Circle maps and C^*-algebras

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Groupoids from circle maps

Assume throughout $\phi : \mathbb{T} \to \mathbb{T}$ is a continuous piecewise monotone map. Define the valency of ϕ at $x \in \mathbb{T}$ by

- $\text{val}(\phi, x) = (+,+)$ if ϕ is increasing at x.
- $\text{val}(\phi, x) = (-,-)$ if ϕ is decreasing at x.
- $\text{val}(\phi, x) = (+,-)$ if x is a local maximum for ϕ.
- $\text{val}(\phi, x) = (-,+)$ if x is a local minimum for ϕ.

Definition Define the following relations

$$x \overset{(n,m)}{\sim} y \Leftrightarrow \phi^n(x) = \phi^m(y), \text{val}(\phi^n, x) = \text{val}(\phi^m, y),$$

$$x \overset{k}{\sim} y \Leftrightarrow \exists n, m \in \mathbb{N} : k = n - m, x \overset{(n,m)}{\sim} y$$

Define the set $\Gamma_\phi^+ = \{(x, k, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} \mid x \overset{k}{\sim} y\}$.

The set Γ_ϕ^+ is an étale locally compact Hausdorff groupoid, which can be identified with a groupoid associated to a pseudo-group $\mathscr{P}^+$ of orientation preserving homeomorphisms on $\mathbb{T}$.

Structure of $C_r^*(\Gamma_\phi^+)$

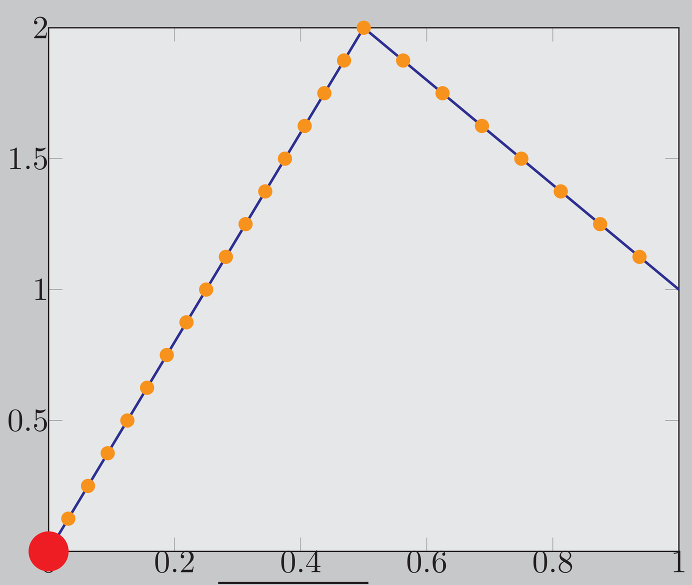
Assume that ϕ is surjective and not locally injective.

Theorem $C_r^*(\Gamma_\phi^+)$ is unital, separable, nuclear and satisfies the universal coefficient theorem by Rosenberg and Schochet [RS].

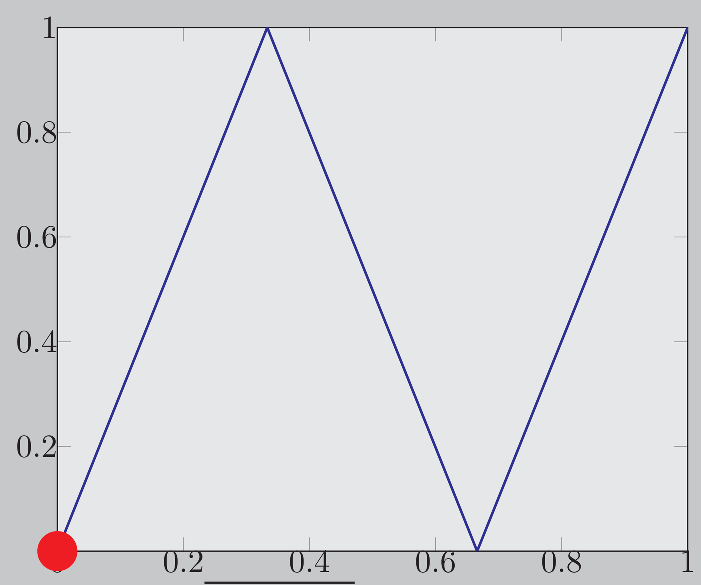
Note, ϕ is transitive if and only if there is a point with dense forward orbit as ϕ is surjective, and ϕ is totally transitive if and only if ϕ is exact.

Theorem If ϕ is transitive, then $C_r^*(\Gamma_\phi^+)$ is purely infinite.

Consider the Γ_ϕ^+-orbit $\Gamma_\phi^+\{x\} = \{y \in \mathbb{T} \,|\, x \overset{k}{\sim} y\}$ of $x \in \mathbb{T}$. Then $C_r^*(\Gamma_\phi^+)$ is simple if and only if the Γ_ϕ^+-orbit of x is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$.



$\overline{\Gamma_\phi^+\{0\}} = \mathbb{T}$



$\overline{\Gamma_\phi^+\{0\}} = \{0\}$

A point $e \in \mathbb{T} \setminus \mathscr{C}$ is an exceptional fixed point if $\phi^{-1}(e) \setminus \mathscr{C} = \{e\}$, where $\mathscr{C} = \mathscr{C}_1$, and $\mathscr{C}_n$ denotes the set of critical points for ϕ^n.

Theorem $C_r^*(\Gamma_\phi^+)$ is simple if and only if ϕ is exact (or, equivalently, totally transitive) and has no exceptional fixed points.

Hence if ϕ is exact and has no exceptional fixed points, then $C_r^*(\Gamma_\phi^+)$ is classified by its K-theory, by the Kirchberg–Phillips theorem [Ph].

Structure of $C_r^*(R_\phi^+)$

Define the equivalence relations $R_\phi^+(n) = \{(x, 0, y) \in \Gamma_\phi^+ \mid x \overset{(n,n)}{\sim} y\}$ and $R_\phi^+ = \{(x, 0, y) \in \Gamma_\phi^+\} = \bigcup_{n\in\mathbb{N}} R_\phi^+(n)$. Then

$$C_r^*(R_\phi^+) = \overline{\bigcup_{n\in\mathbb{N}} \mathbb{D}_n}, \quad \text{where} \quad \mathbb{D}_n = C_r^*(R_\phi^+(n)).$$

Moreover the core $C_r^*(R_\phi^+)$ of $C_r^*(\Gamma_\phi^+)$ is the fixed point algebra of the gauge action on $C_r^*(\Gamma_\phi^+)$ given by $\beta_z(f)(x, k, y) = z^k f(x, k, y)$ for all $f \in C_c(\Gamma_\phi^+)$.

Theorem There are finite dimensional C^*-algebras $\mathbb{A}_n$ and $\mathbb{B}_n$ and $*$-homomorphisms $I_n, U_n : \mathbb{A}_n \to \mathbb{B}_n$ such that

$$\mathbb{D}_n \simeq \{(a, b) \in \mathbb{A}_n \oplus C([0, 1], \mathbb{B}_n) \mid I_n(a) = b(0), U_k(a) = b(1)\}.$$

Markov maps and K-theory

Assume that ϕ is a Markov map, i.e. maps critical points to critical points. We note the following result about Markov maps:

Theorem There is a $k \in \mathbb{N}$ such that for any $j \geq k$ and $x \in \phi(\mathbb{T} \setminus \mathscr{C})$, there exist $y_\pm \in \mathbb{T}$ such that $\phi^j(y_\pm) = x$ and $\text{val}(\phi^j, y_\pm) = (\pm, \pm)$.

The number k is called the order of ϕ. Also simplicity simplifies:

Theorem $C_r^*(\Gamma_\phi^+)$ is simple if and only if ϕ is transitive.

Moreover there is an algorithm for computing the K-theory of $C_r(\Gamma_\phi^+)$. Using work of Katsura [Ka], one can construct a C^*-correspondence E on $C_r^*(R_\phi^+(k))$ whose corresponding C^*-algebra $\mathscr{O}_E$ is isomorphic to $C_r^*(\Gamma_\phi^+)$. This yields a six-term exact sequence

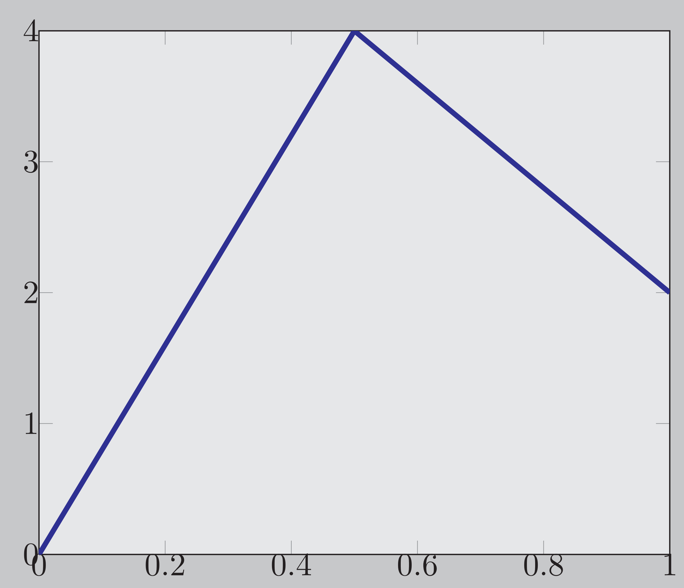
$$\begin{array}{ccccc} K_0(C_r^*(R_\phi^+(k))) & \longrightarrow & K_0(C_r^*(R_\phi^+(k))) & \longrightarrow & K_0(C_r^*(\Gamma_\phi^+)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma_\phi^+)) & \longleftarrow & K_1(C_r^*(R_\phi^+(k))) & \longleftarrow & K_1(C_r^*(R_\phi^+(k))) \end{array}$$

Next, we note that the C^*-algebras $\mathbb{A}_k$ and $\mathbb{B}_k$ are finite dimensional, and that $K_0(\mathbb{A}_k)$ and $K_0(\mathbb{B}_k)$ can be determined by looking at the graph of ϕ. Putting all this together, one obtains short exact sequences

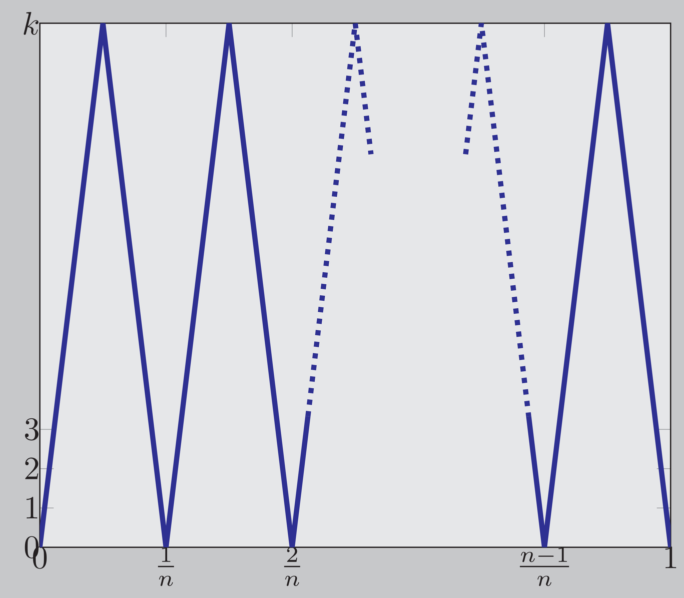
$$0 \longrightarrow \text{coker}(1 - \tilde{A}) \longrightarrow K_0(C_r^*(\Gamma_\phi^+)) \longrightarrow \ker(1 - \tilde{B}) \longrightarrow 0$$

$$0 \longrightarrow \text{coker}(1 - \tilde{B}) \longrightarrow K_1(C_r^*(\Gamma_\phi^+)) \longrightarrow \ker(1 - \tilde{A}) \longrightarrow 0$$

where $\tilde{A}$ and $\tilde{B}$ are restrictions of endomorphisms $A : K_0(\mathbb{A}_k) \to K_0(\mathbb{A}_{k+1})$ and $B : K_0(\mathbb{B}_k) \to K_0(\mathbb{B}_{k+1})$ to $\ker(I_k - U_k)_0$ and $\text{coker}(I_k - U_k)_0$ resp. The maps $\tilde{A}$ and $\tilde{B}$ can also be computed by considering the graph of ϕ. Some examples are shown below.



$K_0(C_r^*(\Gamma_\phi^+)) = \mathbb{Z} \oplus \mathbb{Z}_5,$
$K_1(C_r^*(\Gamma_\phi^+)) = \mathbb{Z}.$



$K_0(C_r^*(\Gamma_\phi^+)) = \mathbb{Z} \oplus \mathbb{Z}_{2nk-1},$
$K_1(C_r^*(\Gamma_\phi^+)) = \mathbb{Z}.$

Further work

There are still several questions that remain open:

- What happens to the whole story if the pseudo-group $\mathscr{P}^+$ is replaced by the pseudo-group $\mathscr{P}$ of all homeomorphisms on $\mathbb{T}$.
- What more can we say about the structure of the core of $C_r^*(\Gamma_\phi^+)$? How does the structure relate to conditions of ϕ?
- Is it possible to relax the requirement that the map ϕ is Markov?

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