-\alpha\left(1-\frac{1}{p}\right)\right) && \text{by assumption 5} \\ &= \alpha - \frac{1}{2}\left(1-\alpha\left(1-\frac{1}{p}\right)\right) \\ &< \alpha_0 && \text{by inequality 2.} \end{aligned}$$

As any element in $\mathcal{O}_{v_F}$ is contained in some finite extension $K \supseteq \mathbb{Q}_p$ with 'large enough' inertia degree e (i.e. such that e satisfies conditions 3–5), we conclude

$$1 + p^{\alpha - \frac{1}{2}(1-\alpha(1-\frac{1}{p}))}\mathcal{O}_{v_F} \subsetneq 1 + p^{\alpha_0}\mathcal{O}_{v_F} \subseteq (F^\times)^p. \tag{8}$$

This contradicts our choice of α_0.

References

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Institut für Mathematische Logik, Einsteinstr. 62, 48149 Münster, Germany
E-mail address: franziska.jahnke@uni-muenster.de

Mathematical Institute, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, UK
E-mail address: koenigsmann@maths.ox.ac.uk