Topological Graphs (Katsura)

A quadruple $E = (E^0, E^1, r, s)$ is called a topological graph if $E^0, E^1$ are locally compact Hausdorff spaces, $r: E^1 \to E^0$ is a continous map, and $s: E^1 \to E^0$ is a local homomorphism.

We think of $E^0$ as a space of vertices, and we think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$.

If $E^0, E^1$ are both countable and discrete, then $E$ is a directed graph.

Let $E$ be a topological graph. An s-section is a subset $S \subset E^1$ such that $s_{|S}$ is a homeomorphism. An e-section is a subset $U \subset E^1$ such that $r_{|U}$ is a homeomorphism. A bisection is a subset $U \subset E^1$ such that $r_{|U}$ and $s_{|U}$ are both homeomorphisms.

Let $E$ be a topological graph. We define
1. $E^2 = E^0 \times E^1$.
2. $E^3 = \{ e \in E^2 \mid \text{there exists a neighbourhood } N \text{ of } e \text{ such that } r^{-1}[N] \text{ is compact} \}$.
3. $E_0 = E^1 \cap E^3$.

$E_0$ is a right $C^*$-algebra generated by $E^3$. $E_0$ is a bisection.

1-cocycle

Let $T$ be a locally compact Hausdorff space and let $N = \{N_i\}_{i \in I}$ be an open cover of $T$. Then for $i, j \in I$, we denote $N_{ij} = N_i \cap N_j$.

A collection of functions $S = \{s_{ij} \in C(N_{ij}, T)\}_{i,j \in I}$ is called a 1-cocycle relative to $N$ if for $i, j, k \in I$, $s_{ik} = s_{ij} \circ s_{jk}$ on $N_{ijk}$.

We think of $s_{ik}$ as a function on $N_{ijk}$ that is constant on $N_{ijk}$.

Bracket Functions

Lemma

Let $T$ be a locally compact Hausdorff space, let $N = \{N_i\}_{i \in I}$ be an open cover of $T$, and let $S = \{s_{ij} \in C(N_{ij}, T)\}_{i, j \in I}$ be a 1-cocycle relative to $N$. Fix $x, x' \in \prod_{i \in I} C(N_i, T)$ with $x_i = s_{ij}(x)$, $x'_i = s_{ij}(x')$, for all $i, j \in I$. Then there is a unique function $\{x_i(x')\} \in C(T)$, such that $\{x_i(x')\}(t) = \pi(t)(x_i(t), x'_i(t))$, if $t \in N_i$.

Twisted Topological Graph Correspondences

Let $E = (E^0, E^1, r, s)$ be a topological graph, let $N = \{N_i\}_{i \in I}$ be an open cover of $E^1$, and let $S = \{s_{ij} \in C(N_{ij}, T)\}_{i, j \in I}$ be a 1-cocycle relative to $N$. We define

$$C_0(E, N, S) = \left\{ x \in \prod_{i \in I} C(N_i, T) \mid x_i = s_{ij} x_j \text{ on } N_{ij}, \text{ for all } x_j \in C_0(N_j) \right\}.$$

For $x \in C_0(E, N, S)$ and $f \in C_0(E^0)$, we define right and left $C_0(E^0)$-actions on $C_0(E, N, S)$ by $(x \cdot f)(s) = x(s) \circ f$ and $(f \cdot x)(r) = f \circ s(x)$, for all $s, r$.

Proposition (Li)

Let $E$ be a topological graph, let $N = \{N_i\}_{i \in I}$ be an open cover of $E^1$, and let $S = \{s_{ij} \in C(N_{ij}, T)\}_{i, j \in I}$ be a 1-cocycle relative to $N$. Fix $x, y \in C_0(E, N, S)$, then there exists a unique function $(x(y))_{(ij)} \in C_0(E^0)$ such that $(x(y))_{(ij)} = s_{ij}(x(y))_{(ij)}$, for all $x \in E^1$. Hence, $(\cdot)_{(ij)}$ is a right $C_0(E^0)$-valued inner product on $C_0(E, N, S)$. Furthermore, the completion $X(E, N, S)$ of $C_0(E, N, S)$ under the $\|\cdot\|_{E^0}$ norm is a $C^*$-correspondence over $C_0(E^0)$.

Twisted Toeplitz Representations (Li)

Let $E$ be a topological graph, let $N = \{N_i\}_{i \in I}$ be a precompact open cover of $E^1$ consisting of $s$-sections, and let $S = \{s_{ij} \in C(N_{ij}, T)\}_{i, j \in I}$ be a 1-cocycle relative to $N$. A twisted Toeplitz representation of $E$ is a $C^*$-algebra $B$ associated with $C_0(E^0)$ over $C_0(E, N, S)$, for each $f \in B$ there exist a finite subset $F \subset I$ and a collection of functions $\{b_{ij} \in F \subset I \}$ such that

1. $\{b_{ij}(x)\}_{i,j \in I}$ covers $r^{-1}(supp(f))$,
2. $\forall i \in I, b_{ij}(x) = 1$ on $r^{-1}(supp(f))$,
3. $\sum_{i \in F} b_{ij}(x) = 1$ on $r^{-1}(supp(f))$, and
4. $x(f) = \sum_{i \in F} b_{ij}(x) x(s_{ij} x_j )$.

This definition is formulated so as to make it easy to check that a given collection $(\psi_i : i \in I)$ is covariant. However, when using covariation of a collection $(\psi_i : i \in I)$ it is true that Condition 4 of the definition holds for every $f \in C_0(E^0)$, every finite subset $F \subset I$, and every collection of functions $\{b_{ij} \in F \subset I \}$ satisfying the first three conditions of the definition.

Result

Theorem (Li)

Let $E$ be a topological graph, let $N = \{N_i\}_{i \in I}$ be a precompact open cover of $E^1$ consisting of $s$-sections, and let $S = \{s_{ij} \in C(N_{ij}, T)\}_{i, j \in I}$ be a 1-cocycle relative to $N$. Then the Toeplitz algebra $T^1(E, N, S)$ is isomorphic with the $C^*$-algebra generated by a universal twisted Toeplitz representation of $E$, and the Cuntz-Pimsner algebra $\mathcal{O}_{E^0,E,N} \subset C^*$ is isomorphic with the $C^*$-algebra generated by a universal covariant twisted Toeplitz representation of $E$.

References