WHEN DOES DEPENDENCE TRANSFER FROM FIELDS TO HENSELIAN EXPANSIONS?

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Abstract. This is an extended abstract for a talk given at the Oberwolfach Workshop *Model Theory: Groups, Geometries, and Combinatorics* (January 2016). The workshop was organized by K. Tent (Münster), F. Wagner (Lyon) and M. Ziegler (Freiburg).

1. Introduction and Motivation

There are many open questions connecting NIP and henselianity, e.g.,

Question 1.1. (1) *Is any valued NIP field (K, v) henselian?*
(2) *Let K be an NIP field, neither separably closed nor real closed. Does K admit a definable non-trivial henselian valuation?*

Both of these questions have been recently answered positively in the special case where 'NIP' is replaced with 'dp-minimal' (cf. Johnson's results in [5]).

The question discussed in this talk is the following:

Question 1.2. *Let K be an NIP field and v a henselian valuation on K. Is (K, v) NIP?*

Known results in this direction were obtained by Delon and Bélair (see [1] for the relevant definitions):

Theorem 1.3. *Let (K, v) be a henselian valued field.*
(1) [2] *If the residue field Kv is NIP (as a pure field) of characteristic* $\mathrm{char}(Kv) = 0$*, then (K, v) is NIP (as a valued field).*
(2) [1, Corollaire 7.6] *Assume that (K, v) is Kaplansky and algebraically maximal of characteristic $p > 0$. If Kv is NIP (as a pure field) then (K, v) is NIP (as a valued field).*

The aim of this talk is to show that if K is an NIP field and v a henselian valuation on K, then (K, v) is NIP or Kv is separably closed. As separably closed fields are NIP, we obtain that the residue field Kv is always NIP (as a pure field).

2. Externally definable sets

Throughout the section, let M be a structure in some language $\mathcal{L}$.

Definition. *Let $N \succ M$ be an $|M|^+$-saturated elementary extension. A subset $A \subseteq M$ is called* externally definable *if it is of the form*

$$\{a \in M^{|\bar{x}|} \mid N \models \varphi(a, b)\}$$

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for some $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and some $b \in N^{|\bar{y}|}$.

The notion of externally definable sets does not depend on the choice of N.

Definition. *The* Shelah-expansion M^{Sh} *is the expansion of M by predicates for all externally definable sets.*

Proposition 2.1 (Shelah, see [7, Chapter 3])**.** *If M is NIP then so is M^{Sh}.*

Example 2.2. *Let (K, w) be a valued field and v be a coarsening of w. Then, there is a convex subgroup $\Delta \leq wK$ such that we have $vK \cong wK/\Delta$. As Δ is externally definable in the ordered abelian group wK, the valuation ring $\mathcal{O}_v$ is definable in $(K, w)^{\mathrm{Sh}}$.*

3. p-henselian valuations

Throughout this section, let K be a field and p a prime. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree (in a fixed algebraic closure). Note that we have

- $K \neq K(p)$ iff K admits a Galois extension of degree p and
- $[K(p) : K] < \infty \implies K = K(p)$ or $p = 2$ and $K(2) = K(\sqrt{-1})$.

Definition. *A valuation v on a field K is called* p-henselian *if v extends uniquely to $K(p)$. We call K* p-henselian *if K admits a non-trivial p-henselian valuation.*

In particular, every henselian valuation is p-henselian for all primes p. Assume $K \neq K(p)$. Then, there is a canonical p-henselian valuation on K: We divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{v \text{ } p\text{-henselian on } K \mid Kv \neq Kv(p)\}$$

and

$$H_2^p(K) = \{v \text{ } p\text{-henselian on } K \mid Kv = Kv(p)\}.$$

One can show that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is non-empty, then there exists a unique coarsest valuation v_K^p in $H_2^p(K)$; otherwise there exists a unique finest valuation $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation* (see [6] for more details).

The following properties of the canonical p-henselian valuation follow immediately from the definition:

- If K is p-henselian then v_K^p is non-trivial.
- Any p-henselian valuation on K is comparable to v_K^p.
- If v is a p-henselian valuation on K with $Kv \neq Kv(p)$, then v coarsens v_K^p.

Theorem 3.1 ([4, Theorem 3.1])**.** *Fix a prime p. Let K be a field with $K \neq K(p)$. In case $\mathrm{char}(K) \neq p$, assume that K contains a primitive pth root of unity. In case $p = 2$ and $\mathrm{char}(K) = 0$, assume further that K is not real. There exists a parameter-free $\mathcal{L}_{ring}$-formula $\phi_p(x)$ independent of K with $\phi_p(K) = \mathcal{O}_{v_K^p}$.*

4. External definability of henselian valuations

Proposition 4.1. *Let (K, v) be henselian such that Kv is neither separably closed nor real closed. Then v is definable in K^{Sh}.*

Proof. (Sketch) Assume Kv is neither separably closed nor real closed. Choose any prime p such that Kv has a finite Galois extension of degree divisible by p^2. We construct some finite extension (L, v') of (K, v) such that v_L^p is $\emptyset$-definable on L in $\mathcal{L}_{\mathrm{ring}}$ and such that v_L^p refines v'. The restriction of v_L^p to K is then definable in $\mathcal{L}_{\mathrm{ring}}$. Thus, v is definable in the Shelah expansion of K. □

Proposition 4.2. *Let (K, v) be henselian such that Kv is real closed. Then v is definable in K^{Sh}.*

Proof. (Sketch) Assume that (K, v) is henselian and Kv is real closed. Using the definability of the 2-henselian valuation, we reduce to the case that vK is 2-divisible. In this case, K is uniquely ordered. By [3, Corollary 3.6] and Beth Definability Theorem, the ordering on K is $\mathcal{L}_{\mathrm{ring}}$-definable. Let $\mathcal{O}_w \subseteq K$ be the convex hull of $\mathbb{Z}$ in K. Then, $\mathcal{O}_w$ is definable in K^{Sh}. By [3, Proposition 2.2], w is the finest henselian valuation ring on K with real closed residue field. In particular, we get $\mathcal{O}_w \subseteq \mathcal{O}_v$ and hence $\mathcal{O}_v$ is also definable in K^{Sh}. □

Corollary 4.3. *Let K be NIP, v henselian on K.*

1. *If Kv is not separably closed, then (K, v) is NIP.*
2. *Kv is NIP as a pure field.*

The question what happens in case Kv is separably closed remains. In particular it would be interesting to know an answer to the following

Question 4.4. *Let (K, v) be henselian, Kv NIP, not perfect. Is (K, v) NIP?*

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