Westfälische Wilhelms-Universität Münster

SFB Workshop: Groups, Dynamical Systems and C^*-algebras
August 20-24, 2013

K-theory for certain crossed products by $\mathbb{Z}^2$-actions

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Introduction

One of the most celebrated results in K-theory for C^*-algebra is the Pimsner Voiculescu exact sequence [4] associated with a C^*-dynamical system $(A, \alpha, \mathbb{Z})$

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{K_0(\alpha)-id} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[K_1(\alpha)-id]{} & K_1(A) \end{array}$$

Hence, the K-theory of the crossed product is determined by $(K_*(A), K_*(\alpha), \mathbb{Z})$ up to an extension problem. One might ask what happens if we look at $\mathbb{Z}^n$-actions.

Question 1: To what extent does $(K_*(A), K_*(\alpha), \mathbb{Z}^n)$ determine the K-theory of the crossed product $K_*(A \rtimes_\alpha \mathbb{Z}^n)$?

Analogously to the case $n = 1$, one may hope that almost all information is contained in the induced action on K-theoretic level.

Question 2: Does it suffice to solve certain extension problems depending only on $(K_*(A), K_*(\alpha), \mathbb{Z}^n)$ in order to determine $K_*(A \rtimes_\alpha \mathbb{Z}^n)$?

An application of the classification done in [1] shows that Question 2 has a negative answer in general. Unfortunately, [1] makes heavy use of the Kirchberg-Phillips classification of Kirchberg algebras [2, 3]. The disadvantage of this approach is that it only states the existence of such $\mathbb{Z}^2$-actions. Therefore, it would be nice to have more concrete examples giving a negative answer to Question 2.

Approach

We give concrete examples of $\mathbb{Z}^2$-actions on unital C^*-algebras for which Question 2 has a negative answer. Considering a crossed product by $\mathbb{Z}^2$ as an iterated crossed product by $\mathbb{Z}$, the Pimsner Voiculescu sequence can be applied twice. A careful study of the second application shows the existence of an unexpected obstruction homomorphism. Together with $(K_*(A), K_*(\alpha), \mathbb{Z}^2)$, this homomorphism determines the K-theory of the crossed product up to extension problems. There is a concrete description of this obstruction homomorphism in terms of Bott-elements in the K-groups of the underlying algebra.

To illustrate this, we would like to find $\mathbb{Z}^2$-actions with non-trivial obstruction homomorphisms. We mainly focus on pointwise inner $\mathbb{Z}^2$-actions, meaning that each automorphism is inner. At first glance, it may seem unplausible that pointwise inner action can lead to non-trivial obstruction homomorphisms. The crucial observation is that a pointwise inner action is not necessarily unitarily implemented. Two unitaries u and v in a unital C^*-algebra define a pointwise inner $\mathbb{Z}^2$-action if and only if uvu^*v^* is a central unitary. It turns out that such an action can only have a non-trivial obstruction homomorphism if uvu^*v^* has full spectrum. This leads to the Heisenberg group C^*-algebra

$$C^*(H_3) := C^*(u, v \text{ unitaries} : uvu^*v^* \text{ is central })$$

as a first candidate.

Results

Let α denote the pointwise inner $\mathbb{Z}^2$-action on $C^*(H_3)$ induced by the unitaries u and v.

Theorem The obstruction homomorphism associated with α is non-trivial, and we have

$$K_*(C^*(H_3) \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C^*(H_3) \otimes C(\mathbb{T}^2)).$$

Furthermore, one can construct many examples of pointwise inner $\mathbb{Z}^2$-action with non-trivial obstructions in the following way.
Let A be a unital separable C^*-algebra and let $u, v \in A$ be unitaries such that uvu^*v^* is central and has full spectrum. As mentioned above, u and v define a pointwise inner $\mathbb{Z}^2$-action on A. Let B be a unital separable C^*-algebra whose K-groups do not vanish. Moreover, let $x \in B$ be a central unitary with full spectrum and $p \in B$ a projection satisfying

$$[x] \neq k[xp + 1 - p] \in K_1(B)$$

for all integers k. Let $C := A *_{C(\mathbb{T})} B$ be the amalgamated free product where uvu^*v^* and x are identified. By construction, uvu^*v^* is a central unitary in C. Thus, u and v define a pointwise inner action α on C extending the action on A.

Theorem The obstruction homomorphism of $(C, \alpha, \mathbb{Z}^2)$ is non-trivial. If $K_*(C) \cong \mathbb{Z}^k$ with $k > 0$, we have

$$K_*(C \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C \otimes C(\mathbb{T}^2)).$$

As an example, consider $A = C^*(H_3)$ and $B = C(\mathbb{T}) \oplus C(\mathbb{T})$. In this situation, $x = z \oplus z$ and $p = 1 \oplus 0$ satisfy the conditions above. The amalgamated free product $A *_{C(\mathbb{T})} B$ can be viewed as the universal unital C^*-algebra for a (non-trivial) obstruction associated with a pointwise inner $\mathbb{Z}^2$-action. This means that $A *_{C(\mathbb{T})} B$ has a universal obstruction in the sense, that every obstruction class can be obtained as the image of this universal obstruction.
Since homotopic $\mathbb{Z}^n$-actions give rise to crossed products with isomorphic K-theory, one might ask if this also holds for pointwise homotopic actions. The next theorem shows that this even fails for pointwise homotopies in the inner automorphisms.

Theorem There is a separable C^*-algebra C with $K_0(C) \cong K_1(C) \cong \mathbb{Z}$ and a pointwise inner $\mathbb{Z}^2$-action α on C, which is pointwise homotopic to the trivial action inside $\mathrm{Inn}(C)$, satisfying

$$K_*(C \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C \otimes C(\mathbb{T}^2)).$$

References

[1] M. Izumi, H. Matui: *$\mathbb{Z}^2$-actions on Kirchberg algebras*, Adv. Math. **224**(2010):355–400.

[2] E. Kirchberg: *The classification of Purely Infinite C^*-algebras using Kasparov's theory*, preprint, 1994, to appear in Fields Institute Communication series.

[3] N. C. Phillips: *A Classification Theorem for Nuclear Purely Infinite Simple C^*-algebras*, Doc. Math. **5**(2000):49–114.

[4] M. Pimsner, D. Voiculescu: *Exact Sequence for K-Groups and Ext-Groups of certain Cross-Product C^*-algebras*, J. Operator Theory 4(1980):93–118.

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funded by the DFG

Groups, Geometry & Actions