# Model theory of perfectoid fields

Lecture series[1] by Franziska Jahnke

The aim of this lecture series was to present model-theoretic material such that the main ideas and methods are accessible to researchers from non-archimedean geometry, thus allowing new exchange and interactions between the two subjects. It took place in form of a three-part lecture series at the conference "Model-theoretic methods in non-archimedean geometry", Münster, Germany, 27th – 31st January 2025.

The plan for the lecture series is as follows:

(1) The classics: AKE theorem ($p \to \infty$)
(2) Model theory of deeply ramified fields ($e \to \infty$)
   - AKE-type results for perfectoid fields
   - almost purity theorem for valuations rings
(3) What's next?

Background material on valued fields can be found in the book by Engler and Prestel [EP05], model-theoretic material can be found in the book by Tent and Ziegler [TZ12].

## 1. Lecture (27/01/2025)

§ 0. **Motivation.** Underlying question—*version 1*: how similar are

$$\mathbb{F}_p(\!(t)\!) = \left\{ \sum_{i=m}^{\infty} a_i t^i : a_i \in \mathbb{F}_p,\, m \in \mathbb{Z} \right\} \quad \text{and} \quad \mathbb{Q}_p = \left\{ \sum_{i=m}^{\infty} a_i p^i : a_i \in \{0, 1, \dots, p-1\},\, m \in \mathbb{Z} \right\}?$$

We consider these two fields as *valued fields*, which are fields endowed with a valuation.

**Definition.** A *valuation* on a field $K$ is a map $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, where $\Gamma$ is an ordered abelian group such that for all $x, y \in K$:

(i) $v(x) = \infty \iff x = 0$
(ii) $v(xy) = v(x) + v(y)$
(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

**Example.** For the above two fields, we set:

- $v_t\left(\sum a_i t^i\right) = \min\{i : a_i \neq 0\}$
- $v_p\left(\sum a_i p^i\right) = \min\{i : a_i \neq 0\}$.

Valuations come equipped with the following objects:

- $\mathcal{O}_v \coloneqq \{x \in K : v(x) \geq 0\}$, the *valuation ring*,

[1] Notes typed by Leo Gitin.

- $\mathfrak{m}_v := \{x \in K : v(x) > 0\}$, the (unique) maximal ideal of $\mathcal{O}_v$,
- $Kv := \mathcal{O}_v/\mathfrak{m}_v$, the *residue field*,
- furthermore, we write $vK$ for $\Gamma$.

**Example.** We have $\mathbb{Q}_p v_p = \mathbb{F}_p = \mathbb{F}_p(\!(t)\!)v_t$ and $v_p\mathbb{Q}_p = \mathbb{Z} = v_t\mathbb{F}_p(\!(t)\!)$.

**Note.** A valuation gives rise to a notion of Cauchy sequences, under which $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p(\!(t)\!), v_t)$ become complete valued fields.

Underlying question—*version 2*: how similar are two complete valued fields $(K, v)$ and $(L, w)$ with $vK = wL$ and $Kv = Lw$? In what follows, we will see two asymptotic answers with slogans $p \to \infty$ and $e \to \infty$, respectively.

In the context of model theory, completeness is not the right notion (in particular when the value group is non-archimedean, i.e., does not embed into the reals). Instead, we need the following:

**Definition.** A valued field $(K, v)$ is *henselian* if for all $f(X) \in \mathcal{O}_v[X]$ and $a \in \mathcal{O}_v$ such that $f(a) \in \mathfrak{m}_v$ and $f'(a) \notin \mathfrak{m}_v$, there exists $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$.

A condition equivalent to henselianty is: $v$ extends uniquely to any algebraic extension $L/K$.

**Hensel's Lemma.** *Every complete $\mathbb{Z}$-valued field is henselian.*

This can be shown via Newton's method.

**Theorem** (Ax-Kochen/Ershov 1965, [AK65, Erš65])**.** *Let $(K, v)$ and $(L, w)$ be henselian valued fields with $\operatorname{char}(Kv) = \operatorname{char}(Lw) = 0$. Then:*

$$(K, v) \underset{\mathcal{L}_{val}}{\equiv} (L, w) \iff Kv \underset{\mathcal{L}_{ring}}{\equiv} Lw \quad \text{and} \quad vK \underset{\mathcal{L}_{oag}}{\equiv} wL$$

We will explain the notation used shortly.

**Fact.** *Any valued field $(K, v)$ has a henselization, i.e., there exists a henselian valued field $(K^h, v^h)$ extending $(K, v)$ such that for all $(L, w) \supseteq (K, v)$ henselian, we can find an embedding $(K^h, v^h)$ into $(L, w)$ with*

$$\begin{array}{ccc} (L, w) & & \\ \subseteq \uparrow & \overset{\exists}{\nwarrow} & \circlearrowleft . \\ (K, v) & \xrightarrow{\subseteq} & (K^h, v^h) \end{array}$$

*Moreover, $vK = v^hK^h$, $Kv = K^hv^h$ and $K^h/K$ is a separable algebraic extension.*

**Example.** Let $K = (\mathbb{Q}(t), v_t)$ with $v_t\big(\sum a_i t^i\big) = \min\{i : a_i \neq 0\}$. Then

$$\begin{array}{ccc} (\mathbb{Q}(\!(t)\!), v_t) & & \\ \subseteq \uparrow & \overset{\exists}{\nwarrow} & \circlearrowleft . \\ (\mathbb{Q}(t), v_t) & \xrightarrow{\subseteq} & (\mathbb{Q}(t)^h, v_t^h) \end{array}$$

Our next steps are:

- Explain the AK/E statement,
- the connection to $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$, and
- prove AK/E.

## § 1. **First-order logic.**

### § 1.1. *Languages and structures.*

**Definition.** A *first-order language* $\mathcal{L}$ is given by

- a set of constant symbols,
- a set of function symbols, each with an arity,
- a set of relation symbols, each with an arity,
- logical symbols for
  - a binary relation $\doteq$,
  - a set of variables $\{v_i : i \in \mathbb{N}\}$,
  - connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$,
  - quantifiers $\forall$, $\exists$.

For convenience of notation, we may further allow additional symbols for variables such as $x$ or $y$.

**Example.** We mentioned the following languages before:

- $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot, -\}$ consist of two constant symbols 0 and 1, two binary function symbols $+$ and $\cdot$, and one unary function symbol $-$.
- $\mathcal{L}_{\text{val}} = \mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$ with an extra unary relation symbol $\mathcal{O}$.
- $\mathcal{L}_{\text{oag}} = \{0, +, -, <\}$, where the last symbol is a binary relation symbol.

**Definition.** An $\mathcal{L}$-*structure* $\mathcal{A}$ consists of a non-empty set $A$ (the *universe*) together with interpretations of each of the symbols in $\mathcal{L}$.

This means that any $n$-ary function symbol $f \in \mathcal{L}$ is interpreted as a (genuine) function

$$f^{\mathcal{A}} : A^n \longrightarrow A.$$

Any constant symbol $c \in \mathcal{L}$ is interpreted as a constant $c^{\mathcal{A}} \in A$ and any $n$-ary relation symbol $R \in \mathcal{L}$ as a relation $R^{\mathcal{A}} \subseteq A^n$.

**Example.** Any ring is naturally an $\mathcal{L}_{\text{ring}}$-structure, but not conversely so.

Roughly speaking, the set of $\mathcal{L}$-*formulas* is recursively defined as follows: it is a set of finite strings of $\mathcal{L}$-symbols which "make sense" (i.e. are well-formed) in such a way, that for any $\mathcal{L}$-structure $\mathcal{A}$ and $\mathcal{L}$-formula $\varphi$, one can substitute elements from $A$ into variables of $\varphi$ that are not bounded by a quantifier and obtain a statement that is either true or false in $\mathcal{A}$.

**Note.** It is crucial that quantifiers may only range over elements (not subsets) of $A$.

**Definition.** An $\mathcal{L}$-*sentence* is an $\mathcal{L}$-formula with no free variables.

**Example.** The following examples give a flavour of what we mean:

- $\exists y\,(y \cdot y \doteq x)$ makes sense in $\mathcal{L}_{\text{ring}}$.
- $y + y$ is not an $\mathcal{L}_{\text{ring}}$-formula, as it cannot be interpreted unambiguously as *true* or *false* in a structure.
- $\sum_{i=0}^{n} v_i \cdot x^i \doteq 0$ is an $\mathcal{L}_{\text{ring}}$-formula (here, the sum $\sum$ and $X^i$ should not be viewed as part of the language; they are solely used as a shorthand for the obvious expressions involving $+$ and $\cdot$ only).
- $\forall v_0 \ldots \forall v_n \exists x \sum_{i=0}^{n} v_i \cdot x^i \doteq 0$ is an $\mathcal{L}_{\text{ring}}$-sentence. Taken as a set of sentences parametrized by $n$, this expresses the property of being algebraically closed.

For us, the following sentences are of particular importance:

- Fix $d, N \in \mathbb{N}$. Then the statement "every homogeneous polynomial of degree $d$ in $N$-many variables has a non-trivial root" is a first-order statement, i.e., it can be written as a first-order sentence. $(*)$
- (J. Robinson) Consider $\varphi(x) \equiv \exists y\, y^2 = 1 + px^2$. Then, for $p \neq 2$:

$$\mathbb{Q}_p \models \varphi(a) \Longleftrightarrow a \in \mathbb{Z}_p = \mathcal{O}_{v_p} \tag{$**$}$$

Here, "$\models$" is the *satisfaction relation*. We write $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ if $\varphi$ holds in $\mathcal{A}$ when we substitute $a_1, \ldots, a_n$ for the free variables occurring in $\varphi$.

§ 1.2. *Elementary equivalences and elementary substructures.*

**Definition.** Given two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \equiv \mathcal{B}$ if the same $\mathcal{L}$-sentences $\varphi$ hold in $\mathcal{A}$ and $\mathcal{B}$, i.e.,

$$\mathcal{A} \models \varphi \Longleftrightarrow \mathcal{B} \models \varphi$$

for all $\varphi$, and call $\mathcal{A}$ and $\mathcal{B}$ *elementary equivalent* in that case.

**Note.** Being algebraically closed is a first-order property while being complete is not (this would require the ability to quantify over Cauchy sequences; more formally, this is a consequence of the Löwenheim–Skolem theorem).

**Definition.** Let $\mathcal{A} \subseteq \mathcal{B}$ be two $\mathcal{L}$-structures (where $\mathcal{A}$ is an $\mathcal{L}$-substructure of $\mathcal{B}$, i.e., all interpretations of symbols restrict from $\mathcal{B}$ to $\mathcal{A}$). Then $\mathcal{A} \preccurlyeq \mathcal{B}$ ($\mathcal{A}$ is an *elementary $\mathcal{L}$-substructure*) if for all $\mathcal{L}$-formulas $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$, we have

$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \Longleftrightarrow \mathcal{B} \models \varphi(a_1, \ldots, a_n).$$

**Note.** This equivalence holds automatically in some cases:

- when $\varphi$ is quantifier-free (qf),
- $\Leftarrow$ holds when $\varphi$ is *universal*, i.e., of the form $\forall x_1 \cdots \forall x_n \psi(x_1, \ldots, x_{n+k})$ with $\psi$ qf,
- $\Rightarrow$ holds when $\varphi$ is *existential*, i.e., of the form $\exists x_1 \cdots \exists x_n \psi(x_1, \ldots, x_{n+k})$ with $\psi$ qf.

§ 1.3. *Ultraproducts.*

**Definition.** Let $X$ be a set. An *ultrafilter* $\mathcal{U}$ on $X$ is a set $\mathcal{U} \subseteq \mathcal{P}(X)$ such that for all $A, B \subseteq X$ we have

- $\emptyset \notin \mathcal{U}$,
- $A \in \mathcal{U}$, $A \subseteq B \subseteq X$, then $B \in \mathcal{U}$,
- $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$,
- $A \subseteq X$ then $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

**Example.** For any fixed $x \in X$, we have that $\mathcal{U}_x = \{A \subseteq X : x \in A\}$ is an ultrafilter, which is called the *principal* ultrafilter at $x$.

One can show, using Zorn's lemma, that any infinite set $X$ admits a non-principal ultrafilter. An ultrafilter is non-principal if and only if it contains all cofinite sets. These are the only kinds of ultrafilters we will consider.

**Definition.** Given a family of $\mathcal{L}$-structures $\{\mathcal{A}_i\}_{i\in X}$, we define

$$\prod_{\mathcal{U}} \mathcal{A}_i := \prod_{i\in X} \mathcal{A}_i \Big/ \sim,$$

where $(a_i)_{i\in X} \sim (b_i)_{i\in X}$ if and only if $\{i \in X : a_i = b_i\} \in \mathcal{U}$. The symbols in $\mathcal{L}$ are interpreted component-wise in the product.

**Łoś's theorem.** *Let $\{\mathcal{A}_i\}_{i\in X}$ be a family of $\mathcal{L}$-structures indexed by $X$ and $\mathcal{U}$ a non-principal ultrafilter on $X$. Then for any $\mathcal{L}$-sentence $\varphi$, we have that $\prod_{\mathcal{U}} \mathcal{A}_i \models \varphi$ if and only if $\varphi$ holds in almost all $\mathcal{A}_i$, i.e., when $\{i \in X : \mathcal{A}_i \models \varphi\} \in \mathcal{U}$.*

**Example.** Let $\mathcal{U}$ be a non-principal ultrafilter on the set of primes $\mathbb{P}$.

- $\prod_{\mathcal{U}} \mathbb{F}_p$ is a field of characteristic 0. The existence of multiplicative inverses is a first-order property that holds in any $\mathbb{F}_p$, so it holds in the ultraproduct. To "not have characteristic $q$" for some prime $q$ is a first-order property satisfied by almost all $p$ (all $p \neq q$).
- Let

$$(K, v) := \prod_{\mathcal{U}} (\mathbb{Q}_p, v_p) \quad \text{and} \quad (L, w) := \prod_{\mathcal{U}} \mathbb{F}_p(\!(t)\!).$$

  One can similarly show that these are henselian valued fields with the same residue field $\prod_{\mathcal{U}} \mathbb{F}_p$ and value group $\prod_{\mathcal{U}} \mathbb{Z}$. In this case, the AKE theorem applies and tells us that $(K, v) \equiv (L, w)$. This was used by Ax and Kochen to prove an asymptotic version of Artin's conjecture (that all $\mathbb{Q}_p$ are $C_2$-fields, i.e., that for all $d, N \in \mathbb{N}$ with $N > d^2$, every homogeneous polynomial of degree $d$ in $N$-many variables has a non-trivial root).

Let us give a few more details regarding the last example. Artin conjectured that $\mathbb{Q}_p$ is a $C_2$-field (for any prime $p$). Before, it was known that:

**Theorem** (Chevalley + Greenberg)**.** *$\mathbb{F}_p(\!(t)\!)$ is a $C_2$-field for any prime $p$.*

As we have explained before, $(K, v) \equiv (L, w)$. Using Łoś's theorem, one can show that for any $\mathcal{L}_{\text{val}}$-sentence $\varphi$, there exists $N = N(\varphi) \gg 0$ such that for any $p \geq N$:

$$(\mathbb{Q}_p, v_p) \models \varphi \iff (\mathbb{F}_p(\!(t)\!), v_t) \models \varphi$$

In other words, the elementary equivalence of ultrapowers implies that the theories of $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p(\!(t)\!), v_t)$ coincide "asymptotically" (as $p \to \infty$).

If we apply this observation to the $C_2$-property using the theorem above, we get:

**Theorem.** *For any fixed degree $d \in \mathbb{N}$, there is $N(d) \gg 0$ such that for any $p \geq N(d)$, the field $\mathbb{Q}_p$ is $C_2(d)$.*

Terjanian constructed counterexamples that show that no $\mathbb{Q}_p$ is a $C_2$-field (without the restriction on $d$) [Ter66, Ter80].

## 2. Lecture (28/01/2025)

Recall the AK/E theorem from the last lecture:

**Theorem** (Ax-Kochen/Ershov 1965)**.** *Let $(K, v)$ and $(L, w)$ be henselian valued fields with* $\mathrm{char}(Kv) = \mathrm{char}(Lw) = 0$*. Then:*

$$(K, v) \underset{\mathcal{L}_{val}}{\equiv} (L, w) \Longleftrightarrow Kv \underset{\mathcal{L}_{ring}}{\equiv} Lw \quad \text{and} \quad vK \underset{\mathcal{L}_{oag}}{\equiv} wL$$

**Corollary.** *Given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{P}$, we have:*

$$\prod_{\mathcal{U}} (\mathbb{Q}_p, v_p) \equiv \prod_{\mathcal{U}} (\mathbb{F}_p(\!(t)\!), v_t).$$

Today, we would like to sketch a prove of this theorem.

§ 1.4. *Ultrapowers.* Given a non-principal ultrafilter $\mathcal{U}$ on a set $X$ and an $\mathcal{L}$-structure $\mathcal{A}$, we define

$$\mathcal{A}^{\mathcal{U}} := \prod_{\mathcal{U}} \mathcal{A},$$

which is just the ultraproduct of $|X|$-many copies of $\mathcal{A}$ over $\mathcal{U}$. By Łoś's theorem, the diagonal embedding

$$\mathcal{A} \longrightarrow \mathcal{A}^{\mathcal{U}}, \quad a \longmapsto [(a)_{i \in X}]$$

is an elementary embedding.

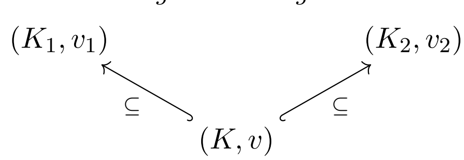
**Theorem** (Keisler-Shelah)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. Then $\mathcal{A} \equiv \mathcal{B}$ if and only if there exists a non-principal ultrafilter $\mathcal{U}$ on some infinite set $X$ such that $\mathcal{A}^{\mathcal{U}} \cong \mathcal{B}^{\mathcal{U}}$.*

We use this theorem as a black box. One of the nice properties of ultrapowers is that they are *saturated*. Given $\{\varphi_i(x) : i \in \mathbb{N}\}$ finitely satisfiable in $\mathcal{A}$ (any finite conjunction is satisfiable), then $\bigwedge_{i \in \mathbb{N}} \varphi_i(x)$ is satisfiable in $\mathcal{A}^{\mathcal{U}}$. [Indeed, assume for all $i \in \mathbb{N}$, we have $\mathcal{A} \models \varphi_0(a_i) \wedge \ldots \wedge \varphi_i(a_i)$. Then $\mathcal{A}^{\mathcal{U}} \models \varphi_i\big([(a_i)_{\mathcal{U}}]\big)$, for all $i \in \mathbb{N}$, by Łoś's theorem.]

**Example.** Let $\Gamma \subseteq \mathbb{R}$ be an ordered abelian group and let $\gamma \in \Gamma \setminus \{0\}$ and $X = \mathbb{N}$. Then $[(i\gamma)_{i \in X}]$ describes an element of $\Gamma^{\mathcal{U}}$ outside of $\langle \gamma \rangle$ (the *convex hull* of $\gamma$, i.e., the smallest convex subgroup containing $\gamma$). Hence $\Gamma^{\mathcal{U}}$ is non-archimedean. This is one of the reasons why, in model theory, one has to study valuations of arbitrary rank.
Moreover, if $\Gamma \ncong \mathbb{Z}$ (i.e., $\Gamma$ is dense in $\mathbb{R}$), then for any $\delta \in \Gamma_{>0}$, $n \in \mathbb{N}$, $\gamma \in \Gamma$, there exists $\gamma_0 \in [0, \delta]$ such that $\gamma - \gamma_0 \in n\Gamma$. This implies that $\Gamma^{\mathcal{U}} / \langle \delta \rangle$ is $n$-divisible for all $n \in \mathbb{N}_{>0}$. (We will need this fact later in our study of perfectoid fields.)

§ 2. **Proof of AKE.** The following is the key ingredient:

**Embedding Lemma.** *Assume we are given a diagram*



*of henselian valued fields with* $\mathrm{char}(Kv) = 0$*,* $v_1K_1/vK$ *torsion-free, and* $(K_2, v_2)$ *being* $|K_1|^+$*-saturated (i.e., for all* $A \subseteq K_2$ *with* $|A| \leq |K_1|$*, any set of* $\mathcal{L}_{val}(A)$*-formulas* $\Phi(x)$ *that is finitely satisfiable in* $K_2$ *is satisfiable in* $K_2$*). Furthermore, assume we are given embeddings*

$$\sigma_k : K_1v_1 \hookrightarrow_{Kv} K_2v_2$$
$$\sigma_\Gamma : v_1K_1 \hookrightarrow_{vK} v_2K_2.$$

*on the level of residue fields and value groups. Then, there exists an embedding*

$$\sigma : (K_1, v_1) \hookrightarrow_{(K,v)} (K_2, v_2)$$

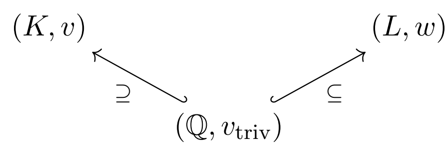
*that induces* $\sigma_k$ *and* $\sigma_\Gamma$*.*

§ 2.1. *Proof sketch of AKE assuming the Embedding Lemma.* This proof is based on Prestel's proof in presented in [PD11, Appendix A]. We want to prove the following statement:

**Theorem** (Ax-Kochen/Ershov 1965)**.** *Let* $(K, v)$ *and* $(L, w)$ *be henselian valued fields with* $\mathrm{char}(Kv) = \mathrm{char}(Lw) = 0$*. Then:*

$$(K, v) \underset{\mathcal{L}_{val}}{\equiv} (L, w) \iff Kv \underset{\mathcal{L}_{ring}}{\equiv} Lw \quad \textit{and} \quad vK \underset{\mathcal{L}_{oag}}{\equiv} wL$$

Based on the assumption that $\mathrm{char}(Kv) = 0$, we start with the diagram:




Using Keisler-Shelah, we may assume without loss of generality that $Kv \cong Lw$ and $vK \cong wL$ (by replacing the valued fields involved with suitable ultrapowers). Replacing again, if necessary, $(L, w)$ by a $|K|^+$-saturated elementary extension $(L_1, w_1)$, we can find an embedding

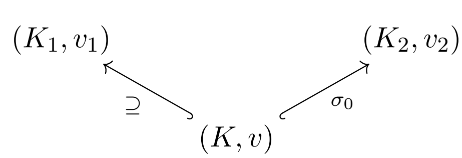
$$\sigma_0 : (K, v) \hookrightarrow_{(K,v)} (L_1, w_1).$$

using the Embedding Lemma (the condition of torsion-freeness is trivial in this case). In particular, for any existential $\mathcal{L}_{\mathrm{val}}$-sentence $\varphi$ (or formula $\varphi(x)$), we have $K \models \varphi \Longrightarrow L \models \varphi$ (or $K \models \varphi(c) \Longrightarrow L \models \varphi(c)$ for any constant $c$ in $K$, more generally). Continuing in this fashion (back-and-forth), we can find an embedding

$$\sigma_1 : (L_1, w_1) \hookrightarrow_{(K,v)} (K_1, v_1).$$

into a sufficiently saturated elementary extension $(K_1, v_1) \succcurlyeq (K, v)$, and so on. In each step, we get a larger class of formulas (with more and more quantifier changes) that is reflected between $(K, v)$ and $(L, w)$. In the end, we obtain $(K, v) \equiv (L, w)$.

§ 2.2. *Proof sketch of the Embedding Lemma (see Prestel-Delzel ???)* We need to complete the diagram:



Let $\sigma_0$ be the partial embedding $(K, v) \hookrightarrow (K_2, v_2)$. Our aim is to extend $\sigma_0$ to all of $(K_1, v_1)$. This will happen in three steps:

*Step 1.* Extend $\sigma_0$ to $\sigma_1$ such that $\mathrm{dom}(\sigma_1) = L \subseteq K_1$ and $Lv_1 = K_1 v_1$.

*Step 2.* Extend $\sigma_1$ to $\sigma_2$ such that $\mathrm{dom}(\sigma_2) = M \subseteq K_1$ and $v_1 M = v_1 K_1$.

*Step 3.* Take care of *immediate extension.* These are extensions that do not extend the value group and residue field.

We will need the following extra ingredient:

**Fundamental (in)equality.** *Let $(K, v)$ be a valued field and let $L/K$ be finite. Then*

$$[L : K] = \sum_{w \supseteq v} \underbrace{e(w/v)}_{(wL:vK)} \underbrace{f(w/v)}_{[Lw:Kv]} p^{d_w},$$

*where*

$$p = \begin{cases} \mathrm{char}(Kw) & \textit{if } \mathrm{char}(Kw) \neq 0 \\ 1 & \textit{otherwise.} \end{cases}$$

*In particular,*

$$\sum_{w \supseteq v} e(w/v) f(w/v) \leq [L : K].$$

Now for the proof.

**Step 1.** If $K_1 v_1 \neq Kv$, consider some $\overline{x} \in K_1 v_1 \setminus Kv$.

If $\overline{x}$ is algebraic over $Kv$, let $f(X) \in \mathcal{O}_v[X]$ be a monic polynomial that reduces to the minimal polynomial $\overline{f}(X) \in (Kv)[X]$ of $\overline{x}$ over $Kv$. By henselianity, $f$ has a root $\alpha_1 \in K_1$, respectively $\alpha_2 \in K_2$. Hence, we can extend $\sigma_0$ to

$$K(\alpha_1) \xrightarrow{\cong} K(\alpha_2)$$

and one can easily check that this isomorphism preserves valuations.

If $\overline{x}$ is transcendental over $Kv$, let $x_1 \in \mathcal{O}_{v_1}$ and $x_2 \in \mathcal{O}_{v_2}$ be elements that reduce to $\overline{x_1} = \overline{x} = \overline{x_2}$. It follows from the fundamental inequality that both $x_1$ and $x_2$ are transcendental over $K$. One can check that $v$ extends uniquely to $K(x_1)$ and $K(x_2)$ as a *Gauss extension*, given by

$$v_1\left(\sum a_i x_1^i\right) = \min_i v(a_i) \quad \text{and} \quad v_2\left(\sum a_i x_2^i\right) = \min_i v(a_i).$$

This then induces an isomorphism of valued fields

$$K(x_1) \xrightarrow{\cong} K(x_2).$$

Thereafter, to restore henselianity, one can further extend to $K(x_1)^h$ by the universal property of henselizations.

After iterating these two constructions (none of which extend the value group), we obtain an embedding $\sigma_1$ of $(L, v_1)$ into $(K_2, v_2)$ which extends $\sigma_0$ and such that $Lv_1 = K_1 v_1$ and $v_1 L = v_1 K_1$.

**Step 2.** Similar (though requires more care).

**Step 3.** Assume $(K_1, v_1)/(K, v_1)$ is immediate. By the fundamental inequality, any $x_1 \in K_1 \setminus K$ must be transcendental. For $a \in K$, choose $b_a \in K$ such that $v(x_1 - a) = v(b_a)$. Then

$$\Pi(x) \coloneqq \{v(b_a) = v_2(x - a) : a \in K\} \cup \{f(x) \neq 0 : f(X) \in K[X] \text{ irred. of deg} > 1\}$$

is realized in $K_2$, say by $x_2$. It turns out that

$$K(x_1) \xrightarrow{\cong} K(x_2)$$

even as valued fields. To show that the valuation is preserved by the isomorphism requires some work and is the content of *Kaplansky theory*.

## 3. Lecture (29/01/2025)

§ 3. **The coarsening method.** For a final time, we recall:

**Theorem** (Ax-Kochen/Ershov 1965). *Let $(K,v)$ and $(L,w)$ be henselian valued fields with* $\mathrm{char}(Kv) = \mathrm{char}(Lw) = 0$. *Then:*

$$(K,v) \underset{\mathcal{L}_{val}}{\equiv} (L,w) \iff Kv \underset{\mathcal{L}_{ring}}{\equiv} Lw \quad \text{and} \quad vK \underset{\mathcal{L}_{oag}}{\equiv} wL$$

**Corollary.** *Let $(K,v)$ and $(L,w)$ be henselian unramified valued fields of characteristic $(0,p)$ (this means that $v(p)$ is minimal positive in $vK$ and the residue field has characteristic $p$). Assume $Kv$ is perfect. Then*

$$(K,v) \equiv (L,w) \iff Kv \equiv Lw \text{ and } vK \equiv wL.$$

**Note.** This yields an axiomatization of $\mathrm{Th}(\mathbb{Q}_p, v_p)$: we have $(K,v) \equiv (\mathbb{Q}_p, v_p)$ if and only if $(K,v)$ is henselian of characteristic $(0,p)$, unramified, $Kv = \mathbb{F}_p$, and $vK \equiv \mathbb{Z}$.

*Proof of Corollary.* "$\Longrightarrow$" Follows easily, as $Kv$ and $vK$ are interpretable in $(K,v)$.
"$\Longleftarrow$" By taking ultrapowers, we may assume without loss of generality that $Kv \cong Lw$ and $vK \cong wL$ and that $(K,v)$, $(L,w)$ are $\aleph_1$-saturated. Let $\Delta_0$ be the minimal convex subgroup of $vK$ (resp. $wL$) containing $v(p)$. This will be isomorphic to $\mathbb{Z}$. We obtain *coarsened* valuations $v_0 : K \twoheadrightarrow vK/\Delta_0$, $w_0 : L \twoheadrightarrow wL/\Delta_0$ and furthermore, a decomposition

$$\begin{array}{ccc} K & & L \\ \Big| v_0 & & \Big| w_0 \\ Kv_0 & & Lw_0 \\ \mathbb{Z}\Big| \overline{v} & & \mathbb{Z}\Big| \overline{w} \\ Kv & & Lw \end{array}$$

into places. By saturation, $(Kv_0, \overline{v})$ and $(Lw_0, \overline{w})$ must be complete $\mathbb{Z}$-valued fields with perfect residue fields. By the functoriality of Witt vectors, this means that

$$Kv_0 \cong \mathrm{Frac}(W[Kv]) \cong \mathrm{Frac}(W[Lw]) \cong Lw_0.$$

Hence, we can conclude by the AKE theorem that $(K, v_0) \equiv (L, w_0)$. At the very least, the fields $K$ and $L$ are elementarily equivalent in the language of rings $\mathcal{L}_{\mathrm{ring}}$. By J. Robinson's definition of the valuation ring in unramified valued fields, cf. (∗∗), it follows that $(K,v) \equiv (L,w)$ (with respect to the original valuations $v$ and $w$). □

What about similar phenomena in other settings?

- Recently, Anscombe and the lecturer treated the case when $Kv$ is imperfect in the above corollary. [AJ22]
- For $\mathrm{char}(K) = \mathrm{char}(L) = p > 0$, these questions are widely open. In particular, it is not known if

$$(\mathbb{F}_p(\!(t)\!), v_t) \equiv (\mathbb{F}_p(t)^h, v_t).$$

**Definition.** We say $(K,v)$ is *algebraically maximal* if for all proper algebraic extensions $(L,w)/(K,v)$, we have that $Kv \subsetneq Lw$ or $vK \subsetneq wL$.

**Example.** $(\mathbb{F}_p(\!(t)\!)^{\mathrm{perf}}, v_t) \not\equiv (L, v_t)$, where $L \supseteq \mathbb{F}_p(\!(t)\!)^{\mathrm{perf}}$ is algebraically maximal and immediate. This is because adjoining an Artin-Schreier root of $\frac{1}{t}$ generates an immediate extension of $\mathbb{F}_p(\!(t)\!)^{\mathrm{perf}}$.

**Theorem** (Kuhlmann, [Kuh16])**.** *Let $(K,v)$ and $(L,w)$ be tame (i.e., algebraically maximal with $p$-divisible value group and perfect residue field). Then whenever $(K,v) \subseteq (L,w)$,*

$$(K,v) \preccurlyeq (L,w) \iff vK \preccurlyeq wL \quad \textit{and} \quad Kv \preccurlyeq Lw$$

*and if* $\mathrm{char}(K) = p$*, moreover*

$$(K,v) \equiv (L,w) \iff vK \equiv wL \quad \textit{and} \quad Kv \equiv Lw$$

## § 4. AKE principles in deeply ramified fields.

**Theorem** (J.-Kartas, [JK25])**.** *Let $(K,v) \subseteq (L,w)$ be two henselian valued fields of residue characteristic $p$ such that $\mathcal{O}_v/p\mathcal{O}_v$ and $\mathcal{O}_w/p\mathcal{O}_w$ are semi-perfect (i.e., the Frobenius map $x \longmapsto x^p$ is surjective). Suppose $\varpi \in \mathfrak{m}_v$ is such that $\mathcal{O}_v[\varpi^{-1}]$ and $\mathcal{O}_w[\varpi^{-1}]$ are algebraically maximal. Then*

$$(K,v) \preccurlyeq (L,w) \iff \mathcal{O}_v/\varpi\mathcal{O}_v \preccurlyeq \mathcal{O}_w/\varpi\mathcal{O}_w \quad \textit{and} \quad (vK, v(\varpi)) \preccurlyeq (wL, w(\varpi)).$$

*If $vK$ and $wL$ are regularly dense (i.e., elementarily equivalent to some $\Gamma \leq \mathbb{R}$ and $\Gamma \not\equiv \mathbb{Z}$), then the value group condition can be omitted.*

This, in particular, applies to perfectoid fields.

**Definition.** A valued field $(K,v)$ of residue characteristic $p > 0$ is *perfectoid* if

(i) $vK$ is archimedean and $p$-divisible,

(ii) $\mathcal{O}_v/p\mathcal{O}_v$ is semi-perfect,

(iii) $(K,v)$ is complete.

For $(K,v)$ perfectoid, $t \in \mathfrak{m}_v \setminus \{0\}$, we get that $\mathcal{O}_v[t^{-1}] = K$ is trivial.

**Main application.** *We have:*

$$\big((\mathbb{F}_p(t)^h)^{\mathrm{perf}}, v_t\big) \preccurlyeq \big(\mathbb{F}_p(\!(t)\!)^{\mathrm{perf}}, v_t\big) \preccurlyeq \big(\mathbb{F}_p\widehat{(\!(t)\!)^{\mathrm{perf}}}, v_t\big)$$

*Proof sketch.* Let $(K,v)$, $(L,w)$ and $\varpi$ be as in the assumption. Without loss of generality (in a non-trivial way), we may assume $(K,v)$ and $(L,w)$ are $\aleph_1$-saturated and $vK$ is $p$-divisible. For exposition, we assume $\mathrm{char}(K) = p$.

Let $\Delta_0$ be the minimal convex subgroup of $vK$ containing $v\varpi$ and $\Delta_\varpi$ be the maximal convex subgroup of $vK$ not containing $v\varpi$. We obtain a decomposition of $v$:

$$K \xrightarrow[v_0]{vK/\Delta_0} K_0 \xrightarrow{\Delta_0/\Delta_\varpi} K_\varpi \xrightarrow[\bar{v}]{\Delta_\varpi} Kv, \qquad K \xrightarrow[v_\varpi]{vK/\Delta_\varpi} K_\varpi$$

One can compute that

$$\mathcal{O}_{v_0} = \mathcal{O}_v[\varpi^{-1}]$$
$$K_\varpi = \mathrm{Frac}((\mathcal{O}_v/\varpi\mathcal{O}_v)_{\mathrm{red}}),$$

and similarly for $L$ and its decomposed places. In particular, we can show that $\mathcal{O}_v/\varpi\mathcal{O}_v \preccurlyeq \mathcal{O}_w/\varpi\mathcal{O}_w$ implies

$$K_\varpi = \mathrm{Frac}((\mathcal{O}_v/\varpi\mathcal{O}_v)_{\mathrm{red}}) \preccurlyeq \mathrm{Frac}((\mathcal{O}_w/\varpi\mathcal{O}_w)_{\mathrm{red}}) = L_\varpi.$$

Moreover, our assumptions on $\mathcal{O}_v[\varpi^{-1}]$ and $\mathcal{O}_w[\varpi^{-1}]$ imply that $v_0$ and $w_0$ are tame, whereas $K_0 \longrightarrow K_\varpi$ and $L_0 \longrightarrow L_\varpi$ are tame by saturation.

From Kuhlmann, we deduce that $(K, v_\varpi) \preccurlyeq (L, w_\varpi)$ and $(K_\varpi, \overline{v}) \subseteq (L_\varpi, \overline{w})$. To get to the right language and obtain $(K, v) \preccurlyeq (L, w)$, one needs to furthermore use an Embedding Lemma for tame fields à la Kuhlmann. □

§ 5. **Tilting and almost purity.** Given a perfectoid field $(K, v)$ with $\mathrm{char}(K, Kv) = (0, p)$, define

$$\mathcal{O}_v^\flat := \varprojlim_{\mathrm{Frob}} \mathcal{O}_v/p\mathcal{O}_v = \varprojlim_{\mathrm{Frob}} \left(\mathcal{O}_v/p\mathcal{O}_v \stackrel{\mathrm{Frob}}{\longleftarrow} \mathcal{O}_v/p\mathcal{O}_v \stackrel{\mathrm{Frob}}{\longleftarrow} \cdots\right)$$

Let $K^\flat = \mathrm{Frac}(\mathcal{O}_v^\flat)$. Then $(K^\flat, v^\flat)$, with $v^\flat$ the valuation on $K^\flat$ corresponding to the valuation ring $\mathcal{O}_v^\flat$, is called the *tilt* and there is $t \in \mathfrak{m}_{v^\flat}$ with

$$\mathcal{O}_v/p\mathcal{O}_v \cong \mathcal{O}_v^\flat/t\mathcal{O}_v^\flat, \;\; Kv \cong K^\flat v^\flat, \;\text{ and }\; vK \cong v^\flat K^\flat.$$

The quintessence is that $K$ and $K^\flat$ are very similar, formalising the idea of letting $e \to \infty$. Tilting allows one to transfer arithmetic properties between $K$ and $K^\flat$, and vice versa [Sch12].

**Theorem** (Fontaine-Wintenberger). *We have:*

$$\mathrm{Gal}(K) \cong \mathrm{Gal}(K^\flat)$$

*as profinite groups. Here,* $\mathrm{Gal}(K) := \mathrm{Gal}(K^{\mathrm{alg}}/K)$ *is the absolute Galois group of* $K$*.*

We approach this model-theoretically by studying ultrapowers of $(K, v)$ and $(K^\flat, v^\flat)$.

**Theorem** (Taming theorem, J.-Kartas). *Let $(k, v)$ be a perfectoid field of mixed characteristic and $(K, v) = (k, v)^{\mathcal{U}}$ an ultrapower. Then there exists a coarsening $\mathcal{O}_u \supseteq \mathcal{O}_v$ on $K$ such that*

- *$(K, u)$ is tame with divisible value group. In particular,* $\mathrm{Gal}(K) \cong \mathrm{Gal}(Ku)$*.*
- *$(k^\flat, v^\flat) \preccurlyeq (Ku, \overline{v})$.*

One can show that this theorem specializes to Fontaine-Wintenberger; it is in fact a non-standard version of the almost purity theorem over perfectoid valuation rings. Furthermore, we have the following consequences:

**Corollary.** *For $(K, v)$, $(L, w)$ perfectoid of mixed characteristic:*

- $(K, v) \equiv (L, w) \Longrightarrow (K^\flat, v^\flat) \equiv (L^\flat, w^\flat)$
- $(K, v) \preccurlyeq (L, w) \Longleftrightarrow (K^\flat, v^\flat) \preccurlyeq (L^\flat, w^\flat)$, *assuming* $(K, v) \subseteq (L, w)$.

Open questions are:

- Can we formalize additional structure (e.g. analytic structure)?
- What about perfectoid rings/perfectoid spaces?
- What are interesting first-order properties to tilt resp. untilt?
(Assume every separable rationally connected variety has a rational point over $K^\flat$. Does the same hold over $K$?—"Untilting Lang-Manin Conjecture")
- If the $C_i$-property holds for $K^\flat$, does it hold for $K$ as well?

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