SFB-workshop: Groups, dynamical systems and C^*-algebras, Münster, August 20-24, 2013

Twisted Topological Graph Algebras

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Topological Graphs(Katsura)

A quadruple $E = (E^0, E^1, r, s)$ is called a *topological graph* if E^0, E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous map, and $s : E^1 \to E^0$ is a local homeomorphism.

We think of E^0 as a space of vertices, and we think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$.

If E^0, E^1 are both countable and discrete, then E is a directed graph.

Let E be a topological graph. An *s-section* is a subset $U \subset E^1$ such that $s|_U$ is a homeomorphism. An *r-section* is a subset $U \subset E^1$ such that $r|_U$ is a homeomorphism. A *bisection* is a subset $U \subset E^1$ such that $r|_U$, $s|_U$ are both homeomorphisms.

Let E be a topological graph. We define
1. $E^0_{\mathrm{sce}} := E^0 \setminus \overline{r(E^1)}$.
2. $E^0_{\mathrm{fin}} := \{v \in E^0 :$ there exists a neighbourhood N of v such that $r^{-1}(\overline{N})$ is compact$\}$.
3. $E^0_{\mathrm{rg}} := E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}$.

1-cocycle

Let T be a locally compact Hausdorff space, let $\mathrm{N} = \{N_i\}_{i \in I}$ be an open cover of T. Then for $i, j \in I$, we denote $N_{ij} := N_i \cap N_j$.

A collection of functions $\mathrm{S} = \{s_{ij} \in C(\overline{N_{ij}}, \mathbb{T})\}_{i,j \in I}$ is called a *1-cocycle* relative to N if for i, j, $k \in I$, $s_{ij}s_{jk} = s_{ik}$ on $\overline{N_{ijk}}$.

Bracket Functions

Lemma

Let T be a locally compact Hausdorff space, let $\mathrm{N} = \{N_i\}_{i \in I}$ be an open cover of T, and let $\mathrm{S} = \{s_{ij}\}_{i,j \in I}$ be a 1-cocycle relative to N. Fix x, $x' \in \prod_{i \in I} C(\overline{N_i})$ with $x_i = s_{ij}x_j$, $x'_i = s_{ij}x'_j$ on $\overline{N_{ij}}$, for all i, $j \in I$. Then there is a unique function $[x|x'] \in C(T)$, such that $[x|x'](t) = \overline{x_i(t)}x'_i(t)$, if $t \in N_i$.

Twisted Topological Graph Correspondences

Let $E = (E^0, E^1, r, s)$ be a topological graph, let $\mathrm{N} = \{N_i\}_{i \in I}$ be an open cover of E^1, and let $\mathrm{S} = \{s_{ij}\}_{i,j \in I}$ be a 1-cocycle relative to N. We define

$$C_c(E, \mathrm{N}, \mathrm{S}) := \Big\{x \in \prod_{i \in I} C(\overline{N_i}) : \ x_i = s_{ij}x_j \text{ on } \overline{N_{ij}},\ [x|x] \in C_c(E^1)\Big\}.$$

For $x \in C_c(E, \mathrm{N}, \mathrm{S})$ and $f \in C_0(E^0)$, we define right and left $C_0(E^0)$-actions on $C_c(E, \mathrm{N}, \mathrm{S})$ by $(x \cdot f)_i = x_i(f \circ s|_{\overline{N_i}})$, and $(f \cdot x)_i = (f \circ r|_{\overline{N_i}})x_i$, for all $i \in I$.

Proposition(Li)

Let E be a topological graph, let $\mathrm{N} = \{N_i\}_{i \in I}$ be an open cover of E^1, and let $\mathrm{S} = \{s_{ij}\}_{i,j \in I}$ be a 1-cocycle relative to N. Fix x, $y \in C_c(E, \mathrm{N}, \mathrm{S})$. Then there exists a unique function $\langle x, y\rangle_{C_0(E^0)} \in C_c(E^0)$, such that $\langle x, y\rangle_{C_0(E^0)}(v) = \sum_{s(e)=v}[x|y](e)$, for all $v \in E^0$. Hence $\langle \cdot, \cdot\rangle_{C_0(E^0)}$ is a right $C_0(E^0)$-valued inner product on $C_c(E, \mathrm{N}, \mathrm{S})$. Furthermore, the completion $X(E, \mathrm{N}, \mathrm{S})$ of $C_c(E, \mathrm{N}, \mathrm{S})$ under the $\|\cdot\|_{C_0(E^0)}$ norm is a C^*-correspondence over $C_0(E^0)$.

Let E be a topological graph. We define $\mathrm{N} := \{E^1\}$, and we define $\mathrm{S} := \{1\}$. Then $X(E, \mathrm{N}, \mathrm{S})$ coincides with the normal graph correspondence $X(E)$ defined by Katsura.

An Annoying Technicality

We can prove that it is fine to work only on the precompact open cover of the edge set consisting of s-sections.

Twisted Toeplitz Representations(Li)

Let T be a locally compact Hausdorff space and N be an open subset of T. We think of $C_0(N) := \{f \in C_0(T) : f(N^c) = 0\}$ as a closed two-sided ideal of $C_0(T)$. Fix $f \in C_0(N)$, and $g \in BC(N)$. We define $f \times g : T \to \mathbb{C}$ by

$$f \times g(t) := \begin{cases} f(t)g(t) & \text{if } t \in N \\ 0 & \text{otherwise,} \end{cases}$$

and in fact $f \times g \in C_0(N)$.

Let E be a topological graph, let $\mathrm{N} = \{N_i\}_{i \in I}$ be a precompact open cover of E^1 consisting of s-sections, and let $\mathrm{S} = \{s_{ij}\}_{i,j \in I}$ be a 1-cocycle relative to N. A *twisted Toeplitz representation* of E in a C^*-algebra B is a collection of linear maps $\{\psi_i : C_0(N_i) \to B\}_{i \in I}$ and a homomorphism $\pi : C_0(E^0) \to B$, such that for all i, $j \in I$, $x \in C_0(N_i)$, $y \in C_0(N_j)$, and $f \in C_0(E^0)$, we have
1. $\psi_i((f \circ r)x) = \pi(f)\psi_i(x)$;
2. $\psi_i(x)^*\psi_j(y) = \pi(\langle x, y\rangle_{C_0(E^0)} \times (s_{ij}|_{N_{ij}} \circ s|^{-1}_{N_{ij}}))$.

Covariant Twisted Toeplitz Representations(Li)

Let E be a topological graph, let $\mathrm{N} = \{N_i\}_{i \in I}$ be a precompact open cover of E^1 consisting of s-sections, let $\mathrm{S} = \{s_{ij}\}_{i,j \in I}$ be a 1-cocycle relative to N, and let $\{\psi_i, \pi\}_{i \in I}$ be a twisted Toeplitz representation of E. We call $\{\psi_i, \pi\}_{i \in I}$ *covariant* if there exists a collection $\mathcal{G} \subset C_c(E^0_{\mathrm{rg}})$ of nonnegative functions generating $C_0(E^0_{\mathrm{rg}})$, and for each $f \in \mathcal{G}$ there exist a finite subset $F \subset I$ and a collection of functions $\{h_i\}_{i \in F} \subset C(E^1, [0, 1])$ such that
1. $\{N_i\}_{i \in F}$ covers $r^{-1}(\mathrm{supp}(f))$;
2. $\mathrm{supp}(h_i) \subset N_i$, for all $i \in F$;
3. $\sum_{i \in F} h_i = 1$ on $r^{-1}(\mathrm{supp}(f))$; and
4. $\pi(f) = \sum_{i \in F} \psi_i\big(\sqrt{h_i(f \circ r)}\big)\psi_i\big(\sqrt{h_i(f \circ r)}\big)^*$.

This definition is formulated so as to make it easy to check that a given collection $\{\psi_i, \pi\}_{i \in I}$ is covariant. However, when *using* covariance of a collection $\{\psi_i, \pi\}_{i \in I}$ it is *true* that Condition 4 of the definition holds for every $f \in C_c(E^0_{\mathrm{rg}})$, every finite subset $F \subset I$, and every collection of functions $\{h_i\}_{i \in F} \subset C(E^1, [0, 1])$ satisfying the first three conditions of the definition

Result

Theorem(Li)

Let E be a topological graph, let $\mathrm{N} = \{N_i\}_{i \in I}$ be a precompact open cover of E^1 consisting of s-sections, and let $\mathrm{S} = \{s_{ij}\}_{i,j \in I}$ be a 1-cocycle relative to N. Then the Toeplitz algebra $\mathcal{T}_{X(E,\mathrm{N},\mathrm{S})}$ is isomorphic with the C^*-algebra generated by a universal twisted Toeplitz representation of E, and the Cuntz-Pimsner algebra $\mathcal{O}_{X(E,\mathrm{N},\mathrm{S})}$ is isomorphic with the C^*-algebra generated by a universal covariant twisted Toeplitz representation of E.

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