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# Universal operator algebras associated to homogeneous varieties

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## The universal operator algebras $\mathcal{A}_I$

### A basic question

Let $p_1, \dots, p_r \in \mathbb{C}[z_1, \dots, z_d]$. Consider the set of equations

$$p_1(z_1, \dots, z_d) = 0$$
$$\vdots$$
$$p_r(z_1, \dots, z_d) = 0.$$

In complex algebraic geometry, the set of solutions $(z_1, \dots, z_d) \in \mathbb{C}^d$ is studied. We ask: Which commuting tuples $(T_1, \dots, T_n) \in \mathcal{B}(\mathcal{H})$ of operators satisfy these equations?

If we require that the tuple $(T_1, \dots, T_d)$ is a row contraction, that is, that $\sum_{i=1}^d T_i T_i^* \le 1$, then there is a universal solution.

### Definition of $\mathcal{A}_I$

Let $I \subset \mathbb{C}[z_1, \dots, z_d]$ be a radical homogeneous ideal. We define $\mathcal{A}_I$ to be the *universal operator algebra generated by a commuting row contraction satisfying the relations in* $I$.

## Some remarks

### Remarks on the definition of $\mathcal{A}_I$

1. $\mathcal{A}_I$ is a *non self-adjoint* operator algebra.
2. The algebra $\mathcal{A}_I$ is generated by a commuting row contraction $S = (S_1, \dots, S_d)$ with

$$p(S) = 0 \quad \text{for all } p \in I,$$

and has the following universal property:
Given any commuting row contraction $T = (T_1, \dots, T_d)$ on a Hilbert space $\mathcal{H}$ satisfying $p(T) = 0$ for all $p \in I$, there is a unique unital completely contractive algebra homomorphism

$$\mathcal{A}_I \to \mathcal{B}(\mathcal{H}), \quad S_i \mapsto T_i \quad (i = 1, \dots, d).$$

3. $\mathcal{A}_I$ can be realized as an algebra of analytic functions on the variety associated to $I$.

### Example

If $d = 1$ and $I = \{0\}$, then $\mathcal{A}_I$ is the disk algebra by von Neumann's inequality.

## The isomorphism problem for the algebras $\mathcal{A}_I$

Kenneth R. Davidson, Christopher Ramsey, and Orr Shalit asked in [1]:

### Question

Let $I, J \subset \mathbb{C}[z_1, \dots, z_d]$ be two radical homogeneous ideals. When are $\mathcal{A}_I$ and $\mathcal{A}_J$ isometrically or topologically isomorphic?

Let
$$V(I) = \{z \in \mathbb{C}^d : p(z) = 0 \text{ for all } p \in I\}$$
be the vanishing locus of $I$.

### Theorem (Davidson, Ramsey, Shalit 2011)

The following are equivalent:
(i) $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic.
(ii) There is a unitary on $\mathbb{C}^d$ which maps $V(J)$ onto $V(I)$.

## Topological isomorphisms

The question about topological isomorphisms is more difficult.

### Theorem (Davidson, Ramsey, Shalit 2011)

Consider the following assertions:
(i) $\mathcal{A}_I$ and $\mathcal{A}_J$ are topologically isomorphic.
(ii) There is an invertible linear map on $\mathbb{C}^d$ which maps $V(J)$ onto $V(I)$ and is isometric on $V(J)$.

Then (i) $\Rightarrow$ (ii) holds. Moreover, (ii) $\Rightarrow$ (i) is true if the geometry of $V(J)$ and $V(I)$ is not too complicated.

### Conjecture (Davidson, Ramsey, Shalit 2011)

The implication (ii) $\Rightarrow$ (i) in the preceding theorem is true in general.

## References

[1] Kenneth R. Davidson, Christopher Ramsey, and Orr Moshe Shalit, *The isomorphism problem for some universal operator algebras*, Adv. Math. **228** (2011), no. 1, 167–218.

[2] Michael Hartz, *Topological isomorphisms for some universal operator algebras*, J. Funct. Anal. **263** (2012), no. 11, 3564–3587.

## Sums of Fock spaces

For a Hilbert space $E$ with $\dim(E) < \infty$, let

$$\mathcal{F}(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

denote the full Fock space.

### A reduction

To establish the conjecture, it suffices to show that for any finite collection of subspaces $V_1, \dots, V_r \subset \mathbb{C}^d$, the algebraic sum

$$\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r) \subset \mathcal{F}(\mathbb{C}^d)$$

is closed.

### Theorem (H. 2012)

Given finitely many subspaces $V_1, \dots, V_r \subset \mathbb{C}^d$, the algebraic sum

$$\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r) \subset \mathcal{F}(\mathbb{C}^d)$$

is closed. Hence, the conjecture holds.

## A key point in the proof of the main result

Projections can be used to determine if algebraic sums are closed. Let $\mathcal{H} = \mathcal{F}(\mathbb{C}^d)$ and let

$$\mathcal{A} = C^*\Big([P_{\mathcal{F}(V_1)}], \dots, [P_{\mathcal{F}(V_r)}]\Big) \subset \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

$\mathcal{A}$ contains the information of whether $\mathcal{F}(V_1) + \ldots + \mathcal{F}(V_r)$ is closed.

### Key lemma

Without loss of generality, we may assume that $V_1 \cap \ldots \cap V_r = \{0\}$. In this case, every irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ satisfies

$$\pi\Big([P_{\mathcal{F}(V_i)}]\Big) = 0 \quad \text{for some } i \in \{1, \dots, r\}.$$

Roughly speaking, the key lemma says that every irreducible representation does not see one of the subspaces. This makes an inductive argument possible.