Operator Algebras

Siegfried Echterhoff
Mathematisches Institut Universität Münster Einsteinstr. 62
48149 Münster, Germany

Contents

0.1 Some Notation

$$\begin{aligned} A^+ &= \text{positive elements}, & P(A) &= \text{positive functionals},\\ A_{sa} &= \text{self adjoint elements}, & \mathcal{S}(A) &= \text{states},\\ U(A) &= \text{unitary elements}, & \mathrm{Pure}(A) &= \text{pure states},\\ \mathrm{Inv}(A) &= \text{invertible elements}, & \mathrm{Max}(A) &= \text{maximal ideals} \end{aligned}$$

1 Spectral Theory and Basic Tools

There are two theorems, which are both refered to as **Gelfand-Naimark Theorem** one holds for commutative C^*-algebras only:

$$A \cong C_0(\hat{A}),$$

while the other:

$$A \cong \pi(A) \subset L(H_\pi),$$

holds for general C^*-algebras. That is any C^*-algebra A can be viewed as a subalgebra of $L(H)$ for some suitable Hilbert space H. Furthermore if A is commutative we can take it to be $C_0(X)$ for some locally compact space X.

Just like the above two theorems carry the same name, also the following two quantities

$$\begin{array}{rrcl} A \text{ commutative}: & \hat{A} & := & \{\chi : A \to \mathbb{C} \,|\, 0 \neq \chi \text{ algebra homomorphism}\} \\ A \text{ general}: & \hat{A} & := & \{[\pi] \,|\, \pi : A \to L(H) \text{ irrep}\} \end{array}$$

are given the same name. If the C^*-algebra is commutative both expressions coincide.

1.1 Banach Algebras

In this section we shall introduce the basic notion of Banach algebra, which shall lead us to the von-Neumann series and finally the result, that the set of invertible elements $\mathrm{Inv}(A)$ are open in the Banach algebra A.

Definition 1.1.1 **Complex Algebra** *A complex algebra is a* $\mathbb{C}$*-vector space* A *with a multiplication*

$$m : A \times A \longrightarrow A$$

that is a bilinear and associative mapping.

Definition 1.1.2 **Banach Algebras (BA)** *A* **normed algebra** *is a normed space* $(A, ||\cdot||)$ *that is a complex algebra such that*

$$||ab|| \leq ||a|| \cdot ||b||.$$

A complete normed algebra is called a Banach algebra (BA).

Lemma 1.1.3 **Continuity of the multiplication** *The relation* $||ab|| \leq ||a|| \cdot ||b||$ *lets the multiplication in a normed algebra be continuous.*

Proof: Let $a_n \to a, b_n \to b$ in $(A, ||\cdot||)$, then

$$||a_n b_n - ab|| \leq ||a_n b_n - a_n b|| + ||a_n b - ab|| \leq ||a_n|| \cdot \underbrace{||b_n - b||}_{\to 0} + \underbrace{||a_n - a||}_{\to 0} \cdot ||b|| \to 0.$$

□

Lemma 1.1.4 **Unitalization** *The unitalization of a normed algebra is the normed unital* ($\mathbb{1}_{A^1} := (0,1)$) *algebra:*

$$A^1 := \{(a,\lambda) \,|\, a \in A, \lambda \in \mathbb{C}\}, \qquad (a,\lambda)(b,\mu) := (ab + \lambda b + \mu a, \lambda\mu), \quad ||(a,\lambda)||_{A^1} := ||a||_A + |\lambda|.$$

Proof: For example

$$||(a,\lambda)(b,\mu)|| = ||ab + \lambda b + \mu a|| + |\lambda\mu| \leq ||a||||b|| + |\lambda|||b|| + |\mu|||a|| + |\lambda\mu| = ||(a,\lambda)|| \cdot ||(b,\mu)||$$

□

Remark 1.1.5 *There are other possible choices for the norm on A^1. For instance, $||(a,\lambda)||_\infty := \max\{||a||, |\lambda|\}$ would be a BA-norm on A^1 which is equivalent to $||\cdot||_{A^1}$ defined above.*

Definition 1.1.6 **Ideals** *A subalgebra $I \subseteq A$ is called a right (left) ideal iff*

$$ia \in I \quad (ai \in I) \quad \forall i \in I, a \in A$$

$I \subseteq A$ is called an ideal iff it is a right and left ideal.

Example 1.1.7

1.) $A \subset A^1$ is an ideal:

$$(a,0)(b,\lambda) = (ab + \lambda a, 0) \in A, \quad (b,\lambda)(a,0) = (ba + \lambda a, 0) \in A.$$

2.) The compact operators $K(B) \subseteq L(B)$ on a Banach space B form an ideal.

3.) Let X be locally compact and Hausdorff, then the following are ideals:

$$I_x := \{f \in C_0(X) \,|\, f(x) = 0\} \subset C_0(X).$$

Lemma 1.1.8 **Quotient Space** *Let $I \subset A$ be an ideal.*

- *Then the following is an algebra:*

$$A/I := \{a + I \,|\, a \in A\} \quad \text{with} \quad (a+I)(b+I) := ab + I.$$

- *If A was Banach, then the following norm lets A/I again be Banach :*

$$||a + I|| := \inf\{||a + b|| \,|\, b \in I\}.$$

Proof: The multiplication is well defined, since for $a' = a + c, b' = b + d$ with $c, d \in I$:

$$(a'+I)(b'+I) := a'b' + I = (a+c)(b+d) + I = ab + \underbrace{(ad + cb + cd)}_{\in I} + I = ab + I =: (a+I)(b+I)$$

If A was Banach, then we know that A/I is a Banach space also. So we only need to show that

$$||(a+I)(b+I)|| \leq ||(a+I)||||(b+I)||.$$

For all $c, d \in I$ we have

$$||(a+I)(b+I)|| = ||ab + (ad + cb + cd) + I|| \leq ||ab + (ad + cb + cd)|| = ||(a+c)(b+d)||$$

and thus

$$||(a+I)(b+I)|| \leq \inf_{c,d \in I} ||(a+c)||||(b+d)|| = ||(a+I)||||(b+I)||.$$

□

Definition 1.1.9 **Invertible Elements** *Let A be a unital algebra, we define*

$$\boxed{\mathrm{Inv}(A) := \{a \in A \,|\, a \text{ invertible}\}}$$

Remark 1.1.10

- $\mathrm{Inv}(A)$ *is a group, since $(ab)^{-1} = b^{-1}a^{-1}$.*

- *If $I \subsetneq A$ is a proper Ideal, then* $\boxed{I \cap \mathrm{Inv}(A) = \varnothing}$ *since*

$$a \in I \cap \mathrm{Inv}(A), \quad \Rightarrow \quad \mathbb{1}_A = aa^{-1} \in I, \quad \Rightarrow \quad A = \mathbb{1}_A A \subseteq I.$$

Theorem 1.1.11 **von – Neumann Series** *Let A be a unital BA and $||a|| < 1$, then we have*

$$(\mathbb{1} - a) \in \mathrm{Inv}(A), \qquad (\mathbb{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof: We know that in a Banach space

$$\sum_{n=0}^{\infty} ||x_n|| \text{ converges} \quad \Rightarrow \quad \sum_{n=0}^{\infty} x_n \text{ converges.}$$

Thus, because of $||a^n|| \leq ||a||^n$, it is clear that the von-Neumann series converges. And we have

$$(\sum_{n=0}^{\infty} a^n)(\mathbb{1} - a) = (\mathbb{1} - a)\sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty} a^n - \sum_{n=1}^{\infty} a^n = a^0 = \mathbb{1}.$$

□

Corollary 1.1.12 **Inv(A) is open** *Let $a \in \mathrm{Inv}(A)$ and*

$$b \in A \quad \text{with} \quad ||a - b|| < ||a^{-1}||^{-1} \quad \Rightarrow \quad b \in \mathrm{Inv}(A).$$

In particular $\mathrm{Inv}(A) \subset A$ *is open.*

Proof: $||a^{-1}(a - b)|| \leq ||a^{-1}|| \cdot ||a - b|| < 1$, thus with Von-Neumann: $(\mathbb{1} - a^{-1}(a - b)) \in \mathrm{Inv}(A)$ and since $b = a(\mathbb{1} - a^{-1}(a - b))$ we have $b^{-1} = (\mathbb{1} - a^{-1}(a - b))^{-1}a^{-1}$. □

1.2 Results from Complex Analysis

In this section, we shall give some central results from complex analysis.

Definition 1.2.1 **Holomorphic Function** *Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is called complex differentiable in $\lambda_0 \in U$, iff*

$$f'(\lambda_0) := \lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0}$$

exists. $f'(\lambda_0)$ is then called the complex derivative of f in λ_0. $f : U \to \mathbb{C}$ is called holomorphic, if it is differentiable in all $\lambda \in U$.

Remark 1.2.2 *All the usual rules of differentiation (product rules, chain rule, etc.) still apply in the complex.*

Theorem 1.2.3 *Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$, then the following are equivalent*

(1) f is holomorphic on U.

(2) To every $\lambda_0 \in U$, there is a $r > 0$ and $a_n \in \mathbb{C}$, $n \in \mathbb{N}$ such that

$$f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n \quad \forall \lambda \in U_r(\lambda_0).$$

Furthermore it holds, that the series in (2) converges on every $U_R(\lambda_0) \subseteq U$ to $f(\lambda)$.

Remark 1.2.4

- *The last statement is very important, since it tells us, that we can maximally extend the radius of convergence in U.*
- *As in real analysis, one can show that the derivative of a function given as a series expansion*

$$f : U_r(\lambda_0) \longrightarrow \mathbb{C}, \quad f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n$$

is obtained just by taking the derivative of each summand, i.e.

$$f'(\lambda) = \sum_{n=1}^{\infty} n a_n(\lambda - \lambda_0)^{n-1}$$

and by induction, we get

$$f^{(k)}(\lambda) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(\lambda - \lambda_0)^{n-k} \quad \forall k \in \mathbb{N}.$$

In particular, f is ∞-times complex differentiable, with

$$f^{(k)}(\lambda_0) = k! a_k, \quad \text{so } a_k = \frac{f^{(k)}(\lambda_0)}{k!}.$$

Thus we have that the above series is the Taylor series of f.

Theorem 1.2.5 **Entire Function** *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic (such a function is called entire), $\lambda_0 \in \mathbb{C}$ arbitrary. Then with $a_n = \frac{f^{(n)}(\lambda_0)}{n!}$, we have*

$$f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n \quad \forall \lambda \in \mathbb{C}.$$

If we choose $\lambda_0 = 0$, then

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

Example 1.2.6 $\exp, \cos, \sin : \mathbb{C} \to \mathbb{C}$ *are entire functions.*

Definition 1.2.7 **Complex Domain** *A complex domain (or simply domain) is a connected open subset of $\mathbb{C}$.*

Definition 1.2.8 *Let $U \subseteq \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic and*

$$N_f := \{\lambda \in U \mid f(\lambda) = 0\}.$$

It then holds that if N_f has an accumulation point in U, then $N_f = U$, i.e.

$$f = 0 \text{ on } U.$$

Proof: Let A be the set of all accumulation points of N_f. Then A is closed in U. That is since f in continuous and thus $f(\lambda_0)$ for every accumulation point λ_0 of N_f and thus also for every point in A.
We now show that A is also open. Since then $U = A \cup U \setminus A$ with $A, U \setminus A$ open, so $A = \varnothing$ or $A = U$ since U is connected.
Let $\lambda_0 \in A$ and $r > 0$ with $U_r(\lambda_0) \subseteq U$ and let $f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n$ be the series expansion of f. We claim that $a_n = 0$ $\forall n \in \mathbb{N}$ (it then follows that $U_r(z_0) \subseteq A$ so A open.)

We assume $a_n \neq 0$ for some $n \in \mathbb{N}$. Let then $n_0 \in \mathbb{N}$ be the minimal a_n such that $a_n \neq 0$. It follows that $n_0 > 0$, since $a_0 = f(\lambda_0) = 0$. We define

$$g : U \longrightarrow \mathbb{C}, \quad g(\lambda) = \begin{cases} a_{n_0}, & \text{if } \lambda = \lambda_0 \\ (\lambda - \lambda_0)^{-n_0}, & \text{if } \lambda \neq \lambda_0. \end{cases}$$

Then g is holomorphic, since g is holomorphic on $U \setminus \{\lambda_0\}$ with the product rule and g holomorphic in λ_0, since $\forall \lambda \in U_r(\lambda)$:

$$g(\lambda) = (\lambda - \lambda_0)^{-n_0} \sum_{n=n_0}^{\infty} a_n (\lambda - \lambda_0)^n = \sum_{n=0}^{\infty} a_{n+n_0} (\lambda - \lambda_0)^n.$$

That is g has a series expansion in $U_r(\lambda_0)$. And thus $g(\lambda_0) = a_{n_0} \neq 0$. Now since g is continuous, there is a $\varepsilon > 0$ such that $g(\lambda_0) \neq 0$ on $U_\varepsilon(\lambda_0)$. Since $\lambda_0 \in A$ there is a sequence $(\lambda_n)_n$ in U with $f(\lambda_n) = 0$, $\lambda_n \to \lambda_0$. For a large enough n, it then follows, that $\lambda_n \in U_\varepsilon(\lambda_0)$, so

$$0 \neq g(\lambda_n) = (\lambda_0 - \lambda_n)^{-n_0} \underbrace{f(\lambda_n)}_{=0} = 0,$$

which is a contradiction. □

Corollary 1.2.9 *Let $U \subseteq \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic. Let $\lambda_n, \lambda_0 \in U$ with $\lambda_n \to \lambda_0 \neq \lambda_n$, $g(\lambda_n) = f(\lambda_n)$ $\forall n \in \mathbb{N}$. It then follows that $f = g$ on U.*

Proof: Apply the last theorem to $h := f - g$. □

Theorem 1.2.10 **Mean Value Property** *Let $f : U \to \mathbb{C}$ be holomorphic and $\lambda_0 \in U$, $r > 0$ with $B_r(\lambda_0) \subseteq U$, then*

$$\boxed{f^{(n)}(\lambda_0) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(\lambda_0 + re^{it}) e^{-int} dt}$$

Proof: Choose a $0 < r < R$ with $U_R(\lambda_0) \subseteq U$, since f is holomorphic, we have

$$f(\lambda) = \sum_{k=0}^{\infty} a_k (\lambda - \lambda_0)^k, \quad a_k = \tfrac{f^{(k)}(\lambda_0)}{k!}, \quad \forall \lambda \in U_R(\lambda_0).$$

Since $r < R$, the series converges uniformly on $B_r(\lambda_0)$. And thus for all $n \in \mathbb{N}_0$

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + re^{it}) e^{-int} dt &= \frac{1}{2\pi} \int_0^{2\pi} (\sum_{k=0}^{\infty} a_k r^k e^{ikt}) e^{-int} dt \overset{\text{uniform conv.}}{=} \frac{1}{2\pi} \sum_{k=0}^{\infty} a_k r^k \int_0^{2\pi} e^{i(k-n)t} dt \\ &= a_n r^n = \frac{f^{(n)}(\lambda_0)}{n!} r^n. \end{aligned}$$

That is since $\int_0^{2\pi} e^{i(k-n)t} dt = (2\pi)\delta_{nk}$. □

Theorem 1.2.11 **Liouville** *Let $f : \mathbb{C} \to \mathbb{C}$ be entire and bounded, then f is constant.*

Proof: We have

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad a_n = \tfrac{f^{(n)}(0)}{n!}, \quad \forall \lambda \in \mathbb{C}.$$

If we we take an arbitrary $r > 0$, then by the mean value property, we have

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(\lambda_0 + re^{it}) e^{-int} dt$$

and thus

$$|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} \underbrace{|f(\lambda_0 + re^{it})|}_{\leq c} dt \leq \frac{c}{r^n}$$

if $c \geq 0$ with $|f(\lambda)| \leq c \ \forall \lambda \in \mathbb{C}$. Since $r > 0$ was arbitrary, it follows that

$$|a_n| \leq \frac{c}{r^n} \underset{r\to\infty}{\longrightarrow} 0$$

so $a_n = 0 \ \forall n > 0$ and thus $f(\lambda) = a_0 = f(\lambda_0) \ \forall \lambda \in \mathbb{C}$. □

Theorem 1.2.12 *Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be holomorphic. If $|f|$ has a local maximum on U, then f is constant.*

Proof: Let w.l.o.g. $\lambda_0 \in U$ be a local maximum of f, then there is a $r > 0$ with $B_r(\lambda_0) \subseteq U$ and $|f(\lambda)| \leq |f(\lambda_0)| \ \forall \lambda \in B_r(\lambda_0)$. Now multiplying f with $\mu = e^{i\varphi}$ for some $\varphi \in [0, 2\pi)$, we can w.l.o.g assume $f(\lambda_0) \geq 0$. We know that

$$f(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + se^{it}) dt \quad \forall 0 \leq s \leq r.$$

We assume that $\exists 0 \leq s \leq r$ and $t_0 \in [0, 2\pi]$. with $f(\lambda_0 + se^{it}) \neq f(\lambda_0)$. We have

$$\mathrm{Re} f(\lambda_0 + se^{it}) \leq |f(\lambda_0 + se^{it})| \leq |f(\lambda_0)| = f(\lambda_0).$$

And it follows that

$$f(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + se^{it}) dt < \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0) dt = f(\lambda_0),$$

which is a contradiction. So we have that $f(\lambda) = f(\lambda_0)$ on $B_r(\lambda_0)$ and thus $f = f(\lambda_0)\mathbb{1}$ on $B_r(\lambda_0)$. And finally, on U, it holds that $f = f(\lambda_0)\mathbb{1}$. □

Corollary 1.2.13 *Let $U \subseteq \mathbb{C}$ be a domain with $\overline{U}$ compact and $f : \overline{U} \to \mathbb{C}$ continuous with $f|_U$ holomorphic. Then the maximum of $|f|$ lies on ∂U.*

Proof: Since $\overline{U}$ is compact, $|f|$ has a maximum on $\overline{U}$. And due to the above theorem, it does not lie on U. □

Lemma 1.2.14 *Let $r > 0$ and $U^r := \{\lambda \in \mathbb{C} \,|\, |\lambda| > r\}$. If then $0 < r < R$ and $f : U^r \to \mathbb{C}$ holomorphic, such that f has a series representation*

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \frac{1}{\lambda^n}, \quad \lambda \in U^R,$$

then the series converges to $f(\lambda)$ on all of $U^r \supseteq U^R$.

Proof: Define $g : U_{1/r}(0) \to \mathbb{C}$ by

$$g(\lambda) := \begin{cases} a_0, & \text{if } \lambda = 0 \\ f(1/\lambda), & \text{if } \lambda \neq 0. \end{cases}$$

Then $g(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \ \forall \lambda \in U_{1/R}(0)$, thus g is holomorphic in 0 and g is holomorphic on $U_{1/r}(0)$ by the chain rule. We have seen, that the series representation for g then also holds on all of $U_{1/r}(0)$. Thus we have

$$f(\lambda) = g(1/\lambda) = \sum_{n=0}^{\infty} a_n \frac{1}{\lambda^n}, \quad |\lambda| > r.$$

□

Theorem 1.2.15 *Let $U \subseteq \mathbb{C}$ be open and $f_n, f : U \to \mathbb{C}$ be functions with f_n holomorphic $\forall n \in \mathbb{N}$ and*

$$||f_n - f||_k = \sup_{x \in K} |f_n(\lambda) - f(\lambda)| \underset{n\to\infty}{\longrightarrow} 0$$

for every compact $K \subset U$. Then also f is holomorphic and $f_n' \to f'$ uniformly on all compact subsets $K \subset U$.

1.3 The Spectrum

The spectrum for a linear operator on an infinite dimensional vector space, is a generalization of the set of eigenvalues of a linear operator F on a finite dimensional vector space V.

$$\begin{aligned} &\lambda \text{ is eigenvalue of } F \in \mathrm{End}(V) \\ &\Leftrightarrow \exists 0 \neq x \in V: \quad Fx = \lambda x \\ &\Leftrightarrow \exists 0 \neq x \in V: \quad (F - \lambda \mathbb{1})x = 0 \\ &\Leftrightarrow \ker(F - \lambda \mathbb{1}) \neq \{0\} \\ &\Leftrightarrow (F - \lambda \mathbb{1}) \text{ is not injective} \\ &\overset{\dim V < \infty}{\Leftrightarrow} (F - \lambda \mathbb{1}) \text{ is not bijective} \\ &\Leftrightarrow (F - \lambda \mathbb{1}) \text{ is not invertible} \end{aligned}$$

Where by the open mapping theorem for $T \in L(E)$:

$$(T - \lambda \mathbb{1}) \text{ bijective} \quad \Leftrightarrow \quad (T - \lambda \mathbb{1}) \in \mathrm{Inv}(L(E)).$$

This motivates the following definition:

Definition 1.3.1 **Spectrum and Resolvent** *Let A be a BA and $a \in A$. The spectrum $\sigma_A(a)$ and the resolvent $R(a)$ of an element a are defined to be the following quatities:*

$$\begin{aligned} \sigma_A(a) &:= \{\lambda \in \mathbb{C} \,|\, (a - \lambda \mathbb{1}) \notin \mathrm{Inv}(\tilde{A})\}, \qquad \tilde{A} := \begin{cases} A^1, & \text{if } \mathbb{1} \notin A \\ A & \text{if } \mathbb{1} \in A \end{cases} \\ R(a) &:= \mathbb{C} - \sigma_A(a). \end{aligned}$$

The **resolvent map** *is defined as:*

$$\begin{aligned} R_a : R(a) &\longrightarrow A \\ \lambda &\longmapsto R_a(\lambda) := (a - \lambda \mathbb{1})^{-1}. \end{aligned}$$

Remark 1.3.2 **Spectrum is closed** *For all $a \in A$, we have*

$$\sigma_A(a) \subset \mathbb{C} \quad \text{is closed,} \qquad R(a) \subset \mathbb{C} \quad \text{is open.}$$

since $\mathrm{Inv}(A) \subset A$ *is open and* $\lambda \mapsto (a - \lambda \mathbb{1})$ *is continuous.*

Theorem 1.3.3 **Compactness of the Spectrum** *Let $\{0\} \neq A$ be a $\mathbb{C}$-BA, then we have*

$$\boxed{\sigma_A(a) \neq \varnothing, \qquad \sigma_A(a) \subseteq \overline{B_{||a||}(0)}}$$

and thus $\sigma_A(a) \subset \mathbb{C}$ is compact.

Proof: We prove only the second assertion. The first needs more work. Let $|\lambda| > ||a||$, then $||\frac{1}{\lambda}a|| < 1$ so with the von Neumann-series, we have

$$\begin{aligned} \exists(\mathbb{1} - \tfrac{1}{\lambda}a)^{-1} \quad &\Leftrightarrow \quad (\mathbb{1} - \tfrac{1}{\lambda}a) \in \mathrm{Inv}(A) \\ &\Rightarrow \quad (a - \lambda \mathbb{1}) = -\lambda(\mathbb{1} - \tfrac{1}{\lambda}a) \in \mathrm{Inv}(A) \\ &\Rightarrow \quad \lambda \in R(a) \ \forall |\lambda| > ||a||. \end{aligned}$$

□

Remark 1.3.4 *In a $\mathbb{R}$-BA $A_{\mathbb{R}}$, we can have $\sigma(A_{\mathbb{R}}) = \varnothing$. E.g.: $\sigma_{\mathrm{Mat}_2(\mathbb{R})}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \varnothing$.*

Definition 1.3.5 <u>**Spectral Radius**</u> *The spectral radius is defined to be*

$$\boxed{\rho(a) := \sup\{|\lambda| \,|\, \lambda \in \sigma_A(a)\}}$$

Lemma 1.3.6 *The spectrum commutes with complex polynomials:*

$$\boxed{p(z) = \sum_{k=0}^{n} a_k z^k, \qquad \Rightarrow \qquad \sigma(p(a)) = p(\sigma(a))}$$

Proof: For any $\alpha \in \mathbb{C}$, one can write

$$p(a) - \alpha\mathbb{1} = c\prod_{i=1}^{n}(a - \lambda_i\mathbb{1})$$

for some $0 \neq c, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, depending on the chosen α. Thus

$$(p(a) - \alpha\mathbb{1}) \in \mathrm{Inv}(A) \qquad \Leftrightarrow \qquad (a - \lambda_i\mathbb{1}) \in \mathrm{Inv}(A)\ \forall i,$$

and

$$\alpha \in \sigma(p(a)) \qquad \Leftrightarrow \qquad \exists \text{ at least one } \lambda_i \in \sigma(a).$$

where the λ_i were chosen to be the zeros of $p(\lambda) - \alpha 1 \Leftrightarrow p(\lambda_i) = \alpha$. □

Lemma 1.3.7 <u>**Spectral Radius Formula**</u> *For the spectral radius ρ it holds that*

$$\boxed{\rho(a) = \lim_{n\to\infty} ||a^n||^{1/n}}$$

Note that here the algebraic quantity $\rho(a)$ is expressed in terms of the topological quantity $||\cdot||$.

Proof: We prove $\rho(a) \leq \underline{\lim}||a^n||^{1/n} \leq \overline{\lim}||a^n||^{1/n} \leq \rho(a)$:

- $\rho(a) \leq \underline{\lim}||a^n||^{1/n}$: If $\lambda \in \sigma(a)$ then $\lambda^n \in \sigma(a^n)$ and thus

  $|\lambda|^n = |\lambda^n| \leq ||a^n||, \quad \Rightarrow |\lambda| \leq ||a^n||^{1/n}, \quad \Rightarrow |\lambda| \leq \underline{\lim}||a^n||^{1/n}, \quad \Rightarrow \rho(a) \leq \underline{\lim}||a^n||^{1/n}.$

- $\overline{\lim}||a^n||^{1/n} \leq \rho(a)$: If there exists a $C_\lambda > 0$ with

  $$|\lambda|^{-n-1}||a^n|| \leq C_\lambda, \quad \forall n$$

  then $||a^n||^{1/n} \leq (C_\lambda|\lambda|)^{1/n}|\lambda|$ and we have

  $$\overline{\lim}||a^n||^{1/n} \leq \overline{\lim}(C_\lambda|\lambda|)^{1/n}|\lambda| \leq \lim_{n\to\infty}(C_\lambda|\lambda|)^{1/n}|\lambda| = |\lambda|$$

  as any $x \in (0,\infty)$ fulfills $x^{1/n} \to 1$. So it follows that $\overline{\lim}||a^n||^{1/n} \leq \rho(a)$.

- We now prove the existence of such a C_λ. In order to do so, we show that $\{\frac{1}{\lambda^n}a^n \,|\, n \in \mathbb{N}\}$ is weakly bounded. As explained, this gives boundedness. Let now $\varphi \in A'$. Consider again the holomorphic function

  $$\varphi_a : R_a \longrightarrow \mathbb{C}, \quad \varphi_a(\lambda) = \varphi(R_\lambda(a)) = \varphi((a - \lambda\mathbb{1})^{-1}).$$

  Since $U^{\rho(a)} := \{\lambda \in \mathbb{C} \,|\, |\lambda| > \rho(a)\} \subseteq R_a$, we have that φ is in particular holomorphic on $U^{\rho(a)}$. If $|\lambda| > ||a||$, then

  $$R_a(\lambda) = (a - \lambda\mathbb{1})^{-1} = -\tfrac{1}{\lambda}(1 - \tfrac{a}{\lambda})^{-1} = -\tfrac{1}{\lambda}\sum_{n=0}^{\infty} a^n \tfrac{1}{\lambda^n} = -\sum_{n=0}^{\infty} a^n \tfrac{1}{\lambda^{n+1}}$$

  (since $||\frac{1}{\lambda}a|| < 1$), and it thus holds that

  $$\varphi_a(\lambda) = -\sum_{n=0}^{\infty}\varphi(a^n)\tfrac{1}{\lambda^{n+1}}, \quad \forall|\lambda| > ||a||.$$

  So the series expansion holds for all $|\lambda| > \rho(a)$. In particular $\varphi(a^n)\frac{1}{\lambda^{n+1}} \to 0$ and thus $\{\varphi(\frac{1}{\lambda^{n+1}}a^n) \,|\, n \in \mathbb{N}\} \subset \mathbb{C}$ is bounded.

□

Lemma 1.3.8 *Let $\varphi : A \to B$ be a unital homomorphism of algebras, then it holds that*

$$\boxed{\sigma_B(\varphi(a)) \subseteq \sigma_A(a)}$$

In particular, if φ is the inclusion map of a subalgebra $A \subseteq B$ with $\mathbb{1}_B \in A$, then $\forall a \in A$:

$$\boxed{\sigma_B(a) \subseteq \sigma_{A\subseteq B}(a)}$$

Proof: If there is a $a^{-1} \in A$, then

$$\varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(\mathbb{1}_A) = \mathbb{1}_B$$

thus $\varphi(a) \in \mathrm{Inv}(B)$ and

$$R_A(a) \subseteq R_B(\varphi(a)) \qquad \Leftrightarrow \qquad \sigma_B(\varphi(a)) \subseteq \sigma_A(a).$$

□

Lemma 1.3.9 *If we have $A \subseteq B$ with $\mathbb{1}_B \in A$, then it holds that*

$$\boxed{\partial\sigma_A(a) \subseteq \sigma_B(a) \subseteq \sigma_A(a)}$$

Proof: Let $\lambda \in \partial\sigma_A(a) \subset \sigma_A(a)$ and $(\lambda_n)_n \subset R_A(a)$ with $\lambda_n \to \lambda$. Assume $\lambda \notin \sigma_B(a)$, then $\lambda \in R_B(a)$ and

$$\underbrace{(a - \lambda_n\mathbb{1})^{-1}}_{\in A\subset B} \longrightarrow \underbrace{(a - \lambda\mathbb{1})^{-1}}_{\in A\subset B}$$

since $A \subset B$ closed. So, since $\lambda \in \sigma_A(a)$ we have a contradiction. □

1.4 The Gelfand-Homomorphism for commutative BAs

The Gelfand-space $\hat{A}$ is a subspace of the (topological) dual A':

$$\boxed{\hat{A} \subset B_1^{A'}(0) \subset A' \subset A^*}$$

The main results of this section hold for commutative BAs: We prove that $\hat{A}$ is locally compact and compact if A is unital. Furthermore we will see that $\hat{A}$ is in bijection with the space of maximal ideals $\mathrm{Max}(A)$.

Definition 1.4.1 $\underline{\textbf{Gelfand – Space}}$ *(Structure Space): Let A be a commutative BA. We define the Gelfand-space to be the quantity*

$$\boxed{\hat{A} := \{\chi : A \to \mathbb{C} \,|\, 0 \neq \chi \text{ algebra homomorphism}\}}$$

Example 1.4.2 *Let $A = C_0(X)$ with X locally compact, then the evaluation map*

$$\begin{aligned} \delta_x : C_0(X) &\longrightarrow \mathbb{C} \\ f &\longmapsto \delta_x(f) := f(x) \end{aligned}$$

is an algebra homomorphism.

Lemma 1.4.3 *Let A be a commutative BA, then $\forall \tilde{\chi} \in \widehat{A^1}$ there is a $\chi \in \hat{A} : \tilde{\chi}|_A = \chi$ and with $\chi_\infty(a + \lambda\mathbb{1}) = \lambda$ we have*

$$\widehat{A^1} = \{\tilde{\chi} \,|\, \chi \in \hat{A}\} \sqcup \{\chi_\infty\}.$$

Proof: Let $\tilde{\chi} : A^1 \to \mathbb{C}$ be a continuation of $\chi : A \to \mathbb{C}$, then

$$\tilde{\chi}(a + \lambda \mathbb{1}) = \tilde{\chi}(a) + \lambda \tilde{\chi}(\mathbb{1}) = \chi(a) + 1.$$

Vice versa: $\tilde{\chi}(a + \lambda \mathbb{1}) := \chi(a) + \lambda$ is a continuation of χ.
Is $\mu \in \widehat{A^1}$ with $\mu|_A = 0$, then $\mu(a + \lambda \mathbb{1}) = \lambda \mu(\mathbb{1}) = \lambda = \chi_\infty(a + \lambda \mathbb{1})$ and so $\mu = \chi_\infty$. $\square$

Lemma 1.4.4 *Let A be a commutative BA, then for $\chi \in \hat{A}$ the following hold:*

- *χ is continuous and $||\chi||_{op} \leq 1$,*
- *$||\chi||_{op} = 1$ if A is unital.*

Proof: Let A be unital, then

$$\chi(a - \chi(a)\mathbb{1}) = \chi(a) - \chi(a) = 0, \quad \forall a.$$

Thus $(a - \chi(a)\mathbb{1}) \notin \mathrm{Inv}(A)$ and therefor $\chi(a) \in \sigma(a)$. We know $\sigma(a) \subseteq B_{||a||}(0)$, so

$$|\chi(a)| \leq ||a||, \qquad \Rightarrow \quad ||\chi||_{op} \leq 1.$$

For A unital we have $\chi(\mathbb{1}) = 1$, which gives $||\chi||_{op} = 1$. $\square$

Remark 1.4.5 *We have shown $\hat{A} \subset B_1^{A'}(0)$ and endow $\hat{A}$ with the weak-* topology τ_{w*} of A'.*

Definition 1.4.6 **Weak $-*$ topology** *(topology of pointwise convergence) τ_{w*} a net $(\chi_n)_n \subset (A', \tau_{w*})$ converges iff*

$$\chi_n(a) \to \chi(a), \quad \forall a \in A.$$

Theorem 1.4.7 *Let A be a commutative BA, then $(\hat{A}, \tau_{w*})$ is*

- *locally compact and*
- *compact if A is unital.*

Proof: Remember Banach-Alaoglou: $B_1^{A'}(0) \subset (A', \tau_{w*})$ is compact. Therefor for A unital we only need to show that

$$\hat{A} \subset (B_1^{A'}(0), \tau_{w*}) \quad \text{closed.}$$

So let $(\chi_n)_n \subset \hat{A}$ be a net with $\chi_n \to \chi \in A'$, then

$$\begin{aligned}
\lim_{n\to\infty} \chi_n(ab) &= \chi(ab) \\
\lim_{n\to\infty} \chi_n(a)\chi_n(b) &= \chi(a)\chi(b) \\
\lim_{n\to\infty} \chi_n(\mathbb{1}) &= 1 = \chi(\mathbb{1})
\end{aligned}$$

and thus $\chi \in \hat{A}$. If A is not unital, then $\hat{A} \subset \widehat{A^1} = \hat{A} \sqcup \{\chi_\infty\}$ open and $\widehat{A^1}$ compact Hausdorff, thus $\hat{A}$ is locally compact. $\square$

Remark 1.4.8 *$\widehat{A^1} = \hat{A} \sqcup \{\chi_\infty\}$ is the one-point compactification.*

Definition 1.4.9 **Gelfand – Transform** *Let A be a commutative BA. The Gelfand-transformation is defined to be*

$$\begin{aligned}
\hat{a} : \hat{A} &\longrightarrow \mathbb{C} \\
\chi &\longmapsto \hat{a}(\chi) := \chi(a).
\end{aligned}$$

Remark 1.4.10 *$\hat{a} \in C_0(X)$ since for $\chi_n \to \chi$ we have:*

$$\hat{a}(\chi_n) := \chi_n(a) \longrightarrow \chi(a) = \hat{a}(\chi).$$

Theorem 1.4.11 *Let A be a commutative BA, then the map*

$$\begin{aligned} \wedge : A &\longrightarrow C_0(\hat{A}) \\ a &\longmapsto \hat{a} \end{aligned}$$

is a continuous, norm decreasing (i.e. $||\hat{a}||_\infty \leq ||a||$) homomorphism of algebras.

Proof: If A is unital, then $\hat{A}$ is compact and there with $C_0(\hat{A}) = C(\hat{A})$. Let now

$$\phi : A^1 \longrightarrow C(\widehat{A^1} = \hat{A} \sqcup \{\chi_\infty\})$$

be the Gelfand-transform for A^1, then

$$\begin{aligned} \phi(a)(\tilde{\chi}) &= \tilde{\chi}(a + 0\mathbb{1}) = \chi(a) = \hat{a}(\chi) \\ \phi(a)(\chi_\infty) &= \chi_\infty(a + 0\mathbb{1}) = 0. \end{aligned}$$

Thus $\hat{a} = \phi(a) \in (\hat{A} \sqcup \{\chi_\infty\})$ with $\phi(a)(\chi_\infty) = 0$. I.e. $\hat{a} \in C_0(\hat{A})$, and since $||\chi|| \leq 1$ for $\chi \in \hat{A}$:

$$|\hat{a}(\chi)| = |\chi(a)| \leq ||a||, \quad \forall a \in A.$$

So it holds that $||\hat{a}||_\infty \leq ||a||$. □

Remark 1.4.12 *The Gelfand-Homomorphism need not be injective or surjective.*
For example take a Banach space A with the multiplication $a \cdot b := 0$ for all $a, b \in A$. Then $\hat{A} = \varnothing$ and $C_0(\hat{A} = \varnothing) = \{0\}$. Here the Gelfand homomorphism is the zero mapping and thus not injective.

Lemma 1.4.13 **Gelfand – Mazur** *Let A be a unital BA, then all non zero element are invertible:*

$$\boxed{\mathrm{Inv}(A) = A - \{0\}, \qquad \Rightarrow \quad A \cong \mathbb{C}}$$

Proof: We know $\sigma(a) \neq \varnothing$ and

$$\lambda \in \sigma(a) \quad \Leftrightarrow \quad a - \lambda\mathbb{1} \notin \mathrm{Inv}(A) \quad \overset{\text{assumption}}{\Leftrightarrow} \quad a - \lambda\mathbb{1} = 0 \quad \Leftrightarrow \quad a = \lambda\mathbb{1}.$$

□

Theorem 1.4.14 *Let A be a commutative, unital BA and $\mathrm{Max}(A)$ the space of maximal ideals in A, then the following hold:*

1.) We have a bijection $\boxed{\hat{A} \cong \mathrm{Max}(A)}$:

$$\begin{aligned} \hat{A} &\longrightarrow \mathrm{Max}(A) \\ \chi &\longmapsto \ker(\chi). \end{aligned}$$

2.) We have $\boxed{a \in \mathrm{Inv}(A) \ \Leftrightarrow \ \hat{a} \in \mathrm{Inv}(C(\hat{A})) \ \Leftrightarrow \ \hat{a}(\chi) \neq 0 \ \forall \chi \in \hat{A}}$

3.) For all $a \in A$ it holds that:

$$\boxed{\sigma(a) = \{\hat{a}(\chi) \,|\, \chi \in \hat{A}\}}$$

Proof:

1.) First: $I_\chi = \mathrm{Im}\chi \in A$ is a maximal ideal for all $\chi \in \widehat{A}$, since for an ideal $J \subseteq A$, with $I_\chi \subseteq J$, we have $\chi(J) \subseteq \mathbb{C}$ which is an ideal in $\mathbb{C}$, due to the isomorphim $A/I_\chi \cong \mathbb{C}$ by $a + I_\chi \to \chi(a)$. Since $\mathbb{C}$ is a field, we have $\chi(J) = \{0\}$ and thus $J \subseteq I_\chi$, i.e. $J = I_\chi$.

- Injectivity: Let $\chi_1, \chi_2 \in \widehat{A}$, with $\chi_1 \neq \chi_2$, then $\ker \chi_1 \neq \ker \chi_2$, since if $\ker \chi_1 = \ker \chi_2 = I$, then $\forall a \in A$:

$$\chi_2(a - \chi_1(a) - \mathbb{1}) = \chi_2(a) - \chi_1(a) = 0, \quad \chi_1(a - \chi_2(a) - \mathbb{1}) = \chi_1(a) - \chi_2(a) = 0$$

so $\chi_1 = \chi_2$. So the map is indeed injective.

- Surjectivity: If $I \subseteq A$ is an arbitrary ideal, then $\mathrm{Inv}(A/I) = A/I \setminus \{0 + I\}$, since if $a \notin I$ and $a + I$ not invertible in A/I, then $\tilde{J} = \{ab + I \,|\, b \in A\}$ is an ideal in A/I, since $1 + I \notin \tilde{J}$. And then $J = (a + I)A \subseteq A$ is an ideal in A with $I \subsetneq J$, which contradicts maximality.
But if $\mathrm{Inv}(A/I) = (A/I) \setminus \{0 + I\}$, then with Gelfand-Mazur $A/I \cong \mathbb{C}$. And with the quotient map $\chi : A \to A/I \cong \mathbb{C}$, we have $\chi \in \widehat{A}$ with $\mathrm{Im}(\chi) = I$.

2.) " $\Rightarrow$ " Is $a \in \mathrm{Inv}(A)$, then $\chi(a) \neq 0\ \forall \chi \in \widehat{A}$, since we have $1 = \chi(a^{-1}a) = \chi(a^{-1})\chi(a)$.

" $\Leftarrow$ " Let now $\chi(a) \neq 0\ \forall \chi \in \widehat{A}$. Assume $a \notin \mathrm{Inv}(A)$, then $J = \langle a \rangle = \{ab \,|\, b \in A\}$ is an ideal in A and thus there exists a maximal ideal $I \subset A$ with $J \subseteq I$. By (1) $\exists \chi \in \widehat{A}$ with $\mathrm{Im}(\chi) = I \supseteq J$. But then $\chi(a) = 0$, since $a \in J \subseteq I$. A contradiction!

3.) First, if $\chi \in \widehat{A}$, then $\chi(a - \chi(a) - \mathbb{1}) = 0$, so $(a - \chi(a) - \mathbb{1}) \notin \mathrm{Inv}(A)$ and $\chi(a) \in \sigma(a)$, thus $\hat{a}(\chi) = \chi(a) \in \sigma(a)\ \forall \chi \in \widehat{A}$.
Let now $\lambda \in \sigma(a)$, then $a - \lambda\mathbb{1} \notin \mathrm{Inv}(A)$ and with (2) there is a $\chi \in \widehat{A}$ such that

$$0 = \chi(a - \lambda\mathbb{1}) = \chi(a) - \lambda\chi(\mathbb{1}) = \chi(a) - \lambda$$

so $\lambda = \chi(a) = \hat{a}(\chi)$.

□

1.5 C^*-Algebras and the Gelfand-Naimark Theorem

Amongst the main results of this sections are the uniqueness theorem for C^*-norm and the fact that all commutative C^*-algebras are symmetric. The most important result however is the Gelfand-Naimark theorem, i.e. that for commutative C^*-algebras $A \cong C_0(\hat{A})$.

Definition 1.5.1 **Involution and $*-$Algebra** *Let A be a $\mathbb{C}$-algebra, then an involution is a mapping $* : A \to A$ such that*

$$(a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad (\lambda a)^* = \overline{\lambda}a^*, \quad (a^*)^* = a.$$

$(A, *)$ is called a $*$-algebra.

Definition 1.5.2 **Banach $- * -$ Algebra** *A Banach -$*$-algebra is a $*$-algebra $(A, *)$ such that*

$$||a^*|| = ||a||.$$

Definition 1.5.3 **C $*-$Algebra** *A C^*-algebra is a Banach-$*$-algebra such that*

$$||a^*a|| = ||a||^2.$$

Example 1.5.4

1.) $(C_0(X), f^* := \overline{f})$ *is a C^*-algebra.*

2.) $(l^1(\mathbb{Z}), ||\cdot||_1, f^*(n) := \overline{f(-n)})$ *is a Banach-$*$-algebra. It is however not a C^*-algebra.*

3.) $(L(H), T^* = \text{adj.op.})$ *is a* C^**-algebra.*

Theorem 1.5.5 *Let* A *be a* C^**-algebra without unity. Then the following is a* C^**-algebra:*

$$(A^1, ||(a,\lambda)|| := ||\Lambda_{(a,\lambda)}||_{op}, (a,\lambda)^* := (a^*, \overline{\lambda})),$$

where

$$\Lambda_{(a,\lambda)}(b) := (a,\lambda) \cdot b = ab + \lambda b.$$

Proof: It is obvious that Λ is a homomorphism of algebras.

- Show: Λ is injective. Let $(a,\lambda) \in A^1$ with $an + \lambda b = 0\ \forall b \in A$. If $\lambda = 0$, then $aa^* = 0$ and thus $a = 0$ since $||a||^2 = ||a^*||^2 = ||aa^*|| = 0$.
  If $\lambda \neq 0$, then $-\frac{1}{\lambda}ab = b\ \forall b \in A$. Then $e := -\frac{1}{\lambda}a$ is a left unity in A, and it follows, that $(ce - c)b = (ce)b - cb = c(eb) - cb = cb - cb = 0\ \forall c, b \in A$. and thus also $(ce - c)(ce - c)^* = 0$ $\forall c \in A$, which gives $ce - c = 0$, i.e. e is a unity in A, which stands in contradiction to the assumptions.

- Set $||(a,\lambda)|| := ||\Lambda_{(a,\lambda)}||_{op}$, then $||(a,0)|| = ||a||\ \forall a \in A$ since with $||\Lambda_{(a,0)}(b)|| = ||ab|| \leq ||a||\,||b||$, we have $||(a,0)|| = ||a||$ and if $a \neq 0$ then due to $||\Lambda_{(a,0)}(\frac{1}{||a||}a^*)|| = \frac{1}{||a||}||aa^*|| = \frac{1}{||a||}||a||^2 = ||a||$, we also have $||(a,0)|| = ||a||\ \forall a \in A$.

- Show $||(a,\lambda)^*(a,\lambda)|| = ||(a,\lambda)||^2\ \forall (a,\lambda) \in A^1$. Let now $\varepsilon > 0$ and $b \in A$ with $||b|| = 1$ and $||ab - \lambda b|| = ||\Lambda_{(a,\lambda)}(b)|| \geq ||(a,\lambda)||(1-\varepsilon)$, it then follows that

  $$\begin{aligned}(1-\varepsilon)^2||(a,\lambda)|| &\leq ||ab - \lambda b||^2 = ||(ab-\lambda b)^*(ab-\lambda b)|| = ||(b^*,0)(a^*,\overline{\lambda})(a,\lambda)(b,0)|| \\ &\leq ||b^*||\,||(a,\lambda)(a,\lambda)||\,||b|| = ||(a,\lambda)^*(a,\lambda)||.\end{aligned}$$

  Since $\varepsilon > 0$, we have

  $$||(a,\lambda)||^2 \leq ||(a,\lambda)^*(a,\lambda)|| \leq ||(a,\lambda)^*||\,||(a,\lambda)||, \qquad (*)$$

  and if we replace (a,λ) by $(a^*,\overline{\lambda})$, then we also get

  $$||(a,\lambda)^*||^2 \leq ||(a,\lambda)^*||\,||(a,\lambda)||.$$

  So we have $||(a,\lambda)^*|| = ||(a,\lambda)||$ and we have equality in $(*)$.

- It remains to prove completeness of A^1. Let $(a_n,\lambda_n)_n$ be a Cauchy series in A^1, then $(\lambda_n)_n$ is a Cauchy series in $A^1/A \cong \mathbb{C}$ w.r.t. the quotient norm (since $A \hookrightarrow A^1$ is an isometry and A is complete, we have $A \subset A^1$ closed). Therewith we have $\lambda_n \to \lambda$ for a $\lambda \in \mathbb{C}$. But then also $(a_n, 0) = (a_n,\lambda_n) - (0,\lambda_n)$ is a Cauchy series. Since $A \cong (A,0)$ and A is complete, there exists an $a \in A$ with $a_n \to a$, and then $(a_n,\lambda_n) \to (a,\lambda)$.

$\square$

Remark 1.5.6 A^1 *with* $||(a,\lambda)|| = ||a|| + |\lambda|$ *is usually not a* C^**-algebra.*

Remark 1.5.7

1.) Let A, B *be* C^**-algebras, then*

$$(A \oplus B, ||(a,b)|| := \max\{||a||, ||b||\})$$

is a C^**-algebra with component wise addition, multiplication and involution.*

2.) If A is unital, then

$$\begin{array}{ccc} A^1 & \xrightarrow{\cong} & A \oplus \mathbb{C} \\ (a,\lambda) & \longmapsto & (a+\lambda\mathbb{1}, \lambda). \end{array}$$

So A^1 is a C^*-algebra with respect to

$$||(a,\lambda)|| := \max\{||a+\lambda\mathbb{1}||, |\lambda|\}.$$

So in any case we can view A^1 as a C^*-algebra.

Definition 1.5.8 **Selfadjoint** *Let A be a C^*-algebra. An element $a \in A$ is called selfadjoint iff*

$$a^* = a.$$

Note that (a^*a) is selfadjoint for all $a \in A$.

Lemma 1.5.9 **Spectral Radius for Selfadjoint Elements** *Let A be a C^*-algebra, then*

$$\boxed{a = a^* \quad \Rightarrow \quad \rho(a) = ||a|| = \sup\{|\lambda| \,|\, \lambda \in \sigma(a)\}}$$

Proof: Assume w.l.o.g. that A is unital (if not then work in A^1). We have

$$a = a^* \quad \Rightarrow \quad ||a^2|| = ||a^*a|| = ||a||^2 \quad \overset{\text{Induction}}{\Rightarrow} \quad ||a^{2^n}|| = ||a||^{2^n}$$

and thus with the spectral radius formula

$$\rho(a) = \lim_{n\to\infty} ||a^n||^{1/n} = \lim_{n\to\infty} ||a^{2^n}||^{1/2^n} = \lim_{n\to\infty} ||a|| = ||a||.$$

□

Corollary 1.5.10 **Norm of a C$*$ Algebra** *The norm of a C^*-algebra is fully determined by the algebraic properties of A:*

$$||a||^2 = ||a^*a|| = \rho(a^*a) \quad \Rightarrow \quad \boxed{||a|| = \sqrt{\rho(a^*a)}}$$

Corollary 1.5.11 **Uniqueness of the C$*$ Norm** *It follows that there is only one norm, namely $||a|| = \sqrt{\rho(a^*a)}$ that turns A into a C^*-algebra.*

Lemma 1.5.12 **Continuity of $*-$Homomorphisms** *Let A be a Banach-$*$-algebra and B a C^*-algebra, then every $*$-homomorphism $\phi : A \to B$ is norm decreasing, i.e.*

$$||\phi(a)||_B \leq ||a||_A.$$

In particular it holds that ϕ is continuous.

Proof: We have $\sigma_B(\phi(a)) \subseteq \sigma_A(a)$ and thus

$$||\phi(a)||^2 = ||\phi(a)^*\phi(a)|| = ||\phi(a^*a)|| = \rho_B(\phi(a^*a)) \leq \rho_A(a^*a) \leq ||a^*a|| \leq ||a||^2.$$

□

Definition 1.5.13 **Symmetric Algebra** *A symmetric algebra is a commutative Banach-$*$-algebra in which for all $\chi \in \hat{A}$ it holds that*

$$\chi(a^*) = \overline{\chi(a)}.$$

Example 1.5.14

1.) $C_0(X)$ *is symmetric, since* $\widehat{C_0(X)} = \{\delta_x \,|\, x \in X\}$ *and* $\delta_x(\overline{f}) = \overline{f}(x) = \overline{\delta_x(f)}$.

2.) $l^1(\mathbb{Z})$ *with* $f^*(n) = \overline{f(-n)}$ *is symmetric, since*

$$\begin{array}{ccc} S^1 & \stackrel{\cong}{\longrightarrow} & \widehat{l^1(\mathbb{Z})} \\ z & \longmapsto & \chi_z, \quad \chi_z(f) = \hat{f}(z). \end{array}$$

Further we have, that

$$\widehat{f^*}(z) = \sum_{n\in\mathbb{Z}} f^*(n)z^n = \sum_{n\in\mathbb{Z}} \overline{f(-n)}z^n \stackrel{\overline{z}=z^{-1}}{=} \overline{\sum_{n\in\mathbb{Z}} f(-n)z^{-n}} \stackrel{n\mapsto -n}{=} \overline{\sum_{n\in\mathbb{Z}} f(n)z^n} = \overline{\hat{f}(z)}$$

and thus $\chi_z(f^*) = \widehat{f^*}(z) = \overline{\hat{f}(z)} = \overline{\chi_z(f)}$.

3.) The Disc algebra with the involution $f^*(z) = \overline{f(\overline{z})}$ *is not symmetric.*

Definition 1.5.15 **<u>Real – and Imaginary Part</u>** *Let A be a C^*-algebra, then define the selfadjoint elements*

$$\boxed{\mathrm{Re}(a) := \tfrac{1}{2}(a + a^*), \quad \mathrm{Im}(a) := \tfrac{1}{2i}(a - a^*), \quad \Rightarrow \quad a = \mathrm{Re}(a) + i\mathrm{Im}(a)}$$

Corollary 1.5.16 *A Banach-$*$-algebra is symmetric iff*

$$a = a^* \quad \Rightarrow \quad \chi(a) \in \mathbb{R}.$$

Proof: If A is symmetric, then

$$\chi(a) = \chi(a^*) = \overline{\chi(a)} \quad \Rightarrow \quad \chi(a) \in \mathbb{R}.$$

Vice versa if $\chi(a) \in \mathbb{R}$, then

$$\chi(b) = \chi(\mathrm{Re}(b)) + i\chi(\mathrm{Im}(b)), \quad \chi(b^*) = \chi(\mathrm{Re}(b)) - i\chi(\mathrm{Im}(b))$$

and thus $\chi(b^*) = \overline{\chi(b)}$. □

Corollary 1.5.17 *Every commutative C^*-algebra is symmetric.*

Proof: Assume w.l.o.g. that A is unital. Let $\chi \in \hat{A}$ and $a = a^* \in A$. Now show that $\chi(a) := x + iy \in \mathbb{R}$, i.e. that $y = 0$.

$$a_t := a + il\mathbb{1}, \; t \in \mathbb{R}, \qquad \Rightarrow \quad a_t^* a_t = (a + il\mathbb{1})^*(a + il\mathbb{1}) = (a - il\mathbb{1})(a + il\mathbb{1}) = a^2 + t^2\mathbb{1}.$$

Further we have $\chi(a_t) = \chi(a) + it = x + i(y + t)$, so

$$x^2 + (y + t)^2 = |\chi(a_t)|^2 \leq ||a_t||^2 = ||a_t^* a_t|| = ||a^2 + t^2\mathbb{1}|| \leq ||a^2|| + t^2$$

and it follows that

$$x^2 + y^2 + 2yt = x^2 + (y + t)^2 - t^2 \leq ||a||^2, \quad \forall t \in \mathbb{R}, \qquad \Rightarrow \quad y = 0.$$

□

Remark 1.5.18 *Not every commutative Banach $*$-algebra is symmetric.*

Theorem 1.5.19 <u>**Gelfand – Naimark (commutative case)**</u> *Let A be a symmetric, commutative Banach $*$-algebra, then*

$$\wedge(A) \subset (C_0(\hat{A}), ||\cdot||_\infty) \quad \text{is dense.}$$

If A is a commutative C^*-algebra, then

$$\begin{array}{rcl} \wedge : A & \xrightarrow{\cong} & C_0(\hat{A}) \\ a & \mapsto & \hat{a} \end{array}$$

*is an isometric *-isomorphism.*

Proof: $\wedge$ is a *-homomorphism since A is symmetric:

$$\widehat{a^*}(\chi) = \chi(a^*) = \overline{\chi(a)} = \overline{\hat{a}(\chi)}.$$

We will see that $\wedge(A)$ strictly separates the points of $\hat{A}$. Since $\chi \neq 0$, there is an $a \in A$ with $\chi(a) = \hat{a}(\chi) \neq 0$. Are now $\chi, \mu \in \hat{A}$ with $\chi \neq \mu$, then there is an $a \in A$ with $\chi(a) - \mu(a) \neq 0$. With Stone-Weierstrass, we have $\wedge(A) \subset C_0(\hat{A})$ dense.
If A is C^*, then

$$||a^2|| = ||a^*a|| = \rho(a^*a) = \sup\{|\lambda \,|\, \lambda \in \sigma(a^*a)\}.$$

Further for all $b \in A$, we have

$$\sigma(b) = \sigma_{A^1}(b) = \{\hat{b}(\chi) \,|\, \chi \in \widehat{A^1}\} = \{\hat{b}(\chi) \,|\, \chi \in \hat{A}\} \sqcup \{\underbrace{\hat{b}(\chi_\infty)}_{=0}\}.$$

It thus follows, that

$$||a||^2 = ||a^*a|| = \rho(a^*a) = ||\widehat{a^*a}||_\infty = ||\overline{\hat{a}}\hat{a}||_\infty = ||\hat{a}||^2_\infty, \qquad \Rightarrow \qquad ||a|| = ||\hat{a}||_\infty.$$

□

1.6 Functional Calculus

We shall develop a very strong tool for normal operators, called functional calculus. What functional calculus establishes, is that for any normal operator a and for any continuous $f : \sigma(a) \to \mathbb{C}$, in some sense (made precise below) one can apply the function to the operator, i.e. $\exists\, f(a) \in A$.
Normal operators are interesting, since $C^*(a)$ is commutative for a normal, and thus one can apply Gelfand-Naimark to $C^*(a) \cong C_0(\widehat{C^*(a)})$.

Definition 1.6.1 *Let A be a C^*-algebra and $S \subset A$ a set, then define*

$$\boxed{C^*(S) := \cap\{B \,|\, B \text{ is } C^* \text{ subalgebra of } A : \ S \subseteq B\}}$$

We always have

$$\boxed{C^*(S) = C^*(S \cup S^*) = \overline{\mathrm{LH}\{a_1 \cdots a_m \,|\, m \in \mathbb{N}\ a_1, \dots, a_m \in S \cup S^*\}}}$$

So $C^*(S)$ is commutative iff $[a,b] = 0$ for all $a, b \in S \cup S^*$.

Definition 1.6.2 <u>**Normal Element**</u> *A normal element is an element $a \in A$ such that*

$$aa^* = a^*a.$$

If a is normal, then $C^*(a), C^*(a, \mathbb{1})$ are commutative.

Theorem 1.6.3 __**Functional Calculus**__ *Let A be a unital C^*-algebra (thus $\sigma(a)$ compact), then (for a normal) there is exactly one $*$-homomorphism such that*

$$\phi : C(\sigma(a)) \longrightarrow A, \quad \text{with } \phi(\mathbb{1}_{\sigma(a)}) = a.$$

Further

$$\phi : C(\sigma(a)) \xrightarrow{\cong} C^*(a, \mathbb{1})$$

*is an isometric *-isomorphism.*

Proof: First show uniqueness. We know that ϕ is automatically continuous with $||\phi|| \leq 1$. Now define

$$P_a := \{p : \sigma(a) \to \mathbb{C} \,|\, p \text{ polynomial in } z, \bar{z}\}$$

that is $p \in P_a \Leftrightarrow p(z) = \sum_{k,m=0}^N \alpha_{k,m} z^k \bar{z}^m$. $P_a \subset C(\sigma(a))$ is a subalgebra that separates the points of $\sigma(a)$, since $\mathbb{1}_{\sigma(a)} \in P_a$. Also $\overline{P_a} = P_a$. And thus with Stone-Weierstraß:

$$P_a \subset (C(\sigma(a)), ||\cdot||_\infty) \qquad \text{is dense.}$$

We have $\phi(\mathbb{1}) = a$ and $\phi(1) = 1$ and thus

$$\phi(p) = \sum_{k,m} \alpha_{k,m} a^k (a^*)^m, \quad \text{if } p(z) = \sum_{k,m} \alpha_{k,m} z^k \bar{z}^m.$$

So ϕ is uniquely determined on the dense subalgebra $P_a \subset C(\sigma(a))$. And thus, since ϕ is continuous, on all of $C(\sigma(a))$.
It is important to observe the following

$$\phi(P_a) \subseteq C^*(a, 1), \quad \Rightarrow \quad \phi(C(\sigma(a))) \subseteq C^*(a, 1).$$

With the Gelfand-Naimark theorem (this is where we need a to be normal, in order to have $C^*(a, \mathbb{1})$ commutative) we have

$$C^*(a, 1) \cong C(\widehat{C^*(a, 1)}).$$

We now claim that $\hat{a} : \widehat{C^*(a, 1)} \to \sigma_B(a)$ is a homeomorphism, for which, due to the continuity of $\hat{a}$ and the compactness of $\widehat{C^*(a, 1)}$ and $\sigma_B(a)$, we only need to prove the injectivity of $\hat{a}$.
Let $\chi_1(a) = \hat{a}(\chi_1) = \hat{a}(\chi_2) = \chi_2(a)$ and thus $\chi_1(a^*) = \overline{\chi_1(a)} = \overline{\chi_2(a)} = \chi_2(a^*)$, so

$$\chi_1(\sum_{k,m} \alpha_{k,m} a^k (a^*)^m) = \sum_{k,m} \alpha_{k,m} \chi_1(a)^k \overline{\chi_1(a)}^m = \sum_{k,m} \alpha_{k,m} \chi_2(a)^k \overline{\chi_2(a)}^m = \chi_2(\sum_{k,m} \alpha_{k,m} a^k (a^*)^m).$$

Since the polynomials are dense, we have $\chi_1 = \chi_2$ and thus injectivity.
We now have the isometric *-isomorphism

$$\varphi_a : C(\sigma_{C^*(a,1)}(a)) \longrightarrow C(\widehat{C^*(a, 1)}), \quad \varphi_a(f) = f \circ \hat{a}.$$

Now with the inverse $\wedge^{-1} : C(\widehat{C^*(a, 1)}) \to C^*(a, 1)$ of the Gelfand-homomorphism we have the isometric *-isomorphism

$$\phi := (\wedge^{-1} \circ \varphi_a) : C(\sigma_{C^*(a,1)}(a)) \longrightarrow C^*(a, 1).$$

Note that

$$\phi(\mathbb{1}_{\sigma_{C^*(a,1)}})(\chi) = (\wedge^{-1} \circ \varphi_a)(\mathbb{1}_{\sigma_{C^*(a,1)}})(\chi) = \wedge^{-1} \hat{a}(\chi) = a(\chi),$$

so indeed $\phi(\mathbb{1}_{\sigma_{C^*(a,1)}}) = a$. Now the only piece missing is to show

$$\sigma_{C^*(a,1)}(a) = \sigma_A(a).$$

We already know that $\sigma_A(a) \subseteq \sigma_{C^*(a,1)}(a)$. We now assume $\exists\lambda \in \sigma_{C^*(a,1)}(a) - \sigma_A(a)$. Then $(a - \lambda 1) \in \mathrm{Inv}(A)$ and $c := (a - \lambda 1)^{-1}$. For $a > ||c||$ define $f : \sigma_{C^*(a,1)}(a) \to \mathbb{C}$ as

$$f(z) := \begin{cases} s, & \text{if } |z - s| \leq \frac{1}{s} \\ |z - s|^{-1}, & \text{if } |z - s| \geq \frac{1}{s}. \end{cases}$$

Note that f is well defined, continuous and $f(\lambda) = s$, which gives $||f||_\infty \geq s$. Let $g := (z - \lambda)f(z)$, which is continuous with $||g||_\infty \leq 1$. Take now $\phi : C^0(\sigma_{C^*(a,1)}(a)) \to C^*(a, 1)$ as defined above, then we get

$$\begin{aligned} ||c|| &< s = f(\lambda) \leq ||_{\sigma_{C^*(a,1)}} = ||\phi(f)|| = ||c(a - \lambda 1)\phi(f)|| = ||c\phi(\mathbb{1} - \lambda 1)\phi(f)|| \\ &= ||c\phi(g)|| \leq ||c||\,||\phi(g)|| = ||c||\,||g||_{\sigma_{C^*(a,1)}} \leq ||c||. \end{aligned}$$

So we have $||c|| < ||c||$, a conradiction! This gives us $\sigma_{C^*(a,1)} = \sigma_A(a)$ and completes the proof. □

Corollary 1.6.4 *Let $C \subseteq A$ be a C^*-subalgebra and $a \in C$ normal, then*

$$\boxed{\sigma_C(a) \cup \{0\} = \sigma_A(a) \cup \{0\}}$$

If A is unital with $\mathbb{1}_A \in C$, then

$$\boxed{\sigma_C(a) = \sigma_A(a) \quad \Rightarrow \sigma_{C^*(a,\mathbb{1})}(a) = \sigma_A(a)}$$

Proof: Let A be unital, then in the proof of the functional calculus theorem, we have seen, that

$$\sigma_C(a) = \sigma_{C^*(a,\mathbb{1})}(a) = \sigma_A(a).$$

In general we have the embedding $C^1 \hookrightarrow A^1$ and the identity

$$\sigma_C(a) \cup \{0\} = \sigma_{C^1}(a) = \sigma_{A^1}(a) = \sigma_A(a) \cup \{0\}.$$

□

Theorem 1.6.5 **<u>Functional Calculus</u>** *Let A be a C^*-algebra (not necessarily unital), then (for $a \in A$ normal) there is exactly one $*$-homomorphism such that*

$$\phi : C_0(\sigma(a)) \longrightarrow A, \quad \text{with } \phi(\mathbb{1}_{\sigma(a)}) = a.$$

Further more it holds that

$$\boxed{\phi : C_0(\sigma(a)) \xrightarrow{\cong} C^*(a)}$$

*is an isometric *-isomorphism.*

Proof: Let $P_a := \{p : \sigma(a) \to \mathbb{C} \,|\, p(z) = \sum_{k,m=0}^N \alpha_{k,m} z^k \bar{z}^m\}$ and set $P_{a,0} = \{p \in P_a \,|\, \alpha_{0,0} = 0\}$. Then $P_{a,0}$ a $*$-subalgebra of $C_0(\sigma(a))$ that strongly separates the points of $\sigma(a) \setminus \{0\}$. ($\mathbb{1}_{\sigma(a)} \in P_{a,0}$ and if $z_1 \neq z_2 \in \sigma(a) \setminus \{0\}$, then $0 \neq \mathbb{1}(z_1) \neq \mathbb{1}(z_2) \neq 0$).
With Stone-Weierstrass we have that $P_{a,0} \subset C_0(\sigma(a))$ is dense w.r.t. $||\cdot||_\infty$. If $\phi : C_0(\sigma(a)) \to A$ is an arbitrary $*$-homomorphism with $\phi(\mathbb{1}) = a$, then as in the prove of 1.6.3, we have that

$$\phi(p) = \sum_{k,m} \alpha_{k,m} a^k (a^*)^m, \quad \forall p = \sum_{k,m} \alpha_{k,m} z^k \bar{z}^m \in P_{a,0}$$

and since ϕ is automatically continuous, we have that ϕ is uniquely determined on $\overline{P_{a,0}} = C_0(\sigma(a))$ and it follows

$$\overline{\phi(C_0(\sigma(a)))} = \overline{\phi(P_{a,0})} = \overline{\{\sum_{k,m} \alpha_{k,m} a^k (a^*)^m \,|\, \dots\}} = C^*(a). \qquad (*)$$

We shall show existence. If A is unital, and $\phi : C(\sigma(a)) \to A$ as in 1.6.3, then $\phi_0 := \phi|_{C_0(\sigma(a))}$ as in the theorem. Since ϕ is an isometry, also ϕ_0 is an isometry and with $(*)$ it follows that $\phi_0 : C_0(\sigma(a)) \to C^*(a)$ is an isometric $*$-isomorphism. If A is not unital, then let $\phi : C_0(\sigma(a)) \to A^1$ be as in the theorem. It then follows that $\phi(C_0(\sigma(a))) = C^*(a) \subseteq A$, which completes the proof.

□

Definition 1.6.6 *For A unital, take $\phi_a : C(\sigma(a)) \longrightarrow C^*(a, \mathbb{1})$ and for A not unital take $\phi_a : C_0(\sigma(a)) \longrightarrow C^*(a)$ both to be the above homomorphisms and set*

$$\boxed{f(a) := \phi_a(f)}$$

Lemma 1.6.7 *Let A, B be unital C^*-algebras and $\psi : A \to B$ a unital *-homomorphism, then for all $a \in A$ normal*

$$\boxed{\psi(f(a)) = f(\psi(a))}$$

Proof: We have $\sigma_B(\psi(a)) \subseteq \sigma_A(a)$, so the formula makes sense! Now look at the following compositions

$$\begin{aligned} \varphi : C(\sigma_A(a)) &\xrightarrow{\phi_a} \quad A \quad \xrightarrow{\psi} B \\ \tilde{\varphi} : C(\sigma_A(a)) &\xrightarrow{Res} C(\sigma_B(\psi(a))) \xrightarrow{\phi_{\psi(a)}} B \end{aligned}$$

where $\phi_a(g) = g(a), \phi_{\psi(a)}(g) = g(\psi(a))$ and Res is the restriction map. Then $\varphi, \tilde{\varphi}$ are $*$-homomorphisms with

$$\varphi(1) = 1 = \tilde{\varphi}(1), \quad \varphi(\mathbb{1}) = \psi(a) = \tilde{\varphi}(\mathbb{1}).$$

So it follows that $\varphi = \tilde{\varphi}$ on P_a and thus $\varphi = \tilde{\varphi}$ on $C(\sigma_A(a))$ since P_a is dense and φ continuous. So finally we have

$$f(\psi(a)) = \phi_{\psi(a)}(Res(f)) = \tilde{\varphi}(f) = \varphi(f) = \psi(f(a)).$$

□

Lemma 1.6.8 *Let A be a unital C^*-algebra, $a \in A$ normal and $f \in C(\sigma(a))$, then*

$$\boxed{\sigma(f(a)) = f(\sigma(a))} \qquad \boxed{\forall g \in C(\sigma(f(a))) : \quad g(f(a)) = (g \circ f)(a)}$$

Proof: The first observation is, that also $f(a)$ is normal, since due to the fact that $C(\sigma(a)) \to f \mapsto f(a)$ is a $*$-homomorphism, we have $f(a)^* = \overline{f}(a)$ and

$$f(a)^* f(a) = \overline{f} f(a) = f\overline{f}(a) = f(a)\overline{f}(a) = f(a)f(a)^*.$$

Further more, because of $C(\sigma(a)) \cong C^*(a, 1) =: B$, we have that

$$\sigma_A(f(a)) \overset{1.6.4}{=} \sigma_B(f(a)) = \sigma_{C(\sigma(a))}(f) = f(\sigma(a)).$$

We now consider the $*$-homomorphism $\phi : C(\sigma(a)) \to A \quad \phi(g) := (g \circ f)(a)$. It holds that $\phi(1) = 1$ and $\phi(\mathbb{1}) = (\mathbb{1} \circ f)(a)$. Due to uniqueness in 1.6.3, we have that ϕ is the functional calculus for $f(a)$, so it holds that

$$(g \circ f)(a) = \phi(g) = g(f(a)), \quad \forall g \in C(\sigma(f(a))).$$

□

Corollary 1.6.9 **Spectrum of a selfadjoint element** *Let $a \in A$ be a selfadjoint element, then the spectrum of a is real and it holds that:*

$$\boxed{\sigma(a) \subseteq [-||a||, ||a||]}$$

and at least one of $\pm||a||$ is in $\sigma(a)$.

Proof: W.l.o.g. let A be unital. Since $a = a^*$, it is normal. We have $\mathbb{1}_{\sigma(a)}(a) = a = a^* = \overline{\mathbb{1}_{\sigma(a)}}(a)$ and thus

$$\mathbb{1}_{\sigma(a)} = \overline{\mathbb{1}_{\sigma(a)}}, \quad \Rightarrow \quad z = \mathbb{1}_{\sigma(a)}(s) = \overline{\mathbb{1}_{\sigma(a)}}(z) = \overline{z}.$$

So $\sigma(a) \subset \mathbb{R}$, but we also know $\sigma(a) \subset B_{||a||}(0)$, thus

$$\sigma(a) \subset \mathbb{R} \cap B_{||a||}(0) = [-||a||, ||a||].$$

We further know that $||a|| = \rho(a) = \max\{|\lambda| \mid \lambda \in \sigma(a)\}$, which concludes the proof. $\square$

Corollary 1.6.10 *Let A, B be C^*-algebras and $\psi : A \to B$ an injective $*$-homomorphism, then*

$$||\psi(a)|| = ||a||,$$

that is they are isometries. And in particular $\psi(A) \subset B$ is a $*$-subalgebra.

Proof: With 1.5.12 we have that $||\psi(a)|| \leq ||a|| \ \forall a \in A$. We now assume that $\exists a \in A$ with $||\psi(a)|| < ||a||$. Let w.l.o.g. $||a|| = 1$, set $c = a^*a$, then c is selfadjoint and $||c|| = ||a||^2 = 1$ as well as

$$\alpha := ||\psi(c)|| = ||\psi(a)^*\psi(a)|| = ||\psi(a)||^2 < ||a||^2 = ||c|| = 1.$$

Since c and $\psi(c)$ are selfadjoint, it holds that $\sigma(\psi(c)) \subseteq [-\alpha, \alpha]$ and $\sigma(c) \subseteq [-1, 1]$ with 1 or -1 in $\sigma(c)$. Define $f : [-1, 1] \to \mathbb{C}$

$$f(t) := \begin{cases} 0, & \text{if } |t| \leq \alpha \\ \frac{|t|-\alpha}{1-\alpha}, & \text{if } |t| \geq \alpha. \end{cases}$$

It then holds that $f(c) \neq 0$ since $f(1) = f(-1) = 1 \neq 0$, so $f \neq 0$ on $\sigma(c)$ and thus $||f(c)|| = ||f||_{\sigma(a)} \neq 0$ and (since $f(c) \neq 0$ and ψ injective) it follows that

$$0 = f(\psi(c)) \overset{1.6.7}{=} \psi(f(c)) \neq 0,$$

which is a contradiction. $\square$

1.7 Positive Elements

In this section the notion of positivity of elements in a C^*-algebra is introduced, which leads to an equivalence relation on the space of selfadjoint elements A_{sa}.

Definition 1.7.1 <u>**Positive functions**</u> *A function $f \in C_0(X)$ for X locally compact is called positive iff:*

$$f(x) \geq 0, \quad \forall x \in X.$$

Due to $\sigma_{C_0(X)}(f) = f(X)$, this is equivalent to $f = \overline{f}$ and thus

$$\sigma_{C_0(X)} \subseteq [0, \infty).$$

Definition 1.7.2 <u>**Positive Element**</u> *A positive element of a C^*-algebra A, is an element $a \in A$ with*

$$\boxed{a \geq 0 \quad :\Leftrightarrow \quad a = a^*, \ \sigma(a) \subseteq [0, \infty)}$$

Remark 1.7.3 *For a selfadjoint $a = a^*$ its square is positive $a^2 \geq 0$, since*

$$\sigma(a^2) = \sigma(a)^2.$$

Lemma 1.7.4 **Positive and negative part** *The positive and negative part of a selfadjoint element $a = a^* \in A$ in a C^*-algebra are elements $a_+, a_- \in A$, such that*

$$a_+, a_- \geq 0, \quad a = a_+ - a_- \quad [a_-, a] = 0 = [a_+, a], \quad a_+a_- = 0 = a_-a_+.$$

They exist for any $a = a^* \in A$ and are unique.

Proof:

- Existence: $a = a^*$ thus $\sigma(a) \subseteq \mathbb{R}$. Define

  $$f_\pm : \sigma(a) \longrightarrow \mathbb{R}, \qquad f_\pm(x) := \max\{\pm x, 0\}, \quad a_\pm := f_\pm(a)$$

  so $a_\pm, a \in C^*(a)$ and since $C^*(a)$ commutative, we have that all products commute. Also note

  $$(f_+ - f_-)(x) = x, \quad \Rightarrow \quad a_+ - a_- = \mathbb{1}(a) = a, \qquad f_+ \cdot f_- = 0, \quad \Rightarrow \quad a_+a_- = 0.$$

- Uniqueness: follows from the uniqueness of the decomposition $f = f_+ - f_-$ which in turn is unique due to $f_+ \cdot f_- = 0$.

$\square$

Lemma 1.7.5 *Let A be a unital C^*-algebra and $a = a^* \in A$, then the following hold:*

1.) If $||1 - a|| \leq 1$, then $a \geq 0$.

2.) If $||a|| \leq 1$ and $a \geq 0$ then $||1 - a|| \leq 1$.

3.) $a \geq 0 \Leftrightarrow || \, ||a||\mathbb{1} - a|| \leq ||a||$.

Proof:

1.) With 1.6.9 we have that $\sigma(1 - a) \subseteq [-1, 1]$ and with 1.6.8 it holds that $\sigma(1 - a) = 1 - \sigma(a)$. For $x \in \mathbb{R}$ we have

$$1 - x \in [-1, 1] \quad \Leftrightarrow \quad x - 1 \in [-1, 1] \quad \Leftrightarrow \quad x \in [0, 2]$$

and thus $\sigma(a) \subseteq [0, 2]$.

2.) If $||a|| \leq 1$ and $a \geq 0 \Rightarrow \sigma(a) \subseteq [0, 1]$, thus $\sigma(\mathbb{1} - a) \subseteq [-1, 1]$.

3.) " $\Rightarrow$ " $a \geq 0 \overset{1.6.9}{\Rightarrow} \sigma(a) \subseteq [0, ||a||]$, and thus with 1.6.8 $\sigma(||a||\mathbb{1} - a) \subseteq ||a|| - [0, ||a||] = [0, ||a||]$ $\overset{1.6.9}{\Rightarrow} ||\,||a||\mathbb{1} - a|| \leq ||a||$.

" $\Leftarrow$ " Let w.l.o.g. $a \neq 0$, using $b = \frac{a}{||a||}$, it follows $||\mathbb{1} - b|| \leq 1$ and thus $b \geq 0$ and thus also $a \geq 0$.

$\square$

Theorem 1.7.6 **Space of positive elements** *For the positive elements $A^+ \subset A$ of a C^*-algebra the following hold*

1.) $A^+ \subset A$ is closed .

2.) A^+ is a positive cone, i.e.:

$$a, b \in A^+, \ \lambda \geq 0, \quad \Rightarrow \quad (a + b), \lambda a \in A^+.$$

3.) $A^+ \cap -A^+ = \{0\}$

4.) For all $a \in A$ there are $a_1, a_2, a_3, a_4 \in A^+$ such that

$$a = (a_1 - a_2) + i(a_3 - a_4).$$

Proof: If A is not unital, then $A^+ = A \cap (A^1)^+$. Thus let w.l.o.g. A be unital.

1.) Let $(a_n)_n \subset A^+$ be a sequence with $a_n \to b \in A$, thus

$$|| \, ||a_n|| \mathbb{1} - a_n|| \to || \, ||b|| \mathbb{1} - b||, \quad ||a_n|| \to ||b||$$

and with part (3) of the last lemma, we get $|| \, ||b|| \mathbb{1} - b|| \leq ||b||$ and $b \geq 0$.

2.) Due to $\sigma(\lambda a) = \lambda \sigma(a)$, we have $\lambda a \geq 0$ if $a \geq 0$ and $\lambda \geq 0$. Let now $a, b \geq 0$ with $a, b \neq 0$. Switching to $\frac{1}{c}a, \frac{1}{c}b$ with $c := \max\{||a||, ||b||\}$, w.l.o.g. we can assume that $||a||, ||b|| \leq 1$ and thus $||\mathbb{1} - a||, ||\mathbb{1} - b|| \leq 1$ which gives

$$||\mathbb{1} - \tfrac{1}{2}(a+b)|| \leq \tfrac{1}{2}||\mathbb{1} - a|| + \tfrac{1}{2}||\mathbb{1} - b|| \leq 1,$$

so we have $\frac{1}{2}(a+b) \geq 0$ and thus also $a + b \geq 0$.

3.) If $a \in A^+ \cap -A^+$, then $a = a^*$ with $\sigma(a) = \{0\}$, but also $||a|| = \sigma(a)$.

4.) Let $b = \mathrm{Re}(a) = \frac{1}{2}(a + a^*)$, $c = \mathrm{Im}(a) = \frac{1}{2i}(a - a^*)$ which are both selfadjoint and thus $b = b_+ - b_-, c = c_+ - c_-$ with $b_+, b_-, c_+, c_- \geq 0$.

$\square$

Lemma 1.7.7 Equivalence Relation on the space of selfadjoint elements *On the space of seladjoint elements $A_{sa} \subset A$ of a C^*-algebra, we define the following equivalence relation*

$$a \leq b \quad :\Leftrightarrow \quad b - a \geq 0.$$

Lemma 1.7.8 *Let A be a $\mathbb{C}$-algebra and $a, b \in A$, then*

$$\boxed{\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}}$$

and if A is a C^*-algebra, then

$$ab \geq 0 \quad \Leftrightarrow \quad ba \geq 0.$$

Proof: Let w.l.o.g. A be unital and let $0 \neq \lambda \in \mathbb{C}$. We need to show that

$$ab - \lambda \mathbb{1} \in \mathrm{Inv}(A) \quad \Leftrightarrow \quad ba - \lambda \mathbb{1} \in \mathrm{Inv}(A).$$

In order to do that, define $u := (ab - \lambda \mathbb{1})^{-1}$, so that

$$abu = (ab - \lambda \mathbb{1})u + \lambda u = \mathbb{1} + \lambda u, \quad \Rightarrow$$

$$(ab - \lambda \mathbb{1})(bua - \mathbb{1}) = b(abu)a - ba - \lambda bua + \lambda \mathbb{1} = b(1 + \lambda u)a - ba - \lambda bua + \lambda \mathbb{1} = \lambda \mathbb{1}.$$

Analogously it holds that

$$(bua - \mathbb{1})(ba - \lambda \mathbb{1}) = b(uab)a - ba - \lambda bua + \lambda \mathbb{1} = b(\mathbb{1} + \lambda u)a - ba - \lambda bua + \lambda \mathbb{1} = \lambda \mathbb{1}.$$

Therefor we have $\frac{1}{\lambda}(bua - 1) = (ba - \lambda \mathbb{1})^{-1}$. $\square$

Lemma 1.7.9 *Let A be a C^*-algebra and $a = a^* \in A$, then the following are equivalent:*

1.) $a \geq 0$

2.) $\exists b \in A$ with $a = b^* b$

3.) $\exists! c \geq 0$ *with* $c^2 = a$ *and* $[a, c] = 0$. *Denote* $c := \sqrt{a}$.

Proof:

1.) $\Rightarrow$ 3.) Since $a \geq 0$ we have $\sigma(a) \subset [0, \infty)$ and the square root is well defined on it, which in turn by functional calculus lets $c := \sqrt{a}$ be well defined. One can also show that c is unique.

2.) $\Leftrightarrow$ 3.) Let $b = c$.

2.) $\Rightarrow$ 1.) Let now $a = b^*b$ for a $b \in A$. Let $a_+, a_- \in A^+$ and $u = \sqrt{a_+}, v = \sqrt{a_-}$ as in (3). It then holds that $u \in C^*(a_+) \subseteq C^*(a), v \in C^*(a_-) \subseteq C^*(a)$, i.e. all u, v, a_+, a_- commute. We further have $vu^2v = v^2u^2 = a_+a_- = 0$ and thus

$$(bv)^*(bv) = vb^*bv = v(u^2 - v^2)v = vu^2v - v^4 = -v^4 \in A^+.$$

Since the square of a selfadjoint element is positive. We set $bv = x + iy$ with selfadjoint x, y, then

$$(bv)(bv)^* = (x + iy)(x - iy) + \underbrace{(x - iy)(x + iy) - (bv)^*(bv)}_{=0} = 2x^2 + 2y^2 + v^4 \geq 0.$$

With 1.7.8 it also follows that $(bv)^*(bv) = -v^4 \geq 0$, but then $v^4 \in A^+ \cap -A^+ = \{0\}$, so $v^4 = 0$ and thus $v^2 = 0$, which gives $a = u^2 - v^2 = u^2 \geq 0$.

$\square$

Remark 1.7.10 **Positivity does not depend on the subalgebra** *Let* $B \subseteq A$ *be a subalgebra with* $a = a^* \in B$, *then* $\sigma_B(a) \cup \{0\} = \sigma_A(a) \cup \{0\}$ *and therefor it holds that*

$$\sigma_B(a) \subset [0, \infty) \quad \Leftrightarrow \quad \sigma_A(a) \subset [0, \infty).$$

In particular for $\hat{a} \in C_0(\sigma_A(a))$:

$$a \geq 0 \quad \Leftrightarrow \quad \hat{a} \geq 0.$$

Definition 1.7.11 **Absolute Value** *We now see, that the following is well defined*

$$|a| := \sqrt{a^*a}.$$

Lemma 1.7.12 **Positive Linear Operators** *Let* $T^* = T \in L(H)$, *then*

$$\boxed{T \geq 0 \quad \Leftrightarrow \quad \langle Tx, x \rangle \geq 0 \; \forall x \in H.}$$

Proof:

"$\Rightarrow$" If $T \geq 0$ then there is a $S \in L(H)$: $T = S^*S$ and thus

$$\langle Tx, x \rangle = \langle S^*Sx, x \rangle = \langle Sx, Sx \rangle \geq 0 \forall x \in H$$

"$\Leftarrow$" From functional analysis, we know that

$$S = S^* \quad \Rightarrow \quad ||S|| = \sup_{||x||=1} |\langle Sx, x \rangle|.$$

Now let $\langle Tx, x \rangle \geq 0$ and w.l.o.g. $||T|| = 1$ (since for $T = 0$ it is trivial). It is now sufficient to show $||\mathbb{1} - T|| \leq 1$. But if $||x|| = 1$, then

$$0 \leq \langle Tx, x \rangle \leq 1 \quad \Rightarrow \quad |\langle (\mathbb{1} - T)x, x \rangle| = \langle x, x \rangle - \langle Tx, x \rangle \leq \langle x, x \rangle = 1.$$

$\square$

Definition 1.7.13 **Unitary Elements** *A unitary element $u \in A$ in a C^*-algebra is an element such that*

$$uu^* = \mathbb{1} = u^*u \quad \Leftrightarrow \quad u^* = u^{-1}.$$

We set

$$U(A) := \{u \in A \,|\, u \text{ unitary}\}.$$

Lemma 1.7.14 **Unitary Linear Operators** *Let $u \in L(H)$, then it holds that*

$$u^* = u^{-1} \quad \Leftrightarrow \quad uH = H, \text{ and } \langle ux, uy\rangle = \langle x, y\rangle .$$

That is a linear operator is unitary iff it is surjective and an isometry.

Proof:

"$\Rightarrow$" If $u = u^{-1}$, so since an inverse exists, it is bijective and further

$$\langle ux, uy\rangle = \langle u^*ux, y\rangle = \langle x, y\rangle .$$

"$\Leftarrow$" Let $uH = H$, and $\langle ux, uy\rangle = \langle x, y\rangle$, then u is bijective and continuous, thus (by the open mapping theorem) invertible. Further

$$\langle u^*ux, y\rangle = \langle ux, uy\rangle = \langle x, y\rangle \quad \Rightarrow \quad \langle u^*ux - x, y\rangle = 0 \;\forall x, y \in H.$$

That is $u^*u = \mathbb{1}$.

□

Theorem 1.7.15 **Polar Decomposition**: *Let A be a unital C^*-algebra, then*

$$a \in \mathrm{Inv}(A) \quad \Rightarrow \quad \exists!\; u \in U(A): \; a = u|a|.$$

Proof: $a \in \mathrm{Inv}(A)$ thus also $a^*a = |a|^2 \in \mathrm{Inv}(A)$. Now take $u := a|a|^{-1}$ and thus

$$u^*u = |a|^{-1}a^*a|a|^{-1}|a|^{-1}|a|^2|a|^{-1} = \mathbb{1}.$$

□

Remark 1.7.16 *If a is not invertible, a polar decomposition does not necessarily exist.*

Lemma 1.7.17 *Let A be a C^*-algebra, $a, b \in A_{sa}$, then*

$$a \leq b \quad \Leftrightarrow \quad x^*ax \leq x^*bx \;\forall x \in A.$$

Further, if A is unital, then

$$a, b \in \mathrm{Inv}(A), \text{ and } 0 \leq a \leq b \quad \Rightarrow \quad 0 \leq b^{-1} \leq a^{-1}.$$

1.8 Approximate Unities and Quotient Spaces

After discussing functional calculus, we now come to a second major tool in the theory of C^*-algebras: approximate unities.

Definition 1.8.1 **Approximate Unity** *An approximate unity in a normed algebra A is a net $(u_n)_n \subset A$, that is both an approximate left unity:*

$$u_n a \longrightarrow a, \;\forall a \in A$$

and a approximate right unity: $au_n \longrightarrow a, \;\forall a \in A$.

Theorem 1.8.2 *Let A be a C^*-algebra and $J \subseteq A$ a dense ideal in A, it then holds that*

1.) There exists an approximate unity $(u_n)_n$ in A with

a) $0 \leq u_n$, $||u_n|| \leq 1$ and for all n: $u_n \in J$

b) $\lambda \leq \mu$ then $u_\lambda \leq u_\mu$.

2.) If A is seperable, then there is a sequence with the above properties 1.a) and 1.b).

3.) Let I be a right ideal ($IA \subseteq I$), then there is a net $(u_n)_n$ in $I \cap A^+$ which fulfills properties 1.a) and 1.b) and $u_n b \to b$ for all $b \in \overline{I}$.

4.) The same as in 3.) holds also for left ideals.

Proof:

1.a) Let $\Lambda := \{F \subseteq J \,|\, F \text{ finite}\}$ with $F_1 \leq F_2 \Leftrightarrow F_1 \subseteq F_2$. For $\lambda = \{x_1, \ldots, x_l\} \in \Lambda$ set

$$v_\lambda = x_1 x_1^* + \cdots + x_l x_l^* \geq 0.$$

With $v_\lambda \geq 0$ we have $\sigma(\frac{1}{l} + v_\lambda) \subseteq [\frac{1}{l}, \infty)$, so $(\frac{1}{l} + v_\lambda)$ is invertible (since $0 \neq \sigma(\frac{1}{l} + v_\lambda)$) in A^1 and we set

$$u_\lambda := v_\lambda(\tfrac{1}{l} + v_\lambda)^{-1} = f_l(v_\lambda), \qquad f_l(t) = t(\tfrac{1}{l} + t)^{-1}.$$

Then $u_\lambda \in J$, since $v_\lambda \in J$ and J is an ideal in A and thus also in A^1. Due to $0 \leq f_l \leq 1$ we have $0 \leq u_\lambda \leq 1$ in A^1 and so $||u_\lambda|| \leq 1$. Further, for $\lambda = \{x_1, \ldots, x_l\}$, we have

$$\sum_{i=1}^{l} [(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* = (u_\lambda - 1)(\sum_{i=1}^{l} x_i x_i^*)(u_\lambda - 1) = (u_\lambda - 1)v_\lambda(u_\lambda - 1) = g_l(v_\lambda)$$

with $g_l(t) = (f_l(t) - 1)^2 t$. It follows, that

$$f_l(t) - 1 = \frac{t}{\frac{1}{l} + t} - 1 = \frac{-\frac{1}{l}}{\frac{1}{l} + t} = \tfrac{1}{l}(\tfrac{1}{l} + t)^{-1}$$

and $(\frac{1}{l} + t)^2 \geq \frac{2}{l}t$, so $\frac{l}{2t} \geq (\frac{1}{l} + t)^{-2}$. We thus have

$$g_l(t) = \tfrac{t}{l^2}(\tfrac{1}{l} + t)^{-2} \leq \tfrac{t}{l^2}\tfrac{l}{2t} = \tfrac{1}{2l}$$

and in A^1 we get

$$0 \leq [(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* \leq g(v_\lambda) \leq \tfrac{1}{2l}\mathbb{1}.$$

So it holds, that

$$||(u_\lambda - 1)x_i||^2 \leq \tfrac{1}{2l} \quad \forall 1 \leq i \leq l.$$

We now get $||u_\lambda x - x|| \to 0 \ \forall x \in J$, since for $\varepsilon > 0$ we take $\lambda_0 = \{x_1, \ldots, x_l\} \in \Lambda$ with $x = x_1$ and $\frac{1}{\sqrt{2l}} < \varepsilon$ it follows, that $||u_\lambda x - x|| < \varepsilon \quad \forall \lambda \geq \lambda_0$.
It now also holds that $||u_\lambda a - a|| \to 0$ for a general $a \in A$, since for $\varepsilon > 0$, there exists an $x \in J$ with $||a - x|| < \frac{\varepsilon}{3}$ and $\lambda_0 \in \Lambda$ with $||u_\lambda x - x|| < \frac{\varepsilon}{3} \quad \forall \lambda \geq \lambda_0$ and we have

$$||u_\lambda a - a|| \leq ||u_\lambda(a - x)|| + ||u_\lambda x - x|| + ||x - a|| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon \quad \forall \lambda \geq \lambda_0.$$

So we can conclude, that $||a u_\lambda - a|| = ||u_\lambda a^* - a^*|| \to 0 \ \forall a \in A$.

1.b) Let $\lambda \leq \mu$, i.e. $\lambda = \{x_1, \dots, x_l\}$, $\mu = \{x_1, \dots, x_l, x_{l+1}, \dots, x_m\}$, then

$$v_\lambda = \sum_{i=1}^{l} x_i x_i^* \leq \sum_{i=1}^{m} x_i x_i^* = v_\mu$$

and thus $\frac{1}{l} + v_\lambda \leq \frac{1}{l} + v_\mu$ and $(\frac{1}{l} + v_\mu)^{-1} \leq (\frac{1}{l} + v_\lambda)^{-1}$, since for a $t \geq 0$ the map $s \mapsto \frac{s}{s+t}$ is monotonically increasing. It follows, that

$$\tfrac{1}{l}(\tfrac{1}{l} + v_\mu)^{-1} \leq \tfrac{1}{m}(\tfrac{1}{m} + v_\mu)^{-1}.$$

Due to $f_m(t) = t(\frac{1}{m} + t)^{-1} = 1 - \frac{1}{m}(\frac{1}{m} + t)^{-1}$, we now conclude, that

$$u_\lambda = f_l(v_\lambda) = 1 - \tfrac{1}{l}(\tfrac{1}{l} + v_\lambda)^{-1} \leq 1 - \tfrac{1}{l}(\tfrac{1}{l} + v_\mu)^{-1} \leq 1 - \tfrac{1}{m}(\tfrac{1}{m} + v_\lambda)^{-1} = u_\mu.$$

2.) If A is separable, then so is J. Let $(x_1, x_2, \dots)$ be a dense sequence in J and set $u_n = u_\lambda$ with $\lambda = \{x_1, \dots, x_n\}$ as in 1.). Then as in 1.) it follows that $(u_n)_n$ is an approximate unity with the desired properties.

3.) Let now Λ be the set of all finite subsets in I and u_λ as in 1.) $\forall \lambda = \{x_1, \dots, x_l\} \in \Lambda$. The proof then proceeds as the proof of 1.).

□

Corollary 1.8.3 *Let $I \subseteq A$ be a closed ideal, then $I = I^*$ and thus I is a $*$-ideal and a C^*-subalgebra.*

Proof: Let $(u_n)_n \subseteq I \cap A^+$ like in the above theorem, then for all $x \in I$, we have

$$||x^* u_n - x^*|| = ||u_n x - x|| \to 0.$$

Since $x^* u_n \in I$, also $x^* \in I$, since I is closed. □

Lemma 1.8.4 *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal. And $(u_\lambda)_\lambda$ like in 3.) above, then*

$$||a + I|| = \lim_{\lambda \to \infty} ||a - u_\lambda a||.$$

Proof: Let $(u_\lambda)_\lambda$ be an approximate unity as in 1.8.2(3). Because of $u_\lambda \geq 0$ and $||a_\lambda|| \leq 1$, we have $||1 - u_\lambda|| \leq 1$ and thus $||a - u_\lambda a|| \leq ||a||$ $\forall \lambda \in \Lambda$. So the following exists

$$\limsup_\lambda ||a - u_\lambda a||.$$

Due to $||u_\lambda b - b|| \to 0$ $\forall b \in I$ it follows for all $b \in I$, that

$$\begin{aligned} \limsup_\lambda ||a - u_\lambda a|| &= \limsup_\lambda ||a - u_\lambda a + b - u_\lambda b|| = \limsup_\lambda ||(1 - u_\lambda)(a + b)|| \\ &\leq \sup_\lambda ||(1 - u_\lambda)(a + b)|| \leq ||a + b||. \end{aligned}$$

We thus have, that

$$\begin{aligned} ||a + I|| &= \inf_{b \in I} ||a + b|| \geq \limsup_\lambda ||a - u_\lambda a|| \geq \liminf_\lambda ||a - u_\lambda a|| \\ &\geq \inf_\lambda ||a - u_\lambda a|| \geq \inf_{b \in I} ||a + b|| = ||a + I||. \end{aligned}$$

So we have equalities everywhere, which proves the claim. □

Lemma 1.8.5 **Quotient $\mathbf{C*-}$Algebra** *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal. Then A/I is a Banach-algebra with*

$$||a+I|| = \inf\{||a+c|| \mid c \in I\}, \quad (a+I)(b+I) = ab+I.$$

Since we now know that $I^* = I$, we have a well defined involution

$$(a+I)^* := a^* + I,$$

turning A/I into a C^*-algebra

Proof: We have to show that $||a^*a+I|| = ||a+I||^2$.

"$\geq$" Let $(u_n)_n$ be a sequence like in 3.), then for all $b \in I$:

$$||(1-u_n)b(1-u_n)|| \leq \underbrace{||(1-u_n)b||}_{\to 0}\underbrace{||1-u_n||}_{\leq 1} \to 0.$$

So with the previous lemma, we have

$$||a+I||^2 = \lim_{n\to\infty}||(1-u_n)a||^2 = \lim_{n\to\infty}||(1-u_n)a^*a(1-u_n)|| = \lim_{n\to\infty}||(1-u_n)(a^*a+b)(1-u_n)|| \leq ||a^*a+b||,$$

so we get $||a+I||^2 \leq \inf_{b\in I}||a^*a+b|| = ||a^*a+I||$.

"$\leq$" Since A/I is a Banach algebra, we also have

$$||a^*a+I|| \leq ||a^*+I||\,||a+I|| = ||a+I||^2.$$

$\square$

Corollary 1.8.6 *If $\phi : A \to B$ ia a homomorphism of C^*-algebras, then $\phi(A) \subseteq B$ is a C^*-subalgebra.*

Proof: That is since $I := \ker\phi$ is a closed ideal and so

$$\tilde{\phi} : A/I \longrightarrow B, \quad \tilde{\phi}(a+I) := \phi(a)$$

is an injective *-homomorphism. Since A/I is a C^*-algebra, $\tilde{\phi}$ is isometric and thus $\phi(A) = \tilde{\phi}(A/I)$ is a C^*-subalgebra. $\square$

Corollary 1.8.7 *Let $B \subseteq$ be a C^*-subalgebra and $I \subset A$ a closed ideal, then*

$$B+I \subseteq A \text{ is a } C^*\text{-subalgebra and } (B+I)/I \cong B/(B\cap I).$$

Lemma 1.8.8 *Let A be a C^*-algebra, then $A^2 = A$.*

Proof: That is since $\exists a_1, a_2, a_3, a_4 \geq 0$:

$$a = (a_1 - a_2) + i(a_3 - a_4).$$

So the square root is defined on positive elements. $\square$

2 States and Representaions of C^*-Algebras

In quantum mechanics, a state of a (quantum) system (e.g. a hydrogen atom) is described by a vector $x \in H$ in some Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Physical quantities, such as position, momentum or energy are assigned (selfadjoint) operators that act on the Hilbert space H. Their eigenvalues constitute the different possible measurement outcomes, when measuring a certain physical quantity. The expectation value for the outcome of a measurement of the physical quantity T of a (quantum) system in the state $x \in H$ is given by $\langle Tx, x \rangle$.
A different approach to quantum (field) theory (QFT) is so called Algebraic QFT (AQFT), in which states are defined as positive functionals φ (on a C^*-algebra A) of unit norm, as below. The bridge between these two descriptions are the so called vector states: $\varphi_x(T) := \langle Tx, x \rangle$ with $||x|| = 1$. So, intuitively speaking, in AQFT one calls a state of T, what in regular quantum mechanics was referred to as the expectation value of T.
So we can define expectation values without making use of Hilbert spaces. The worry is now: do these states give rise to a unique representation (and thus a unique Hilbert space), or to a whole set of inequivalent quantum system, making the new AQFT formulation inherently different to the usual approach? The answer to this question is the so called GNS-construction.

2.1 Positive Linear Functionals and States

Definition 2.1.1 <u>**Positive Functional**</u> *Let A be a C^*-algebra. A linear functional*

$$\varphi : A \longrightarrow \mathbb{C} \text{ is positive} \quad :\Leftrightarrow \quad \varphi(a^*a) \geq 0 \ \forall a \in A.$$

$$P(A) := \{\varphi : A \to \mathbb{C} \,|\, \varphi(a^*a) \geq 0 \ \forall a \in A\}.$$

Definition 2.1.2 <u>**State**</u> *A state is a positive functional of unit norm. The state space is*

$$\mathcal{S}(A) := \{\varphi \in P(A) \,|\, ||\varphi|| = 1\}.$$

Remark 2.1.3 <u>**Semi − definite Hermitian Form**</u> *Any $\varphi \in P(A)$ defines a semi-definite Hermitian form on A:*

$$\langle \cdot, \cdot \rangle_\varphi : A \times A \longrightarrow \mathbb{C}, \qquad \boxed{\langle a, b \rangle_\varphi := \varphi(b^*a)}$$

To see $\overline{\langle a, b \rangle_\varphi} = \langle b, a \rangle_\varphi$, we need the following

Lemma 2.1.4 *For $\varphi \in P(A)$ we have that $\overline{\varphi(a)} = \varphi(a^*)$.*

Proof: Let $x \in A_{sa}$, then it holds that $x = u - v$ with $u, v \in A^+$, so we have $\varphi(x) = \varphi(u) - \varphi(v) \in \mathbb{R}$ since $\varphi(u), \varphi(v) \geq 0$ holds and therefor

$$\varphi(a^*) = \varphi(x - iy) = \varphi(x) - i\varphi(y) = \overline{\varphi(x) + i\varphi(y)} = \overline{\varphi(a)}.$$

□

Lemma 2.1.5 <u>**Continuity of positive Functionals**</u> *Let $\varphi \in P(A)$, then φ is continuous. I.e. $P(A) \subset A'$.*

Proof: Let $a \in A$, then there are $u, v, u', v' \in A_+$ all with norm smaller or equal to $||a||$ such that

$$a = (u - v) + i(u' - v').$$

It thus suffices to show that there exists a $M \geq 0$ such that

$$\varphi(a) \leq M||a|| \quad \forall a \in A_+.$$

We assume this does not hold. Then there is a sequence $(a_n)_n$ in A_+ with $||a_n|| = 1$ and $\varphi(a_n) \geq 2^n$ $\forall n$. If we now set $a := \sum_{n=1}^{\infty} \frac{1}{2^n} a_n \in A_+$, it follows for all $N \in \mathbb{N}$, that

$$N \leq \varphi(\sum_{n=1}^{N} \tfrac{1}{2^n} a_n) \leq \varphi(\sum_{n=1}^{\infty} \tfrac{1}{2^n} a_n) = \varphi(a) < \infty,$$

which is a contradiction. □

Corollary 2.1.6 *Let $\varphi \in P(A)$, then it holds that $|\varphi(a)|^2 \leq ||\varphi|| \varphi(a^* a)$.*

Proof: Let $(u_n)_n$ be an approximate unity like in 2.), then $0 \leq u_\lambda \leq u_\mu$ for all $\lambda \leq \mu$ and $||u_n|| \leq 1$. So

$$\begin{aligned} |\varphi(a)|^2 &\overset{\text{continuity}}{=} \lim_{n\to\infty} |\varphi(u_n a)|^2 \overset{u_n = u_n^*}{=} \lim_{n\to\infty} |\langle a, u_n\rangle_\varphi|^2 \\ &\leq \sup_n \langle u_n, u_n\rangle_\varphi \langle a, a\rangle_\varphi = \sup_n \varphi(u_n^2)\varphi(a^* a) \\ &\overset{||u_n||^2 \leq 1}{\leq} ||\varphi|| \varphi(a^* a). \end{aligned}$$

□

Lemma 2.1.7 *Let A be a C^*-algebra and $\varphi \in A'$, then the following are equivalent:*

(1) φ is positive.

(2) There exists an approximate unity $(u_n)_n$ with $0 \leq u_\lambda \leq u_\mu$ for all $\lambda \leq \mu$ and $||u_n|| \leq 1$, such that

$$||\varphi|| = \lim_{n\to\infty} |\varphi(u_n)|.$$

(3) For every approximate unity like (2), $||\varphi|| = \lim_{n\to\infty} |\varphi(u_n)|$ holds.

Proof: Let w.l.o.g. $||\varphi|| = 1$

"(1) $\Rightarrow$ (3)" Let $(u_\lambda)_\lambda$ be as in (2) and let $\varphi \geq 0$, then $(\varphi(u_\lambda))_\lambda$ is monotonically increasing and bounded by 1 since $\lambda \leq \mu \Rightarrow u_\mu - u_\lambda \geq 0 \Rightarrow \varphi(u_\mu) - \varphi(u_\lambda) = \varphi(u_\mu - u_\lambda) \geq 0$. So $\lim_\lambda \varphi(u_\lambda) \leq 1$ exists. Let now $\varepsilon > 0$ and $a \in A$ with $||a|| = 1$ and $|\varphi(a)|^2 \geq 1 - \varepsilon$. Due to $0 \leq u_\lambda \leq 1$ (in A^1) we have $u_\lambda^2 \leq u_\lambda$, since if $c = \sqrt{u_\lambda}$, then $u_\lambda^2 = c u_\lambda c \leq c \mathbb{1} c = u_\lambda$. We finally have

$$1-\varepsilon \leq |\varphi(a)|^2 = \lim_\lambda |\varphi(u_\lambda a)|^2 \overset{C-S}{\leq} \sup_\lambda \varphi(u_\lambda^2)\varphi(a^* a) \leq \sup_\lambda \varphi(u_\lambda)\varphi(a^* a) \leq \sup_\lambda \varphi(u_\lambda) \overset{\text{monot.}}{=} \lim_\lambda \varphi(u_\lambda).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lim_\lambda \varphi(u_\lambda) = 1 = ||\varphi||$.

"(3) $\Rightarrow$ (2)" is obvious.

"(2) $\Rightarrow$ (1)" We first show, that $\varphi(a) \in \mathbb{R}$ $\forall a \in A_{sa}$. Let $a = a^*$ with $||a|| = 1$ and let $\varphi(a) = x - iy$ with $x, y \in \mathbb{R}$. Assume $y \neq 0$. Then w.l.o.g. $y > 0$ (if not, we just work with $-a$). For all $n \in \mathbb{N}$ we have

$$||a - inu_\lambda||^2 = ||(a + inu_\lambda)a - inu_\lambda|| = ||a^2 + n^2 u_\lambda^2 + in(u_\lambda a - a u_\lambda)|| \leq 1 + n^2 + n||u_\lambda a - a u_\lambda||.$$

Since $\varphi(u_\lambda) \to 1 = ||\varphi||$, it follows that

$$\begin{aligned} |\varphi(a) - in|^2 &= \lim_\lambda |\varphi(a) - in\varphi(u_\lambda)|^2 = \lim_\lambda |\varphi(a - inu_\lambda)|^2 \\ &\overset{||\varphi|| \leq 1}{\leq} \lim_\lambda (1 + n^2 + n \underbrace{||a u_\lambda - u_\lambda a||}_{\to 0}) = 1 + n^2. \end{aligned}$$

Then: $|\varphi(a) - in|^2 = |x - i(y+n)| = x^2 + (y+n)^2 = x^2 + y^2 + n^2 + 2yn$ and it follows that $x^2 + y^2 + 2yn \leq 1\ \forall n \in \mathbb{N}$, which is a contradiction to $y > 0$.
Let now $a \geq 0$ with $||a|| = 1$, then $||\mathbb{1} - a|| \leq 1$ and with 1.7.5 we have that

$$|\varphi(a) - 1| = \lim_\lambda |\varphi(au_\lambda) - \varphi(u_\lambda)| = \lim_\lambda |\varphi((a - \mathbb{1})u_\lambda)| \leq ||(a - \mathbb{1})u_\lambda|| \leq ||a - \mathbb{1}||\,||u_\lambda|| \leq 1.$$

Where we work in A^1 if A is not unital. Finally since $\varphi(a) \in \mathbb{R}$ and $|\varphi(a) - 1| \leq 1$ we have $\varphi(a) \geq 0$.

□

Corollary 2.1.8 *If A is unital and $\varphi \in A'$, then*

$$\boxed{\varphi \geq 0 \quad \Leftrightarrow \quad \varphi(1) = ||\varphi||}$$

Proof: If A is unital $u_n := \mathbb{1}$ is an approximate unity and the claim follows directly from the previous lemma. □

Corollary 2.1.9 *Let $\varphi, \psi \in P(A)$, then we have $||\varphi + \psi|| = ||\varphi|| + ||\psi||$.*

Proof: $\varphi + \psi$ is positive and thus

$$||\varphi + \psi|| = \lim_{n\to\infty} (\varphi + \psi)(u_n) = \lim_{n\to\infty} [\varphi(u_n) + \psi(u_n)] = ||\varphi|| + ||\psi||.$$

□

Theorem 2.1.10 *Let A be a nonunital C^*-algebra, then for every $\varphi \in P(A)$ there is exactly one $\tilde{\varphi} \in P(A^1)$ such that*

$$\tilde{\varphi}|_A = \varphi, \quad ||\tilde{\varphi}|| = ||\varphi||.$$

It follows that

$$\tilde{\varphi}(a + \mu 1) = \varphi(a) + \mu||\varphi||.$$

Proof: Since $\tilde{\varphi}$ is positive, we get $\tilde{\varphi}(1) = ||\tilde{\varphi}||$. If we assume $||\tilde{\varphi}|| = ||\varphi||$, then due to $\tilde{\varphi}|_A = \varphi$ and the linearity of φ, we have $\tilde{\varphi}(a + \mu 1) = \varphi(a) + \mu||\varphi||$.
Now show that $||\tilde{\varphi}|| = ||\varphi||$, since it then follows that $\tilde{\varphi}(1) = ||\tilde{\varphi}||$ and $\tilde{\varphi}$ is positive.
Let $(u_n)_n$ an approximate unity like above, then $||\varphi|| = \lim_{n\to\infty} \varphi(u_n)$ and

$$\begin{aligned} ||\tilde{\varphi}(a + \mu 1)|| &= |\varphi(a) + \mu||\varphi||\,| = \lim_{n\to\infty} |\varphi(au_n) + \mu\varphi(u_n)| = \lim_{n\to\infty} |\varphi(a + \mu 1)u_n| \\ &\leq ||\varphi||\,||(a + \mu 1)u_n|| \leq ||\varphi||\,||a + \mu 1||. \end{aligned}$$

So we have $||\tilde{\varphi}|| \leq ||\varphi||$. Because of $\tilde{\varphi}|_A = \varphi$ we also have $||\tilde{\varphi}|| \geq ||\varphi||$. So indeed $||\tilde{\varphi}|| = ||\varphi||$. □

Example 2.1.11 <u>**States**</u> *on C^*-algebras. Remember: a state was a pos. functional of unit norm.*

1.) $A = C(X)$ for X compact, then states are just probability-measures on X, i.e. positive Radon integrals with

$$\mu_\varphi(X) = \varphi(\mathbb{1}_X) = 1.$$

2.) $A = L(H_\mathbb{C})$ then for every $x \in H_\mathbb{C}$ we have a positive functional $\varphi_x : L(H_\mathbb{C}) \to \mathbb{C}$ defined by

$$\varphi_x(T) := \langle Tx, x\rangle\,, \quad \Rightarrow \quad ||\varphi_x|| = \varphi_x(\mathbb{1}_H) = \langle x, x\rangle$$

so φ_x is a state iff $||x|| = 1$. These states are called **vector states** *of $L(H_\mathbb{C})$.*

Remark 2.1.12 <u>**Convexity of the State Space**</u> *For the state space $\mathcal{S}(A)$ the following hold:*

1.) If A is not unital, then the continuation map

$$\mathcal{S}(A) \longrightarrow \mathcal{S}(A^1), \quad \varphi \mapsto \tilde{\varphi}$$

(defined as above) is an embedding.

2.) $\mathcal{S}(A)$ is convex: For $\varphi_1, \varphi_2 \in \mathcal{S}(A)$ and $\lambda \in [0,1]$:

$$||\lambda\varphi_1 + (1-\lambda)\varphi_2|| = ||\lambda\varphi_1|| + ||(1-\lambda)\varphi_2|| = \lambda||\varphi_1|| + (1-\lambda)||\varphi_2|| = 1.$$

Theorem 2.1.13 *Let A be a C^*-algebra, then for every $a \in A$ normal, there exists a $\varphi \in \mathcal{S}(A)$ with $|\varphi(a)| = ||a||$. Furthermore for every $a \in A$ there is a $\varphi_a \in \mathcal{S}(A)$ such that*

$$\varphi(a^*a) = ||a||^2.$$

Proof: W.l.o.g. let A be unital. Let $B := C^*(\mathbb{1}, a) \cong C(\hat{B})$. Since $\hat{B}$ is compact, there is a $\chi \in \hat{B}$:

$$||a|| = ||\hat{a}||_\infty = |\hat{a}(\chi)| = |\chi(a)|.$$

Since $\chi : B \to \mathbb{C}$ is a $*$-homomorphism, we also have

$$\chi(b^*b) = \overline{\chi(b)}\chi(b) = |\chi(b)|^2 \geq 0, \quad \forall b \in B.$$

We get $\chi \in P(B)$ with $||\chi|| = \chi(1) = 1$, so $\chi \in \mathcal{S}(B)$ with $|\chi(a)| = ||a||$. Above we have already seen, that there is a $\varphi \in \mathcal{S}(B)$ with $\varphi|_B = \chi$. It follows, that $|\varphi(a)| = ||a||$. Since a was general, we can apply the same reasoning to a^*a. □

2.2 Representations, Gelfand-Naimark Theorem and the GNS-Construction

Definition 2.2.1 **Representation** *A representation of a C^*-algebra A on a $\mathbb{C}$-Hilbert space H is a $*$-homomorphism*

$$\pi : A \longrightarrow L(H).$$

A representation is called:

- **Faithful** *iff π is injective.*
- **Nondegenerate** *iff $\overline{\pi(A)H} = H$.*
- **Cyclic** *iff there exists a so called* **cyclic vector** *ξ, that is a $\xi \in H$ such that*

$$\pi(A)\xi \subset H \text{ dense.}$$

- **Irreducible** *iff $\pi(A)H \neq \{0\}$ and for every closed subspace E:*

$$\pi(A)E \subseteq E, \qquad \Rightarrow \qquad E = \{0\}, \text{ or } E = H.$$

Remark 2.2.2 *We have the following implications:*

$$\text{irreducible} \quad \Rightarrow \quad \text{cyclic} \quad \Rightarrow \quad \text{nondegenerate.}$$

Where the first one is part of Schur's lemma and the second is obvious.

Definition 2.2.3 **Equivalence of Representations** *Two representations*

$$\pi : A \longrightarrow L(H), \qquad \tilde{\pi} : A \longrightarrow L(\tilde{H})$$

are called equivalent, if there exists a unitary operator $U : H \to \tilde{H}$ such that for all $a \in A$ the following commutes

$$\begin{tikzcd} H \arrow[r, "U"] \arrow[d, "\pi(a)"'] & \tilde{H} \arrow[d, "\tilde{\pi}(a)"] \\ H \arrow[r, "U"'] & \tilde{H} \end{tikzcd}$$

Remark 2.2.4 *Faithful Representations are isometric $*$-Isomorphisms onto their image:*

$$\pi : A \xrightarrow{\cong} \pi(A) \subseteq L(H_\pi).$$

Thus it is our goal to construct faithful representations of all C^*-algebras.

Remark 2.2.5 *Let $\pi = \oplus_{i\in I}\pi_i$ be a $*$-representation of A on $H = \oplus_{i\in I}H_i$, then*

$$\pi(a) = 0 \quad \Leftrightarrow \quad \sum_{i\in I} \pi_i(a)\xi_i \ \forall \xi = \sum_i \xi_i \in H$$

and

$$\pi(a) = 0 \quad \Leftrightarrow \quad \pi_i(a) = 0 \ \forall i \in I \quad \Leftrightarrow \quad \ker\pi = \cap_{i\in I} \ker\pi_i.$$

So in order to get π to be faithful ($\ker\pi = \{0\}$), *we just need to find sufficiently many π_i, so that for each $a \in A$, there is a π_i such that $\pi_i(a) \neq 0$, since then $\cap_{i\in I}\ker\pi_i = \{0\}$ and $\pi = \oplus_{i\in I}\pi_i$ is a faithful representation.*

Lemma 2.2.6 **Nondegenerate Representations and approximate Unities***: Let $\pi : A \to L(H)$ be a nondegenerate representation and $(u_n)_n$ an approximate unity with $||u_n|| \leq 1$, then*

$$\pi(u_n)\xi \longrightarrow \xi \ \forall \xi \in H.$$

Proof: Let $\eta := \sum_{k=1}^{l} \pi(a_k)\eta_k$, then

$$\pi(u_n)\eta = \sum_{k=1}^{l} \pi(u_n a_k)\eta_k \longrightarrow \sum_{k=1}^{l} \pi(a_k)\eta_k = \eta$$

since $u_n a_k \to a_k$ and π is continuous. Let now $\xi \in H, \varepsilon > 0$, so by assumption, there is a $\eta \in \pi(A)H$ with $||\xi - \eta|| < \frac{\varepsilon}{3}$ and thus

$$||\pi(u_n)\xi - \xi|| \leq \underbrace{||\pi(u_n)(\xi - \eta)||}_{<\varepsilon/3} + \underbrace{||\pi(u_n)\eta - \eta||}_{\to 0} + \underbrace{||\eta - \xi||}_{<\varepsilon/3}.$$

□

Definition 2.2.7 *A $*$-representation $\pi : A \to L(H)$ and a vector $\xi \in H$ define a positive linear functional*

$$\varphi_{\pi,\xi} \in P(A), \quad \varphi_{\pi,\xi}(a) := \langle \pi(a)\xi, \xi \rangle.$$

Lemma 2.2.8 *The above positive linear functional fulfills $||\varphi_{\pi,\xi}|| \leq ||\xi||^2$.*

Proof: We have

$$\varphi_{\pi,\xi}(a^*a) = \langle \pi(a^*a)\xi, \xi\rangle = \langle \pi(a^*)\pi(a)\xi, \xi\rangle = \langle \pi(a)\xi, \pi(a)\xi\rangle \geq 0$$

$$|\varphi_{\pi,\xi}(a)| = |\langle \pi(a)\xi, \xi\rangle| \overset{Cauchy-Schwarz}{\leq} ||\pi(a)\xi||\,||\xi|| \leq ||\pi(a)||\,||\xi||^2 \leq ||a||\,||\xi||^2.$$

Where in the last step we have used that every $*$-homomorphism between C^*-algebras is norm decreasing.

□

Lemma 2.2.9 **Nondegenerate Representations give Vector States** *If $\pi : A \to L(H)$ is nondegenerate, we have $||\varphi_{\pi,\xi}|| = ||\xi||^2$. It thus follows, that*

$$\boxed{||\xi|| = 1 \quad \Leftrightarrow \quad \varphi_{\pi,\xi}(\cdot) := \langle \pi(\cdot)\xi, \xi\rangle \in \mathcal{S}(A)}$$

Proof: By the lemma above, for a nondegenerate representation, we have an approximate unity $(u_n)_n$ with $||u_n|| \leq 1$ and $\pi(u_n)\xi \to \xi \ \forall \xi \in H$, and thus

$$||\varphi_{\pi,\xi}|| = \lim_{n\to\infty} \varphi_{\pi,\xi}(u_n) = \lim_{n\to\infty} \langle \pi(u_n)\xi, \xi\rangle = \langle \xi, \xi\rangle = ||\xi||^2.$$

□

Theorem 2.2.10 **GNS – Construction** *Let A be a C^*-algebra and $\varphi \in \mathcal{S}(A)$, then the following hold:*

- **Existence**: *there exists a cyclic representation $\pi_\varphi : A \to L(H_\varphi)$ and a cyclic vector ξ_φ for π_φ with $||\xi_\varphi|| = 1$ such that*

$$\varphi = \varphi_{\pi_\varphi,\xi_\varphi}, \quad \text{i.e.}: \quad \varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle .$$

- **Uniqueness**: *If $\rho : A \to L(H)$ is another representation, $\eta \in H$ with $\overline{\rho(A)\eta} = H$ and*

$$\varphi(a) = \langle \rho(a)\eta, \eta\rangle ,$$

  then it is equivalent to π_φ, i.e.: $\rho \cong \pi_\varphi$. That is, there is a unitary operator $V : H_\varphi \to H$ with $V\xi_\varphi = \eta$ and

$$\rho(a)V = V\pi_\varphi(a).$$

That is all states are vector states of a unique (up to equivalence) cyclic representation.

Proof:

- Let A be unital and we define

$$H_0 := A/N, \quad N := \{a \in A \,|\, \varphi(a^*a) = 0\},$$

$$\langle \cdot, \cdot\rangle : H_0 \times H_0 \longrightarrow \mathbb{C}, \quad \langle a + N, b + N\rangle := \varphi(b^*a).$$

  Then $\langle \cdot, \cdot\rangle$ is a scalar product on H_0. We define

$$\boxed{H_\varphi := \overline{H_0}^{\langle \cdot,\cdot\rangle}}$$

- For $a \in A$ we define

$$\pi_\varphi(a) : H_0 \longrightarrow H_0, \quad \boxed{\pi_\varphi(a)(b + N) := ab + N}$$

  Because of $0 \leq a^*a \leq ||a||^2\mathbb{1}$ it follows that $b^*a^*ab \leq b^*||a||^2 b = ||a||^2 b^*b$ and thus

$$||\pi_\varphi(a)(b + N)||^2 = \langle ab + N, ab + N\rangle = \varphi(b^*a^*ab) \leq ||a||^2\varphi(b^*b) = ||a||^2||b + N||^2.$$

  So π_φ is a continuous operator on H_0 and thus has a unique continuation to $H_\varphi := \overline{H_0}$. One can easily show that π_φ is a $*$-homomorphism, e.g. $\pi_\varphi(a^*) = \pi_\varphi(a)^*$ since:

$$\langle \pi_\varphi(a^*)(b + N), c + N\rangle = \langle a^*b + N, c + N\rangle = \varphi(c^*a^*b) = \langle b + N, ac + N\rangle = \langle b + N, \pi_\varphi(a)(c + N)\rangle .$$

- Let now
$$\boxed{\xi_\varphi := \mathbb{1} + N \in H_0 \subseteq H_\varphi \quad \Rightarrow \quad \varphi = \varphi_{\pi_\varphi, \xi_\varphi}}$$
since
$$\varphi(a) = \varphi(\mathbb{1}^* a) = \langle a + N, \mathbb{1} + N \rangle = \langle \pi_\varphi(a)(\mathbb{1} + N), \mathbb{1} + N \rangle = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$$
and furthermore $\overline{\pi_\varphi(A)\xi_\varphi} = \overline{H_0} = H_\varphi$ so ξ_φ is a cyclic vector: $1 = ||\varphi|| = \varphi(1) = ||\xi_\varphi||^2$.

- Let now A be nonunital. We consider the continuation $\tilde{\varphi} \in \mathcal{S}(A^1)$, do the same construction as above and take
$$H_\varphi := H_{\tilde{\varphi}}, \quad \pi_\varphi := \pi_{\tilde{\varphi}}|_A.$$
The only property that needs to be varified is that ξ_φ is again a cyclic vector in this context. For this it suffices to show, that $\overline{\pi_\varphi(A)\xi_\varphi} = \overline{\pi_\varphi(A^1)\xi_\varphi}$ holds. So we need to show
$$\xi_\varphi = \pi_{\tilde{\varphi}}(1)\xi_\varphi \in \overline{\pi_\varphi(A)\xi_\varphi}$$
that is since $\pi_{\tilde{\varphi}}(A^1)\xi_\varphi = \pi_\varphi(A)\xi_\varphi + \mathbb{C}\xi_\varphi$. Inserting an approximate unity $(u_\lambda)_\lambda$ with $||u_\lambda|| \leq 1$, we get
$$\begin{aligned} ||\pi_\varphi(u_\lambda)\xi_\varphi = \xi_\varphi||^2 &= ||\pi_\varphi(u_\lambda)\xi_\varphi||^2 - \langle \pi_\varphi(u_\lambda)\xi_\varphi, \xi_\varphi \rangle - \langle \xi_\varphi, \pi_\varphi(u_\lambda)\xi_\varphi \rangle + ||\xi_\varphi||^2 \\ &= \langle u_\lambda + N, u_\lambda + N \rangle - \langle u_\lambda + N, \mathbb{1} + N \rangle - \langle \mathbb{1} + N, u_\lambda + N \rangle + \langle \mathbb{1} + N, \mathbb{1} + N \rangle \\ &= \underbrace{\varphi(u_\lambda^2)}_{\to ||\varphi||} - 2\underbrace{\varphi(u_\lambda)}_{\to ||\varphi||} + \underbrace{\tilde{\varphi}(1)}_{\to ||\varphi||} \to 0. \end{aligned}$$
That is since if $(u_\lambda)_\lambda$ is an approximate unity, then also u_λ^2 is an approximate unity of A. We thus get $\pi_\varphi(u_\lambda)\xi_\varphi \to \xi_\varphi$ and finally $\xi_\varphi \in \overline{\pi_\varphi(u_\lambda)\xi_\varphi}$.

- Uniqueness: Let now $\rho : A \to L(H')$ be another such representation, $\eta \in H'$ with $\varphi = \varphi_{\rho,\eta}$, then define
$$V : H_0 \to H', \quad V(a + N) := \rho(a)\eta.$$
V is linear and isometric, since:
$$\langle V(a + N), V(b + N) \rangle = \langle \rho(a)\eta, \rho(b)\eta \rangle = \langle \rho(b^* a)\eta, \eta \rangle = \varphi(b^* a) = \langle (a + N), (b + N) \rangle$$
and because of $\overline{V(H_0)} := \overline{\rho(A)\eta} = H'$ we know that V has a unitary continuation with
$$V\pi_\varphi(a)(b + N) = V(ab + N) = \rho(ab)\eta = \rho(a)\rho(b)\eta = \rho(a)V(b + N).$$
We conclude $V\pi_\varphi(a) = \rho(a)V$.

□

Theorem 2.2.11 <u>**Gelfand – Naimark**</u> *Let A be a C^*-algebra. Then there is a faithful, nondegenerate $*$-representation $\pi : A \to L(H)$ for some $\mathbb{C}$-Hilbert space H. If A is separable, then H can be chosen to be separable (i.e. $H \cong \mathbb{C}^n, l^2(H)$).*

Proof:
$$\pi := \bigoplus_{\varphi \in \mathcal{S}(A)} \pi_\varphi, \quad H := \bigoplus_{\varphi \in \mathcal{S}(A)} H_\varphi$$

- π is nondegenerate: $\overline{\pi(A)H} = H$, since $\overline{\pi_\varphi(A)H_\varphi} = H_\varphi$.

- π is faithful: Let $0 \neq a \in A$, then there exists a $\varphi \in \mathcal{S}(A)$ with
$$||a||^2 = \varphi(a^* a) = \langle a + N, a + N \rangle = \langle \pi_\varphi(a)\xi_\varphi, \pi_\varphi(a)\xi_\varphi \rangle = ||\pi_\varphi(a)\xi_\varphi||^2.$$
Thus we have $||\pi_\varphi(a)\xi_\varphi|| \geq ||a||||\xi_\varphi||$ which for the operator norm means $||\pi_\varphi(a)|| \geq ||a||$ and thus $||\pi_\varphi(a)|| = ||a||$ since π_φ is a $*$-homomorphism. In particular we have $\pi_\xi(a) \neq 0$.

- Let now A be separable. Choose a dense sequence $(a_n)_n$ in A and $\varphi_n \in \mathcal{S}(A)$ with $||\varphi_n(a_n^* a_n)|| = ||a_n||^2$. Then also $\tilde{\pi} = \oplus_{n\in\mathbb{N}}\pi_{\varphi_n}$ is a faithful representation on the separable Hilbert space $\oplus_{n\in\mathbb{N}} H_{\varphi_n}$. This is shown in the following two steps:
  - $0 \neq a \in A$ be arbitrary, choose a_n with $||a_n - a|| < \frac{||a||}{2}$. It follows that $||\pi_{\varphi_n}(a_n) - \pi_{\varphi_n}(a)|| < \frac{||a||}{2}$ and $||\pi_{\varphi_n}(a_n)|| = ||a_n|| > \frac{||a||}{2}$, so $||\pi_{\varphi_n}(a)|| \geq ||\pi_{\varphi_n}(a_n)|| - ||\pi_{\varphi_n}(a) - \pi_{\varphi_n}(a_n)|| > 0$. And thus also $\pi(a) \neq 0$.
  - Every H_{φ_n} is separable, since $\{a_n + N \,|\, n \in N\}$ is dense in H_{φ_n} ($||a_n + N - a + N||^2 = \varphi((a_n - a)^*(a_n - a)) \leq ||a_n - a||^2$, since $||\varphi|| = 1$) and thus every H_{φ_n} has a countable ONB. The countable union of all these ONB is a countable ONB of H_0.

□

2.3 Pure States, Irreps and Schurs Lemma

In this section, we shall see that pure states are exactly the states belonging to irreducible representations.
In quantum Mechanics, a pure state is a state, that is not a linear combination of other states. This is reflected in the following definition:

Definition 2.3.1 **Pure States** *A state $\varphi \in \mathcal{S}(A)$ is called pure iff for all $\psi_1, \psi_2 \in P(A)$ with $||\psi_i|| \leq 1$ and $\lambda \in [0,1]$ it follows, that*

$$\text{If } \varphi = \lambda\psi_1 + (1-\lambda)\psi_2, \quad \Rightarrow \quad \psi_1 = \varphi = \psi_2.$$

$$\mathrm{Pure}(A) := \{\varphi \in \mathcal{S}(A) \,|\, \varphi \text{ is pure}\}$$

Theorem 2.3.2 **Krein – Milman** *Let (E, τ) be a locally convex $\mathbb{K}$-VS and $\varnothing \neq K \subseteq E$ compact, then*

$$\mathrm{Ext}(K) \neq \varnothing, \quad K \subseteq \overline{\mathrm{conv}(\mathrm{Ext}(K))}.$$

If $K \subseteq E$ is compact and convex, then

$$K = \overline{\mathrm{conv}(\mathrm{Ext}(K))}.$$

Remark 2.3.3 *Together with the next theorem, Krein-Milman not only proves the existence of pure states, but also proves that there are "sufficiently many."*

Theorem 2.3.4 *Let A be a C^*-algebra and $K := \{\psi \in P(A) \,|\, ||\psi|| \leq 1\}$. Then K is a compact and convex subset of A' in the weak *-topology, and*

$$\mathrm{Ext}(K) = \mathrm{Pure}(A) \cup \{0\}, \quad K = \overline{\mathrm{conv}(\mathrm{Pure}(A) \cup \{0\})}.$$

Proof: Let $(\varphi_n)_n$ be a net in K with $\varphi_n \to \varphi \in A'$. It follows that $\varphi(a^*a) = \lim_{n\to\infty} \varphi_n(a^*a) \geq 0$ so $\varphi \in P(A)$ with $||\varphi|| \leq 1$. Thus K is closed and thus, with Banach-Alouglu, compact.
If $\varphi \in \mathrm{Ext}(K)$, it follows that $||\varphi|| = 0$ or $||\varphi|| = 1$. If $||\varphi|| = 1$ then $\varphi \in \mathrm{Pure}(A)$ by the definition of $\mathrm{Pure}(A)$. Thus we have $\mathrm{Ext}(K) = \mathrm{Pure}(A) \cup \{0\}$, and $K = \overline{\mathrm{conv}(\mathrm{Pure}(A) \cup \{0\})}$. □

Theorem 2.3.5 *Let A be unital, then $\mathcal{S}(A)$ is compact and convex with*

$$\boxed{\mathrm{Ext}(\mathcal{S}(A)) = \mathrm{Pure}(A), \quad \mathcal{S}(A) = \overline{\mathrm{conv}(\mathrm{Pure}(A))}}$$

Proof: Analogous to the previous theorem. □

Corollary 2.3.6 *Let A be a C^*-algebra, it then holds that*

$$\boxed{||a||^2 = \sup_{\varphi \in \mathrm{Pure}(A)} \varphi(a^*a)}$$

Proof: If $a \in A$, then with 2.1.13 there is a $\psi \in \mathcal{S}(A)$ with $||a||^2 = \psi(a^*a)$. With 2.3.4 for every $\varepsilon > 0$ there are $\varphi_1, \ldots, \varphi_l \in \mathrm{Pure}(A)$, $\lambda_1, \ldots, \lambda_l \geq 0$ with $\sum_{i=1}^{l} \lambda_i \leq 1$ and

$$\sum_{i=1}^{l} \lambda_i \varphi_i(a^*a) \geq \psi(a^*a) - \varepsilon = ||a||^2 - \varepsilon.$$

But then there is at least one $i \in \{1, \ldots, l\}$ with $\varphi_i(a^*a) \geq ||a||^2 - \varepsilon$. And since $\varepsilon > 0$ the claim follows. □

Lemma 2.3.7 **<u>Characterization of Pure States</u>** *Let $\varphi \in \mathcal{S}(A)$, then $\varphi \in \mathrm{Pure}(A)$ iff for all $\psi \in P(A)$ with $0 \leq \psi \leq \varphi$ there exists $\lambda \in [0,1]$ with $\psi = \lambda\varphi$.*

Proof:

" $\Rightarrow$ " Let $\varphi \in \mathrm{Pure}(A)$ and $0 \leq \psi \leq \varphi$. Then $\varphi = \psi + (\varphi - \psi)$ with $\varphi - \psi \geq 0$ and with 2.1.8 it follows that $1 = ||\varphi|| = ||\psi|| + ||\varphi - \psi||$. If $\psi \neq 0$ and $\psi \neq \varphi$, then set $\lambda := \frac{1}{1-\lambda}(\varphi - \psi)$. It then follows that $||\psi_1|| = ||\psi_2|| = l$ and $\varphi = \lambda\psi_1 + (1-\lambda)\psi_2$, so $\varphi = \psi_1 = \psi_2$, since $\varphi \in \mathrm{Pure}(A)$. We have, that $\varphi = \psi_1 = \frac{1}{\lambda}\psi$, i.e. $\psi = \lambda\varphi$.

" $\Leftarrow$ " It now holds that $0 \leq \psi \leq \varphi \Rightarrow \psi = \lambda\varphi$ for a $\lambda \in [0,1]$. Let $\psi_1, \psi_2 \in \mathrm{Pure}(A)$ with $||\psi_1||, ||\psi_2|| \leq 1$ and $t \in (0,1)$ with $\varphi = t\psi_1 + (1-t)\psi_2$. Due to $1 = ||\varphi|| = t||\psi_1|| + (1-t)||\psi_2||$, it already follows that $1 = ||\psi_1|| = ||\psi_2||$ and further

$$0 \leq t\psi_1 \leq \varphi, \quad 0 \leq (1-t)\psi_2 \leq \varphi.$$

Thus there exist $\lambda, \mu \in [0,1]$ with

$$t\psi_1 = \lambda\varphi, \quad (1-t)\psi_2 = \mu\varphi.$$

Because of $t = ||t\psi_1|| = ||\lambda\varphi|| = \lambda$ (analogously for ψ_2) it holds that $t = \lambda$, $(1-t) = \mu$, so $\psi_1 = \varphi = \psi_2$.

□

Lemma 2.3.8 *Let A be a C^*-algebra, $\pi : A \to L(H)$ a $*$-representation, $\xi \in H$ with $||\xi|| = 1$ and $\varphi \in \mathcal{S}(A)$ defined by*

$$\varphi(a) = \langle \pi(a), \xi, \xi \rangle \quad \Leftrightarrow \quad \varphi = \varphi_{\pi,\xi}.$$

Then the following hold:

(1) Let $T = T^* \in L(H)$ with $0 \leq T \leq \mathbb{1}$ and $[T, \pi(a)] = 0$ for all $a \in A$, then

$$0 \leq \psi_{T,\xi}(\cdot) := \langle \pi(\cdot)T\xi, T\xi \rangle \leq \varphi.$$

(2) If ξ is a cyclic vector, then $T \mapsto \psi_{T,\xi}(\cdot) := \langle \pi(\cdot)T\xi, T\xi \rangle$ is injective.

(3) For all $0 \leq \psi \leq \varphi$ we have $\psi = \psi_{T,\xi}$ for some $0 \leq T \leq \mathbb{1}$ with $[T, \pi(a)] = 0$.

Proof:

(1) Let $T \in L(H)$ with $||T|| \leq 1$. It then holds that

$$\begin{aligned} \psi_T(a^*a) &= \langle \pi(a^*a)T\xi, T\xi \rangle = \langle \pi(a)T\xi, \pi(a)T\xi \rangle = ||\pi(a)T\xi||^2 \\ &= ||T\pi(a)\xi||^2 \leq ||\pi(a)\xi||^2 = \langle \pi(a^*a)\xi, \xi \rangle = \varphi(a^*a). \end{aligned}$$

(2) Let $0 \leq T, T' \leq 1$ with $\psi_T = \psi_{T'}$, we will show that $T^2 = (T')^2$, since it then follows that $T = \sqrt{T^2} = \sqrt{(T')^2} = T'$.
For all $a \in A$ it holds that

$$\left\langle \pi(a)\xi, T^2\xi \right\rangle \overset{T=T^*}{=} \langle T\pi(a)\xi, T\xi\rangle = \langle \pi(a)T\xi, T\xi\rangle = \psi_T(a) = \psi_{T'}(a) = \left\langle \pi(a)\xi, (T')^2\xi \right\rangle.$$

Since $\overline{\pi(A)\xi} = H$, it follows that $T^2\xi = (T')^2\xi$ and then

$$T^2(\pi(a)\xi) = \pi(a)T^2\xi = \pi(a)(T')^2\xi = (T')^2(\pi(a)\xi).$$

I.e. $T^2 = (T')^2$ on $\pi(A)\xi \subseteq H$, and since $\overline{\pi(A)\xi} = H$, the claim follows.

(3) Let $0 \leq \psi \leq \varphi$ and w.l.o.g. $\overline{\pi(A)\xi} = H$ (since if not, we can just work on $\overline{\pi(A)H}$). Set $H_0 := \pi(A)\xi \subseteq H$, then

$$\langle \cdot, \cdot \rangle_\psi : H_0 \times H_0 \longrightarrow \mathbb{C}, \quad \langle \pi(a)\xi, \pi(b)\xi \rangle_\psi = \psi(b^*a)$$

is a semidefinite hermitian form on H_0, with

$$|\langle \pi(a)\xi, \pi(b)\xi \rangle_\psi|^2 = |\psi(b^*a)|^2 \overset{CS}{\leq} \psi(b^*b)\psi(a^*a) \leq \varphi(b^*b)\varphi(a^*a) = ||\pi(b)\xi||^2||\pi(a)\xi||^2.$$

So $\langle \cdot, \cdot \rangle_\psi$ is continuous and has a continuous continuation to H with $|\langle \eta, \tilde{\eta} \rangle_\psi| \leq ||\eta||\,||\tilde{\eta}||$ $\forall \eta, \tilde{\eta} \in H$. By a theorem from functional analysis, there is now a $\tilde{T}^* = \tilde{T} \geq 0$ with $||\tilde{T}|| \leq 1$ with

$$\langle \eta, \tilde{\eta} \rangle_\psi = \left\langle \tilde{T}\eta, \tilde{\eta} \right\rangle, \quad \forall \eta, \tilde{\eta} \in H.$$

In particular

$$\psi(b^*a) = \langle \pi(a)\xi, \pi(b)\xi \rangle_\psi = \left\langle \tilde{T}\pi(a)\xi, \pi(b)\xi \right\rangle, \quad \forall a, b \in A.$$

For arbitrary $a, b, z \in A$ we then have

$$\left\langle \pi(a)\xi, \tilde{T}\pi(z)\pi(b)\xi \right\rangle = \psi((b^*z^*)a) = \psi(b^*(z^*a)) = \left\langle \pi(z^*a)\xi, \tilde{T}\pi(b)\xi \right\rangle = \left\langle \pi(a)\xi, \pi(z)\tilde{T}\pi(b)\xi \right\rangle$$

and since $\overline{\pi(A)\xi} = H$, it follows that $\pi(z)\tilde{T} = \tilde{T}\pi(z)$ $\forall z \in A$. For $A = \tilde{(T)}^{1/2}$, it also holds that $\pi(z)T = T\pi(z)$ $\forall z \in A$, and

$$\psi(b^*a) = \left\langle \pi(a)\xi, \tilde{T}\pi(b)\xi \right\rangle = \langle \pi(a)\xi, T^2\pi(b)\xi \rangle = \langle T\pi(a)\xi, T\pi(b)\xi \rangle = \psi_T(b^*a).$$

Since $A = A^2 = A^*A$, we have $\psi = \psi_T$.

□

Remark 2.3.9 *If $\pi : A \to L(H)$ a $*$-representation, and $E \subseteq H$ an invariant subspace, then*

$$\pi|_E, \pi|_{E^\perp} \quad \text{are } * \text{-representations and :} \quad \pi = \pi|_E \oplus \pi|_{E^\perp}.$$

That is since for $\xi \in E, \eta \in E^\perp$:

$$\langle \pi(a)\xi, \eta \rangle = \langle \xi, \pi(a^*)\eta \rangle = 0, \quad \text{since } \pi(a^*)\eta \in E.$$

Lemma 2.3.10 <u>**Schur**</u> *Let $\pi : A \to L(H)$ be a $*$-representation, then the following are equivalent:*

(1) π is irreducible.

(2) All $0 \neq \xi \in H$ are cyclic vectors: $\overline{\pi(A)\xi} = H$.

(3) If $[T, \pi(a)] = 0$ for all $a \in A$, then $T = \lambda \mathbb{1}$ for a $\lambda \in \mathbb{C}$.

Proof:

"(1) $\Rightarrow$ (2)" First let π be nondegenerate, then $E : \overline{\pi(A)H} \subseteq H$, $\{0\} \neq E$ and $\pi(A)E \subseteq \pi(A)H \subseteq E$, thus $E = H$. If $0 \neq \xi \in H$, then we have for $E_\xi := \overline{\pi(A)\xi}$, that $0 \neq E_\xi$, since $\xi = \lim \pi(u_\lambda)\xi \in E_\xi$. It follows, that

$$\pi(A)E_\xi \subseteq \overline{\pi(A)\pi(A)\xi} = \overline{\pi(A^2)\xi} = \overline{\pi(A)\xi} = E_\xi$$

and thus $E_\xi = H$, i.e. $H = E$.

"(2) $\Rightarrow$ (1)" If $0 \neq E$ is a closed linear subspace with $\pi(A)E \subseteq E$, then $H = \overline{\pi(A)\xi} \subseteq \overline{\pi(A)E} \subseteq \overline{E} = E$ for all $0 \neq \xi \in E$ and so $H = E$.

"(1) $\Rightarrow$ (3)" Let $T \in L(H)$ with $T\pi(a) = \pi(a)T$ $\forall a \in A$, then $\pi(a)T^* = (T\pi(a^*))^* = (\pi(a^*)T)^* = T^*\pi(a)$ $\forall a \in A$. Switching to $\mathrm{Re}(T), \mathrm{Im}(T)$, we can w.l.o.g. assume, that $T = T^*$. We now show that $\sigma(T) = \{\lambda\}$ for a $\lambda \in \mathbb{R}$. It then follows that $T = \lambda\mathbb{1}$, since on $\sigma(T)$ it holds that $\lambda\mathbb{1} = \{\mathrm{id}\}$, so $T = \mathrm{id}(T) = \lambda\mathbb{1}(T) = \lambda\mathbb{1}$.
Since $T\pi(a) = \pi(a)T$ $\forall a \in A$, we also have $f(T)\pi(a) = \pi(a)f(T)$ $\forall a \in A$, $f \in C(\sigma(T))$. We assume that $\exists \mu, \lambda \in \sigma(T)$ with $\mu \neq \lambda$. Then there are $f, g \in C(\sigma(T))$ with $f(\lambda) = g(\mu) = 1$ $f \cdot g = 0$ and it follows that

$$f(T) \neq 0 \neq g(T), \quad f(T)g(T) = (f \cdot g)(T) = 0.$$

Set $E := \overline{f(T)H} \neq \{0\}$, then

$$\pi(A)E \subseteq \overline{\pi(A)f(T)H} = \overline{f(T)\pi(A)H} \subseteq \overline{f(T)H} = E.$$

So $E = H$ since $E \neq 0$ and π irreducible. But we have $g(T)E \subseteq \overline{g(T)f(T)H} = \overline{0H} = \{0\}$ and thus $E \neq H$ (since $g(T) \neq 0$), which is a contradiction.

"(3) $\Rightarrow$ (1)" Let $0 \neq E \subseteq H$ be a closed linear subspace with $\pi(A)E \subseteq E$. Let

$$P : H \longrightarrow E, \quad P\xi := \xi_1, \quad \text{for} \quad \xi = \xi_1 + \xi_2, \quad \xi_1 \in E, \xi_2 \in E^\perp$$

be the orthogonal projection onto E. Then for all $a \in A, \xi_1 \in E, \xi_2 \in E^\perp$ we have

$$P\pi(a)(\xi_1 + \xi_2) = P(\underbrace{\pi(a)\xi_1}_{\in E} + \underbrace{\pi(a)\xi_2}_{\in E^\perp}) = \pi(a)\xi_1 = \pi(a)(P(\xi_1 + \xi_2)).$$

so $P\pi(a) = \pi(a)P$ for all $a \in A$. So it follows, that $P = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$. Due to $P^* = P^2 = P$ we have $\overline{\lambda} = \lambda^2 = \lambda$, i.e. $\lambda \in \{0, 1\}$. Since $P \neq 0$ (since $E \neq 0$), we have $\lambda = 1$, and thus $P = \mathbb{1}$ and $E = \mathbb{1}H = H$.

□

Theorem 2.3.11 Pure States are Irreducible Representations *Let A be a C^*-algebra, $\pi : A \to L(H)$ a $*$-representation, $\varphi \in \mathcal{S}(A)$ and $\xi \in H$ a cyclic vector such that*

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle .$$

Then the following hold

(1) π is irreducible $\Leftrightarrow$ $\varphi \in \mathrm{Pure}(A)$.

(2) If π is irreducible and $0 \neq \xi, \eta \in H$ with $\langle \pi(a)\xi, \xi \rangle = \langle \pi(a)\eta, \eta \rangle$, then $\eta = \lambda\xi$, with $\lambda \in S^1$.

(3) If π, ρ are irreducible and $\xi \in H_\pi$, $\eta \in H_\rho$ and $\langle \pi(a)\xi, \xi \rangle = \langle \pi(a)\eta, \eta \rangle$, then $\pi \cong \rho$.

Proof:

(3) Since π, ρ are irreducible, they are in particular nondegenerate, so $||\xi||^2 = ||\varphi_{\pi,\xi}|| = ||\varphi_{\rho,\eta}|| = ||\eta||^2$. W.l.o.g. we shall assume that $||\xi|| = ||\eta|| = 1$ and thus $\varphi_{\pi,\xi} = \varphi_{\rho,\eta}$ is a state and (3) follows form the GNS-construction.

(1)" ⇒ " Let $0 \neq E \subseteq H$ be a $\pi(A)$ invariant subspace and $P : H \to E$ the orthogonal projection onto E. Then $P\pi(a) = \pi(a)P$ for all $a \in A$ and $0 \leq P \leq 1$. With 2.3.8 we have $0 \leq \psi_P \leq \varphi$. Since φ is pure, with 2.3.7 we have a $\lambda \in [0, 1]$ such that $\psi_P = \lambda\varphi$ and it follows that

$$\psi_P(a) = \langle \pi(a)P\xi, P\xi \rangle = \lambda \langle \pi(a)\xi, \xi \rangle = \left\langle \pi(a)\sqrt{\lambda}\xi, \sqrt{\lambda}\xi \right\rangle .$$

Since ξ is a cyclic vector for π, we have with 2.3.8, that $P = \sqrt{\lambda}\mathbb{1}$ and since $P = P^2$, we have $\sqrt{\lambda} = \lambda \in [0, 1]$. Since $E \neq 0$ we have $\lambda = 1$ and $E = H$.

" ⇐ " With 2.3.7 it is enough to show $0 \leq \psi \leq \varphi$, so that $\psi = \lambda\varphi$ for a $\lambda \in [0, 1]$. With 2.3.8 there is a $0 \leq T \leq 1$ with $T\pi(a) = \pi(a)T\ \forall a \in A$ and $\psi = \psi_T$. Since π is irreducible, it holds that $T = \lambda\mathbb{1}$ for a $\lambda \in \mathbb{C}$ and since $0 \leq T \leq 1$, we have $\lambda \in [0, 1]$, and thus for all $a \in A$:

$$\psi(a) = \psi_T(a) = \langle \pi(a)\lambda\xi, \lambda\xi \rangle = |\lambda|^2 \langle \pi(a)\xi, \xi \rangle = \lambda^2\varphi(a).$$

(2) Using (3), we have a unitary $U : H \to H$ with $U\xi = \eta$ and $\pi(a)U = U\pi(a)$ and thus with Schur $U = \lambda\mathbb{1}$. Unitarity $U^* = U^{-1}$ gives $\bar{\lambda} = \lambda^{-1}$, so $\lambda \in S^1$ and in particular $\eta = \lambda\xi$.

□

Remark 2.3.12 **Summary** *For a C^*-algebra, define:*

$$\begin{aligned} \mathrm{Cycl}(A) &:= \{(\pi, \xi) \,|\, \pi : A \to L(H) \text{ cyclic rep, } \xi \text{ cyclic vec, } ||\xi|| = 1\}, \\ \mathrm{Irrep}(A) &:= \{(\pi, \xi) \,|\, \pi : A \to L(H) \text{ irreducible rep, } \xi \in H,\ ||\xi|| = 1\}. \end{aligned}$$

And we have the following mappings

$$\begin{aligned} \mathrm{Cycl}(A) &\longrightarrow \mathcal{S}(A), \qquad & \mathcal{S}(A) &\xrightarrow{GNS} \mathrm{Cycl}(A) \\ (\pi, \xi) &\longmapsto \varphi_{\pi,\xi} & \varphi &\longmapsto (\pi_\varphi, \xi_\varphi) \end{aligned}$$

where the second is given by the GNS-construction. We now define an equivalence relation on $\mathrm{Cycl}(A)$ *by*

$$(\pi, \xi) \sim (\rho, \eta), \quad :\Leftrightarrow \quad \exists V \in U(H_\pi, H_\rho) :\ V\xi = \eta \text{ and } V\pi(a) = \rho(a)V.$$

So we get the following two bijections

$$\boxed{\mathrm{Cycl}(A)/\sim \quad \longleftrightarrow \quad \mathcal{S}(A)}$$

which follows from the GNS-construction. And by the map $[(\pi, \xi)] \mapsto \varphi_{\pi,\xi}$ we have

$$\boxed{\mathrm{Irrep}(A)/\sim \quad \longleftrightarrow \quad \mathrm{Pure}(A)}$$

with the inverse map $\varphi \mapsto [(\pi_\varphi, \xi_\varphi)]$.

Definition 2.3.13 **Structure Space** *We define the structure space to be*

$$\boxed{\widehat{A} := \{[\pi] \,|\, \pi : A \to L(H) \text{ irrep}\}}$$

Remark 2.3.14 $\widehat{A}$ *is also called the spectrum of A and denoted* $\mathrm{Spec}(A)$.

Theorem 2.3.15 *Let A be a C^*-algebra, it then holds that*

$$\boxed{||a|| = \sup\{||\pi(a)|| \,|\, [\pi] \in \hat{A}\}}$$

Proof:

$$\begin{aligned} ||a||^2 &= \sup_{\varphi \in \mathrm{Pure}(A)} \varphi(a^*a) = \sup_{(\pi,\xi)\in \mathrm{Irrep}(A)} \langle \pi(a^*a)\xi, \xi\rangle = \sup_{(\pi,\xi)\in \mathrm{Irrep}(A)} \langle \pi(a)\xi, \pi(a)\xi\rangle \\ &= \sup_{(\pi,\xi)\in \mathrm{Irrep}(A)} ||\pi(a)\xi||^2 \overset{||\xi||=1}{\leq} \sup_{[\pi]\in\hat{A}} ||\pi(a)||^2 \leq ||a||^2 \end{aligned}$$

So we get $||a|| \leq \sup_{[\pi]\in\hat{A}} ||\pi(a)||$, but since π is a *-homomorphism, we always have $||\pi(a)|| \leq ||a||$. □

Corollary 2.3.16 *Let $\pi \in [\pi] \in \hat{A}$, then the following is a faithful, nondegenerate representation of A:*

$$\tau := \oplus_{[\pi]\in\hat{A}} \pi : A \longrightarrow L(\oplus_{[\pi]\in\hat{A}} H_\pi).$$

2.4 The Spaces $\hat{A}$ and Prim(A)

Definition 2.4.1 *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal, then define*

$$\begin{aligned} \widehat{A_I} &:= \{[\pi] \in \hat{A} \,|\, \pi(I) \neq 0\}, \\ \widehat{A_{A/I}} &:= \{[\pi] \in \hat{A} \,|\, \pi(I) = 0\}. \end{aligned}$$

So we have

$$\hat{A} = \widehat{A_I} \sqcup \widehat{A_{A/I}}.$$

Theorem 2.4.2 *We have the following bijections*

$$\begin{array}{ccc} \widehat{A_I} \overset{\cong}{\longrightarrow} \hat{I}, & \qquad & \widehat{A/I} \overset{\cong}{\longrightarrow} \widehat{A_{A/I}} \\ [\pi] \longmapsto [\pi|_I] & & [\pi] \longmapsto [\pi \circ q] \end{array}$$

with the quotient map q.

Proof: We only prove the injectivity of $\widehat{A_I} \to \hat{I}$. Its surjectivity is a consequence of the next lemma. The bijectivity of the second map is left as an exercise.
It holds that $\pi : A \to L(H)$ is irreducible with $\pi(I) \neq 0$ and thus $0 \neq E := \overline{\pi(I)H}$ is a $\pi(A)$ invariant closed subspace of H since $\pi(A)\overline{\pi(I)H} \subset \overline{\pi(A)\pi(I)H} = \overline{\pi(AI)H} \subseteq \overline{\pi(I)H}$. Now since π is irreducible, it follows that $\overline{\pi(I)H} = H$.
We now show that $\pi|_I : I \to L(H)$ is irreducible. Let $T \in L(H)$ with $T\pi(b) = \pi(b)T \ \forall b \in I$. We show that $T = \lambda \mathbb{1}_H$ (which with Schur gives the irreducibility of $\pi|_I$). For all $a \in A, b \in I$, we have

$$T\pi(a)(\pi(b)\xi) = T\pi(ab)\xi \overset{ab\in I}{=} \pi(ab)T\xi = \pi(a)(\pi(b)T\xi) = \pi(a)(T\pi(b)\xi) = \pi(a)T(\pi(b)\xi).$$

Since $\overline{\pi(I)H} = H$, we get $T\pi(a) = \pi(a)T \ \forall a \in A$ and because of the irreducibility of π, with Schur, we get that $T = \lambda\mathbb{1}_H$ for a $\lambda \in \mathbb{C}$.
Now show that if $\pi, \rho : A \to L(H_\pi), L(H_\rho)$ are irreducible with $\pi(I) \neq 0 \neq \rho(I)$, then

$$\pi \cong \rho \quad \Leftrightarrow \quad \pi|_I \cong \rho|_I.$$

Once we have this result, it is obvious, that the map $\widehat{A_I} \to \widehat{I}$, $[\pi] \mapsto [\pi|_I]$ is well defined and injective.

" $\Rightarrow$ " If $V : H_\pi \to H_\rho$ is an equivalence for π and ρ, then it is also an equivalence for $\pi|_I$ and $\rho|_I$.

" $\Leftarrow$ " Let $V : H_\pi \to H_\rho$ be unitary with $V\pi(b) = \rho(a)V \ \forall a \in A$. Let $a \in A$, then $\forall b \in I, \xi \in H_\pi$:

$$V\pi(a)(\pi(b)\xi) = V\pi(ab)\xi = \rho(ab)V\xi = \rho(a)(\rho(b)V)\xi = \rho(a)(V\rho(b))\xi = \rho(a)V(\rho(b)\xi)$$

and since $\overline{\pi(A)H} = H$, the claim follows.

□

Remark 2.4.3 *With the above bijections we have*

$$\boxed{\hat{A} \cong \widehat{I} \sqcup \widehat{A/I} \text{ as sets}}$$

Lemma 2.4.4 *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal, then for a nondegenerate $*$-representation $\pi : I \to L(H)$ there exists exactly one continuation*

$$\tilde{\pi} : A \to L(H), \quad \tilde{\pi}|_I = \pi.$$

Proof: If $\tilde{\pi}$ is such a continuation, then for $a \in A, b \in I$ we have

$$\tilde{\pi}(a)(\pi(b)\xi) = \tilde{\pi}(ab)\xi = \pi(ab)\xi.$$

Since $\overline{\pi(I)H} = H$ we get uniqueness. For a $\eta = \sum_{i=1}^m \pi(b_i)\xi_i \in \pi(A)H$ it holds that

$$\tilde{\pi}(a)\eta = \sum_{i=1}^m \pi(ab_i)\xi_i.$$

We now show that this defines a well defined operator $\tilde{\pi}(a) \in L(H)$, such that $\tilde{\pi} : A \to L(H)$ is a $*$-representation of A.

1. Well defined: Let η be as before and $(u_n)_n$ an approximate unity for I, then:

$$\tilde{\pi}(a)\eta = \sum_{i=1}^m \pi(ab_i)\xi_i = \lim_{n\to\infty} \sum_{i=1}^m \pi(au_nb_i)\xi_i = \lim_{n\to\infty} \pi(au_n) \sum_{i=1}^m \pi(b_i)\xi_i = \lim_{n\to\infty} \pi(au_n)\eta.$$

   Thus $\tilde{\pi}(a)\eta$ does not depend on the specific representation of η. I.e. $\tilde{\pi}(a)$ is well defined on $\pi(A)H$. For $\eta \in \pi(A)H$ we then have

$$\begin{aligned} ||\tilde{\pi}(a)\eta||^2 &= \lim_{n\to\infty} ||\pi(au_n)\eta||^2 \le \sup_n ||\pi(au_n)\eta||^2 \le \sup_n ||\pi(au_n)||^2||\eta||^2 \\ &\le \sup_n ||au_n||^2||\eta||^2 \le ||a||^2||\eta||^2. \end{aligned}$$

   Thus $\tilde{\pi}(a)$ is continuous and therefor has a unique continuation to $H = \overline{\pi(A)H}$.

2. $\tilde{\pi} :\to L(H)$ is a $*$-representation: linearity and multiplicativity are easy to show. We only prove $\tilde{\pi}(a^*) = \tilde{\pi}(a)^*$.
   Let $\xi, \eta \in \pi(A)H$, then

$$\langle \tilde{\pi}(a)\xi, \eta \rangle = \lim_{n\to\infty} \langle \pi(au_n)\xi, \pi(u_n)\eta \rangle = \lim_{n\to\infty} \langle \pi(u_n)\xi, \pi(a^*u_n)\eta \rangle = \langle \xi, \tilde{\pi}(a^*)\eta \rangle .$$

   So $\tilde{\pi}$ is indeed a $*$-representation.

3. Finally if $b \in I, \eta \in \pi(I)H$, then

$$\tilde{\pi}(b)\eta = \lim_{n\to\infty} \pi_{bu_n}\eta = \pi(b)\eta,$$

   thus $\tilde{\pi}|_I = \pi$.

□

Definition 2.4.5 **<u>Hull Ker</u>** *Let A be a C^*-algebra and $E \subseteq \hat{A}$ a subset, then define*

$$\begin{aligned} \ker E &:= \cap_{[\pi]\in E} \ker \pi, \\ \mathrm{hull}(J) &:= \{\rho \in \hat{A} \,|\, J \subseteq \ker \rho\}. \end{aligned}$$

Remark 2.4.6 *We have already shown, that for commutative A, we always have*

$$\overline{E} = \text{hull}(\ker E), \quad \forall E \subseteq \hat{A}.$$

Theorem 2.4.7 **Hull – Kernel Topology** *(Jacobsen Topology or Zariski Topology) for a (not neccessaily commutative) C^*-algebra A is the unique topology on $\hat{A}$ such that*

$$\boxed{\overline{E} = \text{hull}(\ker E), \quad \forall E \subseteq \hat{A}}$$

Proof: Let $\mathcal{A} := \{\text{hull}(I) \,|\, I \subseteq A \text{ closed ideal}\}$.

- We first show that $\mathcal{A}$ fulfills the axioms for closed subsets of a topological space.
  1. $\text{hull}(\varnothing) = \hat{A}$ and $\text{hull}(A) = \varnothing$, so $\varnothing, A \in \mathcal{A}$.
  2. Show: If $\{I_i \,|\, i \in \Lambda\}$ is a system of closed ideals in A, then there is a closed ideal $J \subseteq A$ with
  $$\cap_{i\in\Lambda}\text{hull}(I_i) = \text{hull}(J).$$
  This is fulfilled by $J := \overline{\text{LH}\{\cup_{i\in\Lambda} I_i\}}$.
  3. Show $\text{hull}(I_1) \cup \text{hull}(I_2) \in \mathcal{A}$. Consider the closed ideal $I := I_1 \cap I_2$ for which we have $I_1 \cap I_2 = I_1 \cdot I_2$ since $I = I^2 \subseteq I_1 \cdot I_2$.
- Uniqueness: It now follows that $\rho \in \text{hull}(I_1) \cup \text{hull}(I_2) \Leftrightarrow \rho(I_1) = 0$ or $\rho(I_2) = 0$, thus $\rho(I_1 \cdot I_2) = \rho(I) = 0$, so $\rho \in \text{hull}(I)$.
  Vice versa, if $\rho \in \text{hull}(I)$ and e.g. $\rho(I_2) \neq 0$, then $\rho(I_1) = 0$, since $\rho(I_2) \neq 0 \Rightarrow \rho|_{I_2} \in \text{Irrep}(I_2)$ and thus $\overline{\rho(I_2)H} = H$.
  If also $\rho(I_1) \neq 0$, applying the same argument, we would get $\overline{\rho(I_1)H} = H$ and thus
  $$0 = \overline{\rho(I)H} = \overline{\rho(I_1 \cdot I_2)H} = \overline{\rho(I_1)\rho(I_2)H} = H,$$
  a contradiction. So there is only one topology on $\hat{A}$, such that the elements of $\mathcal{A}$ form the closed subsets of $\hat{A}$.
- We also show that if $E \subseteq \hat{A}$, then
  $$\rho \in \overline{E} \quad \Leftrightarrow \quad \rho \in \text{hull}(\ker E).$$
  Since $\ker E \subseteq A$ is a closed ideal, we have that $\text{hull}(\ker A) \subset \hat{A}$ closed with $E \subseteq \text{hull}(\ker E)$. Vice versa if $F = \text{hull}(I) \subset \hat{A}$ is any closed subset with $E \subseteq F$, then $I = \ker F \subseteq \ker E$ and thus $\text{hull}(\ker E) \subseteq \text{hull}(I) = F$. So we conclude that $\text{hull}(\ker E)$ is the smallest closed subset in $\hat{A}$ that contains E.

$\square$

Remark 2.4.8 **Closed Ideals** *If $I \subseteq A$ is a closed ideal, then*

$$\text{hull}(I) = \widehat{A_{A/I}} \cong \widehat{A_I}.$$

This follows from the last result of the last section, which gives

$$\cap_{\sigma\in\widehat{A/I}} \ker\sigma = \ker(\oplus_{\sigma\in\widehat{A/I}}\sigma) = \{0 + I\} \subseteq A/I.$$

Thus for the quotient map it holds, that

$$\cap_{\sigma\in\text{hull}(I)} \ker\sigma = \cap_{\sigma\in\widehat{A/I}} \ker(\sigma \circ q) = \ker q = I.$$

Remark 2.4.9 **Closed Sets** *The closed sets in $\widehat{A}$ are of the following form:*

$$\boxed{K \subset \widehat{A} \text{ closed} \quad \Leftrightarrow \quad K = \mathrm{hull}(I) \cong \widehat{A/I}}$$

for some closed ideal $I \subseteq A$. The open sets are exactly the complements.

Remark 2.4.10 *The hull-kernel topology on $\hat{A}$ does not necessarily fulfill any of the separation axioms:*

1. *The hull-kernel topology on $\hat{A}$ is generally not Hausdorff. For example take $A = L(H)$, with $H = l^2(\mathbb{N})$. Then $\mathbb{1}_A : L(H) \to L(H)$ is irreducible and*

$$\overline{\{\mathbb{1}_A\}} = \mathrm{hull}(\ker \mathbb{1}_A) = \mathrm{hull}(\{0\}) = \widehat{L(H)}.$$

Since $K(H) \subsetneq L(H)$ is a closed ideal, there exists an irreducible representation $\rho \in \widehat{L(H)/K(H)}$. We then have

$$\rho \neq \mathbb{1}_A, \quad \rho \in \overline{\{\mathbb{1}_A\}} = \widehat{L(H)}.$$

But points in a Hausdorff space need to be closed.

2. *The hull-kernel topology on $\hat{A}$ need not even be T_0.*

Definition 2.4.11 **Primitive Ideals and Prim(A)** *Let A be a C^*-algebra. An Ideal $P \subset A$ is called primitive iff*

$$\exists\, \pi \in \mathrm{Irrep}(A) : \ P = \ker \pi.$$

We set

$$\mathrm{Prim}(A) := \{P \subset A \,|\, P \text{ is primitive ideal}\}.$$

And there is a surjective map

$$\begin{aligned} \phi : \hat{A} \ &\longrightarrow \ \mathrm{Prim}(A). \\ [\pi] \ &\longmapsto \ \ker \pi \end{aligned}$$

The topology on Prim(A) *is defined by*

$$U \subset \mathrm{Prim}(A) \text{ is open iff } \phi^{-1}(U) \subseteq \hat{A} \text{ open}$$

and is also refered to as hull-kernel topology.

Remark 2.4.12 *An equivalent definition of the topology on* Prim(A) *is by defining*

$$\ker E := \cap_{Q \in E} Q, \quad \mathrm{hull}(J) = \{P \in \mathrm{Prim}(A) \,|\, J \subseteq P\}$$

and again demanding

$$\overline{E} = \mathrm{hull}(\ker E), \quad \forall E \subseteq \mathrm{Prim}(A).$$

Theorem 2.4.13 Prim(A) *is always a T_0 space and the following are equivalent*

1. *$\hat{A}$ is a T_0 space.*

2. *If $\pi, \rho \in \mathrm{Irrep}(A)$ with $\ker \pi = \ker \rho$, then $\pi \cong \rho$.*

3. *$\phi : \hat{A} \to \mathrm{Prim}(A)$ is a homeomorphism.*

Proof: We only show that $\mathrm{Prim}(A)$ is always a T_0 space and leave the rest as an exercise. Let $P_1 \neq P_2 \in \mathrm{Prim}(A)$, then $\mathrm{hull}(P_1) \neq \mathrm{hull}(P_2)$, since otherwise

$$P_1 = \ker(\mathrm{hull}(P_1)) = \ker(\mathrm{hull}(P_2)) = P_2.$$

Now if $P_2 \in \mathrm{hull}(P_1)$, then $P_2 \supset P_1$. Since $P_2 \neq P_1$ we have $P_2 \not\subseteq P_1$ and thus

$$P_1 \neq \mathrm{hull}(P_2).$$

So for $U := \mathrm{Prim}(A)/\mathrm{hull}(P_2)$ we have $P_1 \in U$ and $P_2 \neq U$. □

Theorem 2.4.14 <u>**(locally) compact**</u> $\mathrm{Prim}(A)$ *and* $\hat{A}$ *are always locally compact. If* A *is unital, then* $\mathrm{Prim}(A)$ *and* $\hat{A}$ *are compact.*

Proof: We only prove the statement for unital algebras. And since $\mathrm{Prim}(A)$ is the continuous image of $\hat{A}$, it is enough to prove the claim for $\hat{A}$.
We will prove that $\hat{A}$ hat the finite intersection property for closed sets, i.e. if $\{E_i \,|\, i \in \Lambda\}$ a system of closed subsets of $\hat{A}$ with nonempty intersection $\cap_{i\in F} E_i \neq \varnothing$, then for any finite subset $F \subset \Lambda$, then also $\cap_{i\in\Lambda} E_i \neq \varnothing$.
Now let $I_i \subseteq A$ a closed ideal with $E_i = \mathrm{hull}(I_i)$. Assume $\cap_{i\in\Lambda} E_i = \varnothing$. In a previous proof, we have already shown that

$$\cap_{i\in\Lambda} E_i = \cap_{i\in\Lambda} \mathrm{hull}(E_i) = \mathrm{hull}(J) \quad \text{with } J = \overline{\mathrm{LH}\{\cup_{i\in\Lambda} I_i\}}.$$

Because of $\varnothing = \mathrm{hull}(J) \cong \widehat{A/J}$ we get that $A/J = \{0\}$ so $A = J$. Then $\tilde{J} = \mathrm{LH}\{\cup_{i\in\Lambda} I_i\} \subset A$ is a dense ideal and since A is unital we already have that $\tilde{J} = A = \mathrm{hull}(A) = \varnothing$. A contradiction! □

2.5 C^*-Algebras of Compact Operators

In this section we shall study a class of operators for which the irreducible representations determine all of the representation theory. Every $*$-representation is given as a direct sum:

$$A \subseteq K(H), \qquad \Rightarrow \qquad A = \bigoplus_{\rho\in\hat{A}} K(H_\rho).$$

Definition 2.5.1 <u>**Projection**</u> *Let* A *be a* C^**-algebra.* $p \in A$ *is called a projection iff*

$$p = p^2 = p^*.$$

Lemma 2.5.2 <u>**Spectral Decomposition**</u> *If* $T \in K(H)$ *with* $T = T^*$*, then*

$$\boxed{T = \sum_{\lambda\in\sigma_p(T)} \lambda P_\lambda, \qquad H = \bigoplus_{\lambda\in\sigma_p(T)} E_\lambda}$$

with $\sigma_p(T) = \{\lambda \in \mathbb{R} \,|\, \lambda \text{ eigenvalue of } T\}$ *and* $P_\lambda : H \to E_\lambda$ *the orthogonal projection onto the eigenspace* $E_\lambda := \ker(T - \lambda\mathbb{1})$*. It then further holds, that*

$$\boxed{\dim E_\lambda < \infty \ \forall \lambda \neq 0, \quad E_\lambda \perp E\mu, \ P_\lambda P_\mu = 0 \text{ for } \mu \neq \lambda}$$

The point spectrum $\sigma_p(T)$ *is either finite or countable and* $\sigma_p(T)$ *has an accumulation point that is not 0. That is since if* $\{\lambda_n \,|\, n \in \mathbb{N}\}$ *is a counting of* $\sigma_p(T) \setminus \{0\}$*, then* $\lambda_n \to 0$ *(if* $\sigma_p(T)$ *is not finite).*

Lemma 2.5.3 <u>**Point Spectrum of Compact Operators**</u> *If* $T \in K(H)$ *with* $T = T^*$*, then*

$$\sigma(T) \cup \{0\} = \sigma_p(T) \cup \{0\}, \qquad P_\lambda = 1_{\{\lambda\}}(T) \ \forall \lambda \in \sigma(T) \cup \{0\}$$

where $1_{\{\lambda\}}(\mu) = \delta_{\lambda\mu}$.

Proof: We shall only prove the first assertion. It trivially holds that $\sigma_p(T) \subseteq \sigma(T)$. We thus take $0 \neq \mu \in \sigma(T) \setminus \sigma_p(T)$ and the idea is to prove that $(T - \mu \mathbb{1})$ is bijective, since then it is invertible and $\mu \notin \sigma(T)$, which completes the prove.

- **Injectivity:** Let $0 \neq \xi \in H$, then $\xi = \sum_{\lambda \in \sigma_p(T)} \xi_\lambda$ with $\xi_\lambda \in E_\lambda$ and $\xi_\lambda \neq 0$ for at least one λ. It follows that $P_\lambda \xi = \xi_\lambda$ and
$$(T - \mu \mathbb{1})\xi = \sum_{\lambda \in \sigma_p(T)} (\lambda \xi_\lambda - \mu \xi_\lambda) = \sum_{\lambda \in \sigma_p(T)} (\lambda - \mu)\xi_\lambda.$$
This gives $||(T - \mu \mathbb{1})\xi||^2 = \sum_{\lambda \in \sigma_p(T)} |\lambda - \mu|^2 ||\xi_\lambda||^2 > 0$, since $(\lambda - \mu) \neq 0 \ \forall \lambda$ and $||\xi_\lambda||^2 > 0$ for at least one λ. So we see $\ker(T - \mu \mathbb{1}) = \{0\}$.

- **Surjectivity:** Again take $\xi = \sum_{\lambda \in \sigma_p(T)} \xi_\lambda \in H$ and define
$$\eta := \sum_{\lambda \in \sigma_p(T)} \frac{1}{\lambda - \mu} \xi_\lambda, \qquad \Rightarrow \qquad (T - \mu \mathbb{1})\eta = \xi.$$
The sum is well defined, since μ is not an accumulation point of $\sigma_p(T)$ and we know that $\{\frac{1}{|\lambda - \mu|} \mid \lambda \in \sigma_p(T)\}$ is bounded. So the sum converges in H.

$\square$

Corollary 2.5.4 *Let $A \subseteq K(H)$ be a C^*-subalgebra and $a = a^* \in A$. Let $0 \neq \lambda \in \sigma(a)$, then the orthogonal projections onto E_λ are in the subalgebra:*
$$a = \sum_{0 \neq \lambda \in \sigma(a)} \lambda P_\lambda, \quad \text{with } P_\lambda \in A \ \forall \lambda.$$

Proof: This is an immediate consequence of $P_\lambda = 1_{\{\lambda\}}(a) \in A$. $\square$

Remark 2.5.5 *If $p \in K(H)$ is an arbitrary projection, then $dim(p(H)) < \infty$. That is since*
$$p|_{p(H)} = \mathbb{1}_{p(H)} : p(H) \longrightarrow p(H)$$
and thus $B_1^{p(H)}(0) = p(B_1^{p(H)}(0)) \subseteq p(B_1^H(0))$ is compact. Subsets of the unit ball in an infinite dimensional Hilbert space are compact iff they are finite dimensional.

Remark 2.5.6 *If $A \subseteq L(H)$, then being a projection is equivalent to being an orthogonal projection. Due to $p = p^2 = p^*$, we have $p \geq 0$ for every projection $p \in A$.*

Lemma 2.5.7 *Let A be a C^*-algebra and $p, q \in A$. The following are equivalent*

(1) $q \leq p$

(2) $qp = pq = q$

(3) *(if $A \subseteq L(H)$)* $q(H) \subseteq p(H)$

Proof: With Gelfand-Naimark: w.l.o.g. we can assume that $A \subseteq L(H)$, then
$$a \geq 0 \quad \Leftrightarrow \quad \langle a\xi, \xi \rangle \geq 0 \ \ \forall \xi \in H.$$

$(1) \Rightarrow (3)$ $q \leq p \Leftrightarrow \langle q\xi, \xi \rangle \leq \langle p\xi, \xi \rangle \ \forall \xi \in H$. We further assume $\exists \xi \in q(H)$ with $\xi \notin p(H)$. We decompose $\xi = \xi_1 + \xi_2$ with $\xi_1 \in p(H)$, $\xi_2 \in p(H)^\perp$. It then follows, that
$$||\xi||^2 = ||\xi_1||^2 + ||\xi_2||^2$$
with $||\xi_2||^2 \neq 0$ and
$$||\xi||^2 = \langle \xi, \xi \rangle = \langle q\xi, \xi \rangle \leq \langle p\xi, \xi \rangle = \langle \xi_1, \xi_1 + \xi_2 \rangle = ||\xi_1||^2 < ||\xi||^2,$$
which is a contradiction.

(3) $\Rightarrow$ (1) is obvious.

(3) $\Leftrightarrow$ (2) is left as an exercise.

$\square$

Definition 2.5.8 Minimal projection *A minimal projection in a C^*-algebra A is an element $0 \neq p \in A$ for which for every other projection $0 \neq q \in A$, we have*

$$q \leq p \qquad \Rightarrow \qquad q = p.$$

Example 2.5.9

1.) One dimensional projections in $L(H)$.

2.) Let $A := \{\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \mid T \in M_2(\mathbb{C})\} \subseteq M_4(\mathbb{C})$, then $P = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$ is a minimal projection in A, since $M_2(\mathbb{C}) \cong A$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is minimal in $M_2(\mathbb{C})$.

Lemma 2.5.10 *Let $A \subseteq K(H)$ be a C^*-subalgebra. If $0 \neq p \in A$, then*

$$\boxed{p \text{ minimal} \quad \Leftrightarrow \quad pAp = \mathbb{C}p}$$

Proof:

"$\Leftarrow$" Let $pAp = \mathbb{C}p$ if now q is a projection in A with $q \leq p$, then

$$q = pq = pqp \in pAp = \mathbb{C}p.$$

So we have $q = \lambda p$ for some $\lambda \in \mathbb{C}$ and we have either $q = 0$ or $q = p$.

"$\Rightarrow$" Let now p be minimal. We will show that if $a \in A$, then $pap = \lambda p$ for some $\lambda \in \mathbb{C}$. Decomposing $a = \mathrm{Re}(a) + i\mathrm{Im}(a)$, we can assume w.l.o.g. that $a = a^*$. We then also have $(pap)^* = pap$ and

$$pap = \sum_{0 \neq \lambda \in \sigma(pap)} \lambda P_\lambda.$$

If we can prove $P_\lambda \leq p \ \forall 0 \neq \lambda \in \sigma(pap)$, we will get $P_\lambda = p \ \forall \lambda$ and the claim is proved since then $\sigma(pap) \setminus \{0\} = \{\lambda\}$ for some λ and thus $pap = \lambda p$. $P_\lambda \leq p$ is seen as follows: We know

$$p(H)^\perp = \ker p \subseteq \ker(pap) \subseteq \ker(P_\lambda) = P_\lambda(H)^\perp$$

and thus $P_\lambda(H) \subseteq p(H)$, i.e. $P_\lambda \leq p$.

$\square$

Lemma 2.5.11 Projections are sums of minimal projections *Let $A \subseteq K(H)$ be a C^*-subalgebra, then every projection $p \in A$ is a* **finite** *sum of minimal projections $p_i \in A$:*

$$p = p_1 + \cdots + p_l, \qquad p_i p_j = 0 \ \forall i \neq j.$$

Proof: We prove the lemma by induction on the dimension $n := \dim(p) := \dim(p(H)) < \infty$.

- $\underline{n = 1:}$ Clearly for $n = 1$ the projection p is minimal.
- $\underline{n \to n+1:}$ If p is minimal, we are done. If not, then there exists another projection $q \in A$ with $0 \neq q \neq p$ and $q \leq p$ (since if such a q did not exist p would be minimal). We then have $q(H) \subsetneq p(H)$ and $\dim(q) < \dim(p) = n+1$, i.e. $\dim(q) \leq n$. Furthermore $p - q$ is the projection onto $q(H)^\perp \cap p(H)$ and we also have $\dim(p-q) \leq n$. So we can apply the induction hypothesis to $p - q$ and q, which gives the desired decomposition of $p = (p-q) + q$.

□

Theorem 2.5.12 *If $A \subset K(H)$ is irreducible (i.e. the representation $\mathbb{1}_A$ is irreducible), then it holds that $A = K(H)$.*

Proof: The strategy of the proof is proving the first point of the following

1) A contains all projections p of rank 1, i.e. $\dim(p) = 1$.

2) A thus contains all projections.

3) With the spectral decomposition $T^* = T = \sum_{0\neq\lambda\in\sigma(T)} \lambda P_\lambda \in K(H)$, we get that A contains all self adjoint compact operators.

4) The decomposition $T = \mathrm{Re}(T) + i\mathrm{Im}(T) \in A + iA = A$, shows that all operators are in A.

- We show, that there exists a minimal projection $p \in A$
The existence of some projection $\tilde{p} \in A$ is assured since $A \neq \{0\}$ due to irreducibility and for $0 \neq a^* = a \in A$ we have $\sigma(a) \neq \{0\}$ and for $0 \neq \lambda \in \sigma(a)$ we have $0 \neq P_\lambda \in A$. We know that every projection is an orthogonal sum of minimal projection in A, in particular minimal projections in A exist.

- For the minimal projection $p \in A$, it holds that $\dim(p) = 1$.
Let $0 \neq \xi \in p(H)$ and let $\eta \in p(H)$ arbitrary with $\eta\perp\xi$, we will show that $\eta = 0$. Since p is minimal: $pAp = \mathbb{C}p$, thus for all $a \in A$:

$$\langle a\xi, \eta\rangle = \langle ap\xi, p\eta\rangle = \langle pap\xi, \eta\rangle = \langle \lambda p\xi, \eta\rangle = \langle \lambda\xi, \eta\rangle \overset{\xi\perp\eta}{=} 0.$$

Since $A \subset L(H)$ is irreducible, we have $\overline{A\xi} = H$, so we get $\eta \in H^\perp = \{0\}$.

- It remains to show that every projection $q \in K(H)$ of rank 1 lies in A.
Let $\eta \in q(H)$ with $||\eta|| = 1$, then $q\xi = \langle \xi, \eta\rangle\, \eta\ \forall\xi \in H$. Then take a p as above. So for a $\xi \in p(H)$ with $||\xi|| = 1$, due to $\overline{A\xi} = H$, there exists a sequence $(a_n)_n$ in A with $a_n\xi \to \eta$ and $||a_n|| = 1\ \forall n \in \mathbb{N}$. We then have $a_npa_n^* \in A\ \forall n \in \mathbb{N}$ and

$$\begin{aligned} ||(a_npa_n^* - q)v|| &= ||a_np(a_n^*v) - \langle v, \eta\rangle\, \eta|| = ||a_n(\langle a_n^*v, \xi\rangle\, \xi) - \langle v, \eta\rangle\, \eta|| \\ &= ||\, \langle a_n^*v, \xi\rangle\, a_n\xi - \langle v, \eta\rangle\, \eta|| = ||\, \langle v, a_n\xi\rangle\, a_n\xi - \langle v, \eta\rangle\, \eta|| \\ &= ||\, \langle v, a_n\xi - \eta\rangle\, a_n\xi - \langle v, \eta\rangle\, (a_n\xi - \eta)|| \\ &\leq 2||v||||a_n\xi - \eta||. \end{aligned}$$

Where the last inequality holds due to $||a_n\xi|| = ||\eta|| = 1$. It follows, that

$$||a_npa_n^*|| \leq 2||a_n\xi - \eta|| \longrightarrow 0$$

and thus $q \in A$.

□

Definition 2.5.13 **Simple algebra** *A simple algebra A is a C^*-algebra in which the only closed ideals are $\{0\}$ and A.*

Lemma 2.5.14 *$K(H)$ is simple.*

Proof: If $0 \neq I \subseteq K(H)$ is an ideal, then $I \subseteq K(H)$ is irreducible, since $\mathbb{1}_{K(H)}$ is irreducible and thus also $\mathbb{1}_{K(H)}|_I :\to K(H)$. It then follows that $I = K(H)$. □

Corollary 2.5.15 **Irreducible subalgebras contain no or all compact operators** *If $B \subseteq L(H)$ is an irreducible C^*-subalgebra with $B \cap K(H) \neq \{0\}$. Then $K(H) \subseteq B$.*

Proof: $\{0\} = B \cap K(H)$ is a closed ideal in B. Since $\mathbb{1} : B \to L(H)$ is irreducible, $\mathbb{1}_{B\cap K(H)}$ is too. With the last lemma, we get $B \cap K(H) = K(H)$. $\square$

Lemma 2.5.16 *Let $A \subseteq K(H)$ be a C^*-subalgebra with $\overline{AH} = H$ and let $p \in A$ be a minimal projection. Let $\xi \in p(H)$ with $||\xi|| = 1$ and $H_0 := \overline{A\xi} \subseteq H$, then*

$$A|_{H_0} = K(H_0).$$

Proof: Let $T \in L(H_0)$ with $Ta = aT\ \forall a \in A|_{H_0}$. We will show that $T = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$, since then, due to Schur, we get $A|_{H_0}$ is irreducible. Then the above lemma lets us conclude that $A|_{H_0} = K(H_0)$.
We shall show $T = \langle T\xi, \xi\rangle\, \mathbb{1}$, for which we define $\mathbf{T} := (T - \langle T\xi, \xi\rangle\, \mathbb{1})$. It follows, that

$$\langle \mathbf{T}\xi, \xi\rangle = \langle T\xi, \xi\rangle - \langle T\xi, \xi\rangle \langle \xi, \xi\rangle \overset{||\xi||^2=1}{=} 0.$$

Now let $a, b \in A$ and observe

$$pb^*\mathbf{T} = pb^*|_{H_0}\mathbf{T} = \mathbf{T}pb^*|_{H_0}.$$

Since p is minimal $\exists \mu \in \mathbb{C}$: $pb^*ap = \mu p$ and we get

$$\langle \mathbf{T}a\xi, b\xi\rangle = \langle \mathbf{T}ap\xi, bp\xi\rangle = \langle pb^*\mathbf{T}ap\xi, \xi\rangle = \langle \mathbf{T}pb^*ap\xi, \xi\rangle = \mu \langle \mathbf{T}\xi, \xi\rangle = 0$$

So, since $\overline{A\xi} = H_0$, we have indeed proved that $\mathbf{T} = 0$ and thus $T = \lambda\mathbb{1}$ for $\lambda = \langle T\xi, \xi\rangle \in \mathbb{C}$. $\square$

Example 2.5.17 *Let $H = \mathbb{C}^4$, $A = \{\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \mid T \in M_2(\mathbb{C})\} \subseteq M_4(\mathbb{C})$,*

$P = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$ *and $\xi = e_1$, then $H_0 = \mathrm{span}_{\mathbb{C}}\{e_1, e_2\} \cong \mathbb{C}^2$ and $A|_{H_0} \cong M_2(\mathbb{C})$.*

We now come to one of the central theorems on the decomposition of $A \subseteq K(H)$ into irreps:

Theorem 2.5.18 **Decomposition into Irreps** *Let $A \subseteq K(H)$ such that $\overline{AH} = H$ and let $\pi : A \to L(\tilde{H})$ be any nondegenerate representation of A, then there exist irreducible representations $\pi_i : A \to L(H_i)$ such that*

$$\boxed{\tilde{H} \cong \bigoplus_{i\in I} H_i, \qquad \pi \cong \bigoplus_{i\in I} \pi_i}$$

Furthermore: it holds that each π_i is equivalent to an irreducible sub-representation of $\mathbb{1} : A \hookrightarrow K(H)$.

Proof:

- First, there is a minimal projection $p \in A$ with $\pi(p) \neq 0$, since otherwise all projections q would be zero $\pi(q) = 0$ in $L(\tilde{H})$, since every projection is a sum of minimal ones. And finally we would get $\pi(A) = 0$, due to the spectral decomposition theorem $\pi(a) = \sum_\lambda \lambda\pi(P_\lambda) = 0$ for all $a = a^* \in A$ and the decomposition $T = \mathrm{Re}(T) + i\mathrm{Im}(T)$ for any $T \in A$.

- For a minimal projection $p \in A$ with $\pi(p) \neq 0$, we know that $pAp = \mathbb{C}p$ and thus there is a linear functional $f : A \to \mathbb{C}$ such that $pap = f(a)p\ \forall a \in A$. Choose an $\eta \in \pi(p)\tilde{H}$ with $||\eta|| = 1$ and a $\xi \in pH$ with $||\xi|| = 1$. Set

$$\tilde{H}_0 := \overline{\pi(A)\eta}, \qquad H_0 := \overline{A\xi}.$$

- Now if $\pi_0 : A \to L(\tilde{H}_0)$ is a sub-representation of π on $\tilde{H}_0$ and $\mathbb{1}_0 : A \to L(H_0)$ is the sub-representation of $\mathbb{1}$ on H_0, then $\pi_0 \cong \mathbb{1}_0$. This is proved as follows: define

$$V|_{A\xi} : A\xi \overset{\text{dense}}{\subseteq} H_0 \longrightarrow \tilde{H}_0, \qquad Va\xi := \pi(a)\eta.$$

Then for all $a\xi, b\xi \in A\xi$ we have

$$\begin{aligned}
\langle Va\xi, Vb\xi\rangle &= \langle \pi(a)\eta, \pi(b)\eta\rangle = \langle \pi(b^*a)\eta, \eta\rangle \\
&= \langle \pi(b^*a)\pi(p)\eta, \pi(p)\eta\rangle = \langle \pi(pb^*ap)\eta, \eta\rangle \\
&= \langle \pi(f(b^*a)p)\eta, \eta\rangle = f(b^*a)\langle \eta, \eta\rangle = f(b^*a) \\
&= \langle f(b^*a)\xi, \xi\rangle = \langle pb^*ap\xi, \xi\rangle \\
&= \cdots = \langle a\xi, b\xi\rangle .
\end{aligned}$$

So V is unitary and due to

$$Va(b\xi) = V(ab)\xi = \pi(ab)\eta = \pi(a)\pi(b)\eta = \pi_0(a)V(b\xi)$$

we have $V\mathbb{1}_0(a) = \pi_0(a)V \ \forall a \in A$, thus $\pi_0 \cong \mathbb{1}_0$.

- All together we have shown

  (1) Every nontrivial representation $\pi : A \to L(\tilde{H})$ has an irreducible sub-representation (since we know that $\mathbb{1}_0(A) = A|_{H_0}$ is irreducible).

  (2) Every such representation is equivalent to an irreducible sub-representation of $\mathbb{1} : A \hookrightarrow K(H)$

A simple application of Zorn's lemma to the sub-representations of π that can be written as a direct sum of irreducible representations, proves the claim.

$\square$

Definition 2.5.19 *Let α be a cardinal number, I a set of cardinality α and H a Hilbert space. Set*

$$\boxed{\alpha \cdot H := \bigoplus_{i\in I} H, \qquad \Rightarrow \qquad \text{for } n \in \mathbb{N}: \quad n \cdot H = \bigoplus_{i=1}^{n} H}$$

If $\pi : A \to L(H)$ is a $*$-representation, then set

$$\boxed{\alpha \cdot \pi := \oplus_{i\in I}\pi : A \longrightarrow L(\alpha \cdot H)}$$

These constructions do not depend on the choice of I (up to equivalence).

Lemma 2.5.20 *Every nondegenerate $*$-representation $\pi : K(H) \to L(\tilde{H})$ is equivalent to $\alpha \cdot \mathbb{1}$ for some α.*

Proof: According to the last lemma $\pi \cong \oplus_{i\in I}\pi_i$ and π_i is equivalent to an irreducible sub-representation of $\mathbb{1} : K(H) \to K(H)$. But we know that $\mathbb{1}$ itself is irreducible, thus $\pi_i \cong \mathbb{1}$ $\forall i \in I$ and we have

$$\oplus_{i\in I}\pi_i \cong \oplus_{i\in I}\mathbb{1} = \alpha \cdot \mathbb{1},$$

which proves the claim. $\square$

Corollary 2.5.21 **<u>Classification of $*-$Isos of K(H)</u>** *Let $\varphi : K(H) \to K(H)$ be an arbitrary $*$-automorphism of $K(H)$ (and thus a $*$-isomorphism), then there is a unitary operator $U \in L(H)$ such that $\forall T \in K(H)$:*

$$\boxed{\varphi(T) = \mathrm{Ad}(U)(T) := UTU^*}$$

Furthermore if $U = V \in L(H)$ is unitary with $\mathrm{Ad}(U) = \mathrm{Ad}(V)$, then there exists a $\lambda \in S^1 \subset \mathbb{C}$ such that $U = \lambda V$. So for the unitary operators $U(H)$ on H:

$$\mathrm{Aut}(K(H)) \cong P(U(H)) := U(H)/S^1.$$

Proof: A $*$-automorphism $\varphi : K(H) \to K(H) \subset L(H)$ is also an irreducible representation, since $K(H) = \varphi(K(H))$ is irreducible in $L(H)$. Thus $\varphi \cong \mathbb{1}$, i.e. there is a unitary $U : H \to H$ with $\varphi(T)U = U\mathbb{1}(T)\ \forall T \in K(H)$, thus $\varphi(T) = UTU^*$.
If $V : H \to H$ is another such operator, with $VTV^* = UTU^*$, then $V^*UT = TV^*U$, i.e. V^*U commutes with all $T \in K(H)$ and thus with Schur there exists a $\lambda \in \mathbb{C}$ with $V^*U = \lambda\mathbb{1}$, thus $U = \lambda V$, and since V^*U is unitary we have $|\lambda| = 1$. $\square$

Corollary 2.5.22 $\underline{\mathbf{* - Irreps\ send\ subalgebras\ of\ K(H)\ to\ K(H')}}$ *If $A \subseteq K(H)$ is a $*$-subalgebra, then for every irreducible representation $\pi : A \to L(\tilde{H})$ it holds that $\pi(A) = K(\tilde{H})$*

Proof: By working in $H_0 := \overline{AH}$ w.l.o.g. we can assume that $\overline{AH} = H$. According to the above decomposition theorem, there exists a closed A invariant subspace $H_1 \subseteq H$ with $\pi \cong \mathbb{1}|_{H_1}$. Due to $a|_{H_1}$ compact $\forall a \in A \subseteq K(H)$, we have that $\{a|_{H_1} \,|\, a \in A\} \subseteq K(H_1)$ is an irreducible subalgebra and it follows that $A|_{H_1} \cong K(H_1)$.
If now $U : \tilde{H} \to H_1$ is unitary with $U\pi = \mathbb{1}|_{H_1}U$, then it follows that

$$\pi(A) = U^*(A|_{H_1})U = U^*K(H_1)U = K(\tilde{H}).$$

That is since $\forall T \in L(H_1)$ it holds that $T \in K(H_1) \Leftrightarrow U^*TU \in K(\tilde{H})$. $\square$

Lemma 2.5.23 *Let $A \subseteq K(H)$ with $\overline{AH} = H$ and let $p \in A$ be a minimal projection. Then for $\rho \in \hat{A}$ it holds that*

(1) $\rho(p) = 0$ or $\rho(p)$ is a projection of rank 1.

(2) There is exactly one $\rho \in \hat{A}$ with $\rho(p) \neq 0$.

(3) If $\rho(p) \neq 0$, then $\overline{\rho(ApA)} = \rho(A) = K(H_\rho)$ and $\tau(ApA) = \{0\}\ \forall \tau \in \hat{A} \setminus \{\rho\}$.

Proof:

(1) We have already shown that if $\rho : A \to L(H_\rho)$ is irreducible and $p \in A$ minimal with $\rho(p) \neq 0$, then $\rho \cong \mathbb{1}|_{\overline{A\xi}}$, where $\xi \in p(H_\rho)$ is arbitrary with $||\xi|| = 1$. Now, because of

$$pA\xi = pAp\xi = \mathbb{C}p\xi = \mathbb{C}\xi,$$

we have that $\mathbb{1}|_{\overline{A\xi}}$ is a rank 1 projection. Due to $\rho \cong \mathbb{1}|_{\overline{A\xi}}$ we get the same result for ρ.

(2) If τ is another irreducible representation with $\tau(p) \neq 0$, then as above, it follows that $\tau \cong \mathbb{1}|_{\overline{A\xi}} \cong \rho$, i.e. $[\tau] = [\rho] \in \hat{A}$.

(3) It holds that

$$\overline{\rho(ApA)} = \overline{\rho(A)\rho(p)\rho(A)} = \overline{K(H_\rho)\rho(p)K(H_\rho)} =: I.$$

Since $\rho(p) \neq 0$ we have that $0 \neq I$ is a closed ideal in $K(H_\rho)$ and it follows that $I = K(H_\rho)$. If now $\tau \in \hat{A} \setminus \{\rho\}$, then (2) gives us $\tau(p) = 0$, we also get $\tau(ApA) = \tau(A)\tau(p)\tau(A) = \{0\}$. $\square$

Remark 2.5.24 *If $A \subseteq K(H)$ is a C^*-subalgebra and $\pi : A \to L(H)$ is an arbitrary $*$-representation, then we have the decomposition*

$$\pi = \oplus_{i \in I}\pi_i, \qquad \pi_i \in \mathrm{Irrep}(A).$$

If $\rho \in \hat{A}$, $I_\rho := \{i \in I \,|\, \pi_i \cong \rho\}$ and $n_\rho = |I_\rho|$, it holds, that

$$\pi \cong \oplus_{\rho \in \hat{A}}(\oplus_{i \in I_\rho}\rho) \cong \oplus_{\rho \in \hat{A}} n_\rho \rho.$$

If now $p \in A$ is a minimal projection and $\rho \in \hat{A}$ with $\rho(p) \neq 0$ (which exists since $\cap_{\rho \in \hat{A}} \ker \rho = \{0\}$), then by the last lemma, it follows that $\rho(p)$ is a projection of rank 1. Furthermore, due to $\tau(p) = 0\ \forall \tau \in \hat{A} \setminus \{\rho\}$, it follows that

$$\dim(\pi(p)H) = \dim((n_\rho \rho)(p)(n_\rho H_\rho)) = n_\rho.$$

Thus the cardinality n_ρ of ρ in π is uniquely determined, i.e.

$$\oplus_{\rho \in \hat{A}} n_\rho \rho \cong \oplus_{\rho \in \hat{A}} m_\rho \rho \qquad \Leftrightarrow \qquad n_\rho = m_\rho\ \forall \rho \in \hat{A}.$$

Definition 2.5.25 **Index set** *An index set is a topological space I with discrete topology.*

Definition 2.5.26 **Direct Sum** *If I is an index set and A_i a C^*-algebra for every $i \in I$. Define the direct sum*

$$\bigoplus_{i \in I} A_i := \{(a_i)_{i \in I} \,|\, a_i \in A_i\, \forall i,\ [i \mapsto ||a_i||] \in C_0(I)\}.$$

Where addition, multiplication and involution being defined component wise. The direct sum is given the norm

$$||(a_i)_{i \in I}||_\infty := \sup_{i \in I} ||a_i||.$$

Remark 2.5.27 *In the following $\hat{A}$ is an index set, so in particular it does not carry its usual Jacobson topology.*

We now come to the main theorem of this section.

Theorem 2.5.28 **Decomposition of subalgebras of the compact operators** *Let $A \subseteq K(H)$ a C^*-subalgebra with $\overline{AH} = H$. Let further $\mathbb{1} \cong \oplus_{\rho \in \hat{A}} n_\rho \rho$ be the decomposition of $\mathbb{1}A \hookrightarrow K(H) \subset L(H)$ as above, then $0 \neq n_\rho \in \mathbb{N}\ \forall \rho \in \hat{A}$ and*

$$\begin{array}{ccc} A & \xrightarrow{\cong} & \bigoplus_{\rho \in \hat{A}} K(H_\rho). \\ a & \longmapsto & (\rho(a))_{\rho \in \hat{A}} \end{array}$$

Proof: Let $\rho \in \hat{A}$, then there is a minimal projection $p \in a$ with $\rho(p) \neq 0$ since ρ is irreducible. So if now $0 \neq p \in A$ is minimal, then

$$0 \neq n_\rho = \dim(pH) = \text{rank}\,(p) \in \mathbb{N}$$

since p compact. Consider now the representation

$$\pi := \oplus_{\rho \in \hat{A}} \rho : A \longrightarrow L(\bigoplus_{\rho \in \hat{A}} H_\rho)$$

which is injective. We shall now show

$$\pi(A) \cong \bigoplus_{\rho \in \hat{A}} \subseteq L(\bigoplus_{\rho \in \hat{A}} H_\rho).$$

This we will acomplish in two steps:

(1) $\pi(A) \subseteq \bigoplus_{\rho \in \hat{A}} K(H_\rho)$, i.e. $[\rho \mapsto ||\rho(a)||] \in C_0(\hat{A})\ \forall a \in A$.

(2) $\pi(A) \subseteq \bigoplus_{\rho \in \hat{A}} K(H_\rho)$ is dense.

(1) We will first show that if $p \in A$ is an arbitrary projection, then $[\rho \mapsto ||\rho(p)||] \in C_0(\hat{A})$. We know that there are minimal projections $p_1, \dots, p_l \in A$ such that

$$p = p_1 + \cdots + p_l.$$

According to the last lemma, to every p_i there corresponds a unique $\rho_i \in \hat{A}$ with $\rho_i(p_i) \neq 0$. It then follows that

$$\rho(p) = 0 \ \forall \rho \notin \{\rho_1, \dots, \rho_l\}, \quad \text{in particular} \quad [\rho \mapsto ||\rho(p)||] \in C_0(\hat{A}).$$

Let now $a = a^* \in A$, then $a = \sum_{0 \neq \lambda \in \sigma(T)} \lambda P_\lambda$. Since $\pi(P_\lambda) \in \bigoplus_{\rho \in \hat{A}} K(H_\rho)$ and the sum is norm convergent, we conclude that also

$$\pi(a) \in \bigoplus_{\rho \in \hat{A}} K(H_\rho).$$

With $a = \mathrm{Re}(a) + i\mathrm{Im}(a)$ the first claim follows.

(2) We shall show that for an arbitrary $b = (b_\rho)_\rho \in \bigoplus_{\rho \in \hat{A}} K(H_\rho)$ and an $\varepsilon > 0$, there exists an $a \in A$ with $||\pi(a) - b|| < \varepsilon$.
First, due to $[\rho \mapsto ||b_\rho||] \in C_0(\hat{A})$, there exists a finite $F \subseteq \hat{A}$ with $||b_\rho|| < \frac{\varepsilon}{2} \ \forall \rho \notin F$.
Now we show that for every $\rho \in F$ there is a $a_\rho \in A$ with $||\rho(a_\rho) - b_\rho|| < \varepsilon$, and $\tau(a_\rho) = 0$ $\forall \tau \ncong \rho$. Because of the last lemma, for every ρ we have a minimal projection $p \in A$ such that $\rho(p) \neq 0$. We then also have $\overline{\rho(ApA)} = K(H_\rho)$, and thus the existence of an $a_\rho \in ApA$ with $||\rho(a_\rho) - b_\rho|| < \varepsilon$. Further, also with the previous lemma, we follow that $\tau(ApA) = 0$ $\forall \tau \ncong \rho$. Now

$$a := \sum_{\rho \in F} a_\rho \qquad \Rightarrow \qquad ||\rho(a) - b_\rho|| < \epsilon \ \forall \rho \in \hat{A}.$$

This concludes the proof of (2).

Last but not least, observe that images of $*$-representations are closed, so with (1) + (2) + injectivity of π we conclude $\pi(A) = \bigoplus_{\rho \in \hat{A}} K(H_\rho)$. □

Definition 2.5.29 <u>**GCR and CCR**</u> *A C^*-algebra is called*

GCR, iff $\pi(A) \cap K(H_\pi) \neq \{0\} \ \forall \pi \in \mathrm{Irrep}(A)$,

CCR, iff $\pi(A) = K(H_\pi) \ \forall \pi \in \mathrm{Irrep}(A)$.

Remark 2.5.30

- *It follows for GCR-algebras that $K(H_\pi) \subseteq \pi(A)$.*
- *CCR stands for "completely continuous representation", the "G" in GCR stands for "generalized."*
- *CCR-algebras are also called liminary algebras and GCR-algebras are also called postliminary algebras.*

Theorem 2.5.31 *Let A be a GCR-algebra, then $\widehat{A}$ is a T_0-space and the map $\widehat{A} \to \mathrm{Prim}(A)$ $\pi \mapsto \ker(\pi)$ is a homeomorphism. Further if A is GCR, then*

$$A \text{ is CCR} \quad \Leftrightarrow \quad \widehat{A} \text{ is a } T_0-\text{space (i.e. points are closed)}.$$

Proof: Let $\pi, \rho \in \widehat{A}$ with $\ker \pi = \ker \rho = J$, we need to show that $\pi \cong \rho$. Note that $\widehat{A/J} \ni \{\pi, \rho\}$, so w.l.o.g. $\ker \pi = \ker \rho = \{0\}$, $\pi : A \to L(H_\pi)$ $K(H_\pi) \subseteq \pi(A)$. Set $I := \pi^{-1}(K(H_\pi)) \subseteq A$ a closed ideal and since $\ker \pi = \ker \rho = \{0\}$ we have that $\rho(I) \neq \{0\}$. Now observe

$$\pi : I \xrightarrow{\cong} K(H_\pi) \quad \Rightarrow \quad \pi_I \cong \rho|_I \quad \Rightarrow \quad \pi \cong \rho \text{ on } A.$$

" $\Rightarrow$ " Let now A be a CCR. Let $\pi \in \widehat{A}$, $\rho \in \overline{\pi} \Leftrightarrow \ker \pi \subseteq \ker \rho$ and if $J = \ker \rho \Rightarrow \rho \in \widehat{A/J}$. If A is a CCR, then $\pi(A) = \pi(A/J) \cong K(H_\pi)$ and thus $\pi \cong \rho$.

" $\Leftarrow$ " $\widehat{A}$ be a T_1-space. Let $\pi \in \widehat{A}$. We will assume $K(H_\pi) \subsetneq \pi(A)$. Define $I := \pi^{-1}(K(H_\pi)) \subsetneq A$ which is a closed ideal and $\Rightarrow \{0\} \neq A/I \Rightarrow \widehat{A/I} \neq \varnothing$. Then $\ker \rho \subseteq I \subsetneq \ker \pi \Rightarrow \rho \in \overline{\pi}$ with $\rho \cong \pi$.

□

Theorem 2.5.32 *If A is seperable, then the following are equivalent:*

(1) $\widehat{A} \to \mathrm{Prim}(A)$ *is bijective ($\leftrightarrow$ is homeomorphism).*

(2) A is a GCR-algebra.

(3) A is a Typ-I-algebra.

3 Locally Compact Groups, Group Algebras and Universal Algebras

The unitary representations of a locally compact group G are in bijection with the $*$-representations of its C^*-group algebra $C^*(G)$.

3.1 Locally compact Groups and the Haar Integral

Definition 3.1.1 **Topological Group** *A topological group is a group G with a topology τ, such that the following maps are continuous*

$$m : G \times G \longrightarrow G, \qquad I : G \longrightarrow G.$$
$$(g, h) \longmapsto gh \qquad g \longmapsto g^{-1}$$

If (G, τ) is locally compact, one speaks of a locally compact group.

Example 3.1.2

- *Any (finite dimensional) Lie group:* $\mathbb{R}^n, GL(n, \mathbb{R}), SL(n, \mathbb{R}), O(n), U(n), \ldots$
- *Discrete groups*

Remark 3.1.3 *If G is a topological group, then the following hold*

(1) If $\mathfrak{U}$ is a neighborhood basis of $\mathbb{1}_G \in G$, then

$$\{xU \mid U \in \mathfrak{U}\}$$

is a neighborhood basis of $x \in G$. This follows directly from the fact that $g \mapsto gx$, $g \mapsto xg$ are homeomorphisms $G \to G$.

(2) If V is a neighborhood of $\mathbb{1}_G \in G$, then there is a neighborhood U of $\mathbb{1}_G \in G$, such that

$$U = U^{-1}, \quad U^2 \subseteq V.$$

If G is locally compact, one can choose U to be compact. That is since, with the continuity of the multiplication, we can choose a $\tilde{U}$ of $\mathbb{1}_G$ with $m(\tilde{U} \times \tilde{U}) = \tilde{U}^2 \subseteq V$ and then set $U := \tilde{U} \cap \tilde{U}^{-1}$.

(3) *If H is another topological group and $\varphi : G \to H$ a homomorphism, then*

$$\varphi \text{ continuous} \quad \Leftrightarrow \quad \varphi \text{ continuous in } \mathbb{1}_G.$$

This is seen by taking $g \in G$, $V \subseteq H$ a neighborhood of $h := \varphi(g)$. Then there is a neighborhood W of $\mathbb{1}_H$ with $hW \subseteq V$. Since φ is continuous in $\mathbb{1}_g$, there exists a neighborhood U of $\mathbb{1}_G$ with $\varphi(U) \subseteq W$. It then holds that

$$\varphi(gU) = \varphi(g)\varphi(U) = h\varphi(U) \subseteq hW \subseteq V.$$

Lemma 3.1.4 **Existence of modular functions** *Let G be a locally compact group and $f \in C_c(G)$, then f is uniformly continuous, i.e. there exists a neighborhood U of $\mathbb{1}_G$ with*

$$|f(gh) - f(g)|, |f(hg) - f(g)| < \varepsilon \quad \forall g \in G, h \in U.$$

Proof: Let $K := \operatorname{supp}(f)$ and $\varepsilon > 0$. For $x \in K$ choose neighborhoods V_x of $\mathbb{1}_G$ with $|f(xh) - f(x)| < \varepsilon \; \forall h \in V_x$. Further choose a neighborhood $U_x = U_x^{-1}$ of $\mathbb{1}_G$ with $U_x^2 \subseteq V_x$. Since K is compact, there exists $x_1, \ldots, x_l \in K$ with $K \subseteq \cup_{i=1}^l x_i U_{x_i}$. Now set $\tilde{U} := \cap_{i=1}^l U_{x_i}$. Then $\tilde{U}$ is a neighborhood of $\mathbb{1}_G$.
If $g \in G$ arbitrary and $h \in U$, then $g \in K \Rightarrow \exists i$ with $g \in x_i U_{x_i} \Rightarrow gh \in x_i U_{x_i} \tilde{U} \subseteq x_i V_{x_i}$, and thus also $g, gh \in x_i V_{x_i}$. Analogously: If $gh \in K$, then $\exists i$ such that $gh \in x_i U_{x_i} \Rightarrow$ $g \in x_i U_{x_i} \tilde{U}^{-1} \subseteq x_i U_{x_i}^2 \subseteq x_i V_{x_i}$, thus also $g, gh \in x_i V_{x_i}$. We have

$$|f(gh) - f(g)| \leq |f(gh) - f(x_i)| + |f(x_i) - f(g)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Similarly we find a neighborhood W of $\mathbb{1}_G$ with $|f(hg) - f(g)| < \varepsilon \; \forall h \in \tilde{W}$. Finally define

$$U := \tilde{U} \cap \tilde{W}.$$

$\square$

Definition 3.1.5 **Haar – integral** *Let G be a locally compact group. A (left) Haar-integral on G is a nontrivial left invariant, positive Radon-integral*

$$I : C_c(G) \longrightarrow \mathbb{C}.$$

That is $\forall f \in C_c(G)$ and $h \in G$ it holds that

$$I({}_h f) = I(f), \quad {}_h f(g) := f(h^{-1}g).$$

Remark 3.1.6 *The above condition $I({}_h f) = I(f)$ is also written as*

$$\int_G f(h^{-1}g)dg = \int_G f(g)dg.$$

Example 3.1.7

(1) *The Lebesgue-integral is a Haar-integral on $\mathbb{R}^n$.*

(2) *On $GL(n, \mathbb{R})$ we can define a Haar-integral by*

$$I(f) := \int_{GL(n,\mathbb{R})} f(A) \frac{1}{|A|^n} dA$$

With the Lebesgue-measure dA on $M_n(\mathbb{R})$.

(3) On a discrete group G the counting measure

$$I(f) := \sum_{x \in G} f(x)$$

defines a Haar-integral on G.

The following we shall state without proof.

Theorem 3.1.8 *Every locally compact group G has a Haar-integral $I : C_c(G) \to \mathbb{C}$, which is unique up to multiplication with a $c > 0$. That is if $J : C_c(G) \to \mathbb{C}$ is another Haar-integral, then $\exists c > 0 : I = cJ$.*

Definition 3.1.9 **Measurable subset** *A subset $A \subseteq G$ is called measurable, iff $\mathbb{1}_A$ is locally integrable. I.e. $\mathbb{1}_{A \cap K} K$ locally integrable for all $K \subseteq G$ compact. We then write*

$$\mu_I(A) := \int_G \mathbb{1}_A dg \in [0, \infty].$$

Remark 3.1.10

(1) If V is a compact neighborhood of $\mathbb{1}_G$ then $\mu_I(V) > 0$.
Since if $K \subseteq G$ compact, then there are $x_1, \ldots, x_l \in G$ with $K \subseteq \cup_{i=1}^{l} x_i V$ and thus

$$\mu_I(K) \subseteq \sum_{i=1}^{l} \mu_I(x_i V) = \sum_{i=1}^{l} \mu_I(V) = l \mu_I(V).$$

(2) Due to $\mu_I(V) > 0$, it follow for $0 \neq f \in C_c^+(G)$, that $I(f) > 0$.
If $f(x) \neq 0$, then there is a neighborhood V of $\mathbb{1}_G$, $\varepsilon > 0$ with $\varepsilon > 0$ and with $\varepsilon \mathbb{1}_{xV} \leq f$, thus $I(f) \geq I(\varepsilon \mathbb{1}_{xV}) = \varepsilon \mu_I(V) > 0$.

Remark 3.1.11 *Let $I : C_c(G) \to \mathbb{C}$ be a Haar-integral on a locally compact group G. For $h \in G$, we can define*

$$\tilde{I}(f) := I(f_h) = \int_G f(gh) dg, \quad f_h(g) := f(gh).$$

This is another left invariant Radon-integral on G. That is since

$$\tilde{I}({}_l f) = \int_G {}_l f(gh) dg = \int_G f(l^{-1} gh) dg = \int_G f_h(l^{-1} g) dg = \int_G f_h(g) dg = \tilde{I}(f).$$

Due to the uniqueness of the Haar-integral, we know that there is a positive constant $\Delta(h) > 0$ with

$$I(f) = \int_G f(g) dg = \Delta(h) \int_G f(gh) dg = \Delta(h) \tilde{I}(f) \quad \forall f \in C_c(G).$$

Definition 3.1.12 **Modular function** *The function $\Delta : G \to \mathbb{R}^+$, $h \mapsto \Delta(h)$ is called modular function of the group G.*

Definition 3.1.13 **Unimodular group** *A group G for which $\Delta = 1$ is called unimodular.*

Example 3.1.14

(1) Every abelian group is unimodular, since left and right multiplication coincide.

(2) Every discrete group is unimodular, since the counting measure is right invariant.

(3) Every compact group is unimodular.

(4) The (ax+b) group G is defined as

$$G := \{\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0\} \subset GL(2, \mathbb{R})$$

with the Haar-integral

$$I(f) = \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} f\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \frac{1}{|a|} db da \qquad \Rightarrow \qquad \Delta\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} = |a|.$$

Theorem 3.1.15 *The modular function $\Delta : G \to \mathbb{R}^+$ is a continuous homomorphism.*

Proof: Choose $f \in C_c^+$ with $\int_G f(g)dg = 1$, then

$$\begin{aligned} \Delta(hl) \int_G f(ghl)dg &= \int_G f(g)dg = \Delta(l) \int_G f(gl)dg = \Delta(l) \int_G f_l(g)dg \\ &= \Delta(l)\Delta(h) \int_G f_l(gh)dg = \Delta(l)\Delta(h) \int_G f(ghl)dg. \end{aligned}$$

and it follows that $\Delta(hl) = \Delta(h)\Delta(l)$. In order to prove continuity, choose an arbitrary neighborhood V of $\mathbb{1}_G$ and set $\mu := \mu_I(KV) \in (0, \infty)$ with $K = \text{supp}(f)$.
Let $\varepsilon > 0$ and choose a neighborhood $U = U^{-1}$ of $\mathbb{1}_G$ with $U \subseteq V$, such that

$$|f(gh) - f(g)| < \frac{\varepsilon}{\mu}, \quad \forall h \in U.$$

We get that

$$\begin{aligned} |\Delta(h^{-1}) - \Delta(\mathbb{1}_G)| &= |\Delta(h^{-1}) - 1| = |\Delta(h^{-1}) \int_G f(g)dg - \int_G f(g)dg| \\ &= |\int_G f(gh) - f(g)dg| \leq \int_G |f(gh) - f(g)|dg \\ &= \int_{KV} |f(gh) - f(g)|dg \leq \frac{\varepsilon}{\mu}\mu = \varepsilon. \end{aligned}$$

With $U = U^{-1}$ it follows, that $|\Delta(h^{-1}) - 1| < \varepsilon \; \forall h \in U$. □

Corollary 3.1.16 *Every compact group G is unimodular.*

Proof: Since $\Delta : G \to \mathbb{R}^+$ is a continuous homomorphism, we have that $\Delta(G) \subset \mathbb{R}^+$ is a compact subgroup of $\mathbb{R}^+$. Thus we have $\Delta(G) = \{1\}$. □

Lemma 3.1.17 *Let G be a locally compact group, then*

$$\int_G f(g)dg = \int_G f(g^{-1})\Delta(g^{-1})dg \quad \forall f \in L^1(G).$$

Proof: We will show that the above equation holds for all $f \in C_c(G)$. The full statement then follows by approximation.

- We show that the following is a Haar-integral:

$$J : C_c(G) \to \mathbb{C} \quad J(f) := \int_G f(g^-1)\Delta(g^{-1})dg.$$

This is due to

$$\begin{aligned} J({}_hf) &= \int_G {}_hf(g^{-1})\Delta(g^{-1})dg = \int_G f(h^{-1}g^{-1})\Delta(g^{-1})dg \\ &\overset{g \to gh^{-1}}{=} \Delta(h^{-1}) \int_G f(g^{-1})\Delta(hg^{-1})dg = \int_G f(g^{-1})\Delta(g^{-1})dg \\ &= J(f). \end{aligned}$$

So J is a left invariant Radon-integral and the uniqueness of the Haar-integral gives us: $I(f) = cJ(f) \; \forall f$ and some $c > 0$.

- We will see that $c = 1$. Let $0 \neq f \in C_c^+(G)$ Set $\tilde{f}(g) := f(g) + f(g^{-1})\Delta(g^{-1}) \in C_c^+(G)$. So $\tilde{f} \neq 0$ and $\tilde{f}(g^{-1})\Delta(g^{-1}) = \tilde{f}(g)\ \forall g \in G$, it follows $cJ(\tilde{f}) = I(\tilde{f}) = J(\tilde{f})$, thus indeed $c = 1$. $\square$

Definition 3.1.18 **Convolution and Involution** *Let G be a locally compact group. We define the convolution and involution for $f_1, f_2 \in L^1(G)$*

$$\boxed{f_1 * f_2 := \int_G f_1(h) f_2(h^{-1}g) dh, \quad f^*(g) := \overline{f(g^{-1})}\Delta(g^{-1})}$$

Theorem 3.1.19 **Involution** *Let G be a locally compact group, then the above convolution and involution turns $(L^1(G), ||\cdot||_1)$ into a Banach-$*$-algebra.*

Proof: Let $f_1, f_2 \in L^1(G)$, then $f_1 \odot f_2(h,g) = f_1(h)f_2(g)$ is integrable over $G \times G$, where the product integral is the Haar-integral on $G \times G$. consider the transformation

$$G \times G \longrightarrow G \times G, \quad (h,g) \longmapsto (h, h^{-1}g).$$

This conserves the integral on $C_c(G \times G)$ (by Fubini) and thus also on $L^1(G \times G)$. So we have that

$$(h,g) \longmapsto f_1(h)f_2(h^{-1}g)$$

is integrable on $G \times G$ and, again with Fubini, we have the existence of

$$f_1 * f_2 := \int_G f_1(h)f_2(h^{-1}g)dh$$

for almost every $g \in G$. We find that

$$\begin{aligned} ||f_1 * f_2|| &= \int_G |\int_G f_1(h)f_2(h^{-1}g)dh|dg \\ &\leq \int_G \int_G |f_1(h)f_2(h^{-1}g)|dhdg \overset{\text{Fubini}}{=} \int_G \int_G |f_1(h)f_2(h^{-1}g)|dgdh \\ &\overset{g \to hg}{=} \int_G |f_1(h)|dh \int_G |f_2(g)|dg = ||f_1||_1 ||f_2||_1. \end{aligned}$$

The equation $||f^*||_1 = ||f||_1$ follows from the last lemma. Everything else (associativity, etc) follows from lengthy calculations. $\square$

Theorem 3.1.20 *$L^1(G)$ is commutative $\Leftrightarrow$ G is abelian.*

Theorem 3.1.21 *$L^1(G)$ is unital $\Leftrightarrow$ G is discrete.*

Proof: We only prove " $\Leftarrow$ ": Let G be discrete and

$$\delta_1(g) := \begin{cases} 1, & \text{if } g = 1\!\!1_G \\ 0, & \text{other.} \end{cases}$$

Then, for all $g \in G$, it follows that

$$(f * \delta_1)(g) = \sum_{h \in G} f(h)\delta_1(h^{-1}g) = f(g) = (\delta_1 * f)(g).$$

$\square$

Lemma 3.1.22 *Let $1 \leq p < \infty$ and $f \in L^p(G)$, then there is a $\varepsilon > 0$ and a neighborhood $V = V^{-1}$ of $1\!\!1_G$ with*

$$||_g f - {}_{g_0} f||_p < \varepsilon \quad \forall g \in g_0 V$$

($||f_g - f_{g_0}||_p < \varepsilon \quad \forall g \in V g_0$ respectively). In particular the mapping $G \to L^p(G)$, $g \mapsto {}_g f$ ($g \mapsto f_g$ resp.) is continuous.

Proof: Let $f \in C_c(G)$, $W \subseteq G$ an arbitrary compact neighborhood of $\mathbb{1}_G$ and let $K = W\text{supp}(f)$, so we know that f is uniformly continuous, i.e. there is a neighborhood $V = V^{-1}$ of $\mathbb{1}_G$ with $V \subseteq W$ and $|f(g^{-1}h) - f(h)| < \frac{\varepsilon}{\mu(K)^{1/p}}$ for all $g \in V, h \in G$. Let now $g_0 \in G$ be arbitrary and $g \in g_0V$, then

$$||_gf - {}_{g_0}f||_p^p = \int |f(g^{-1}h) - f(g_0^{-1}h)|^p dg \overset{h \to g_0 h}{=} \int_G |f((g_0^{-1}g)^{-1}h) - f(h)|^p dh < \int_G \frac{\varepsilon^p}{\mu(K)} dh = \varepsilon^p.$$

Where "$<$" holds due to $\text{supp}({}_{g_0^{-1}g}f - f) \subseteq K$ since $V \subseteq W$.
If now $f \in L^p(G)$, choose $\varphi \in C_c(G)$ with $||f - \varphi||_p < \frac{\varepsilon}{3}$ and further choose a V as above with $||_g\varphi - {}_{g_0}\varphi||_p < \varepsilon \ \forall g \in g_0V$. For all $\forall g \in g_0V$, we then have

$$||_gf - {}_{g_0}f||_p \leq ||_gf - {}_g\varphi||_p + ||_g\varphi - {}_{g_0}\varphi||_p + ||_{g_0}\varphi - {}_{g_0}f||_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

And $||f_g - f_{g_0}||_p < \varepsilon \quad \forall g \in Vg_0$ follows in the same manner. □

Theorem 3.1.23 *For functions $f_1, f_2 : G \to \mathbb{C}$ the following hold:*

(1) If $f_1, f_2 \in C_c(G)$, then also $f_1 * f_2 \in C_c(G)$ and $\text{supp}(f_1 * f_2) \subseteq \text{supp}(f_1)\text{supp}(f_2)$.

(2) If $f_1 \in L^1(G), f_2 \in L^\infty(G)$, then $f_1 * f_2$ exists and $f_1 * f_2 \in C^b(G)$.

(3) For all $f \in L^1(G)$ and $\varepsilon > 0$ there exists a neighborhood $V = V^{-1}$ of $\mathbb{1}_G$ with

$$||f * \varphi - f||_1, ||\varphi * f - f||_1 < \varepsilon \quad \forall \varphi \in C_c^+(G) \text{ with } \text{supp}(\varphi) \subseteq V \text{ and } \int_G \varphi(g)dg = 1.$$

Proof:

(2) $f_1 \in L^1(G), f_2 \in L^\infty(G) \overset{\text{Hoelder}}{\Rightarrow} h \mapsto f_1(h)f_2(h^{-1}g)$ is integrable $\forall g \in G$ and thus $f_1 * f_2(g)$ exists for all $g \in G$ and it holds that

$$f_1 * f_2(g) = \int_G f_1(h)f_2(h^{-1}g) \overset{h \to gh}{=} \int_G f_1(gh)f_2(h^{-1})dh.$$

It follows, that

$$\begin{aligned} |f_1 * f_2(g) - f_1 * f_2(g_0)| &\leq \int_G |f_1(gh) - f_1(g_0h)||f_2(h^{-1})|dh \\ &\overset{\text{Hoelder}}{\leq} ||_{g^{-1}}f_1 - {}_{g_0^{-1}}f_1||_1||f_2||_\infty \underset{g \to g_0}{\longrightarrow} 0. \end{aligned}$$

(1) Continuity follows from (2) and if

$$0 \neq f_1 * f_2(g) = \in_G f_1(h)f_2(h^{-1}g)dh$$

then there exists an $h \in G$ with $h \in \text{supp}(f_1)$ and $h^{-1}g \in \text{supp}(f_2)$, thus

$$g = h(h^{-1}g) \in \text{supp}(f_1)\text{supp}(f_2).$$

(3) There is a neighborhood $V_1 = V_1^{-1}$ of $\mathbb{1}_G$ with $||_xf - f||_1 < \varepsilon \ \forall x \in V_1$. If now $\varphi \in C_c^+(G)$ with $\text{supp}(\varphi) \subseteq V_1$ and $\int_G \varphi(g)dg = 1$, then

$$\begin{aligned} ||\varphi * f - f||_1 &= \int_G |\varphi * f(g) - f(g)|dg = \int_G |(\int_G \varphi(h)f(h^{-1}g)dh) - f(g)|dg \\ &= \int_G |\int_G (\varphi(h)f(h^{-1}g) - f(g))dh|dg \overset{\text{Fubini}}{\leq} \int_{V_1} \varphi(h) \int_G |f(h^{-1}g) - f(g)|dgdh \\ &= \int_{V_1} \varphi(h)||_hf - f||_1 dh < \varepsilon \int_{V_1} \varphi(h)dh = \varepsilon. \end{aligned}$$

Analogously there exists a neighborhood $V_2 = V_2^{-1}$ of $\mathbb{1}_G$ with $||f * \varphi - f||_1 < \varepsilon \ \forall \varphi \in C_c^+(G)$ with $\text{supp}(\varphi) \subseteq V_2$ and $\int_{V_2} \varphi(g)dg = 1$. Setting $V = V_1 \cap V_2$ concludes the proof.

□

Corollary 3.1.24 *Let G be a locally compact group, then there is a net $(\varphi_\lambda)_\lambda$ in $C_c^+(G)$ with $||\varphi_\lambda * f - f||_1 \to 0$, i.e. $(\varphi_\lambda)_\lambda$ is an approximate unity.*

Proof: Set $\Lambda := \{V \,|\, V = V^{-1} \text{ neighborhood of } \mathbb{1}_G\}$, which is orderes by $V_1 \geq V_2 \;\Leftrightarrow\; V_1 \subseteq V_2$ and for every $V \in \Lambda$ we choose a $\varphi_V \in C_c^+(G)$ with $\mathrm{supp}(\varphi_V) \subseteq V$ and $\int_G \varphi_V(g)dg = 1$. The rest follows from (2) of the last theorem. □

3.2 Unitary Representations and C^*-Group Algebras

The main result of this section is the bijection

$$\text{unitary representations of } G \quad \leftrightarrow \quad \text{nondegenerate } *-\text{representations of } L^1(G).$$

Definition 3.2.1 **Strongly continuous** *A strongly continuous homomorphism is a unitary representation*

$$\pi : G \longrightarrow U(H_\pi)$$

of the locally compact group G. Strongly continuous means that for all $\xi \in H_\pi$ the map

$$g \longmapsto \pi(g)\xi$$

is continuous.

Example 3.2.2

(1) The trivial representation $\mathbb{1}_G : G \to U(\mathbb{C})$ $\mathbb{1}_G(g) = 1$ $\forall g \in G$.

(2) The **left regular representation**

$$\lambda : G \longrightarrow U(L^2(G)), \quad (\lambda(g)\xi)(h) = \xi(g^{-1}h).$$

This is strongly continuous and $\lambda(g) \in U(L^2(G))$, since for all $\xi, \eta \in L^2(G)$:

$$\langle \lambda(g)\xi, \lambda(g)\eta \rangle = \int_G \xi(g^{-1}h)\overline{\eta(g^{-1}h)}dh \overset{h \to gh}{=} \int_G \xi(h)\overline{\eta(h)}dh = \langle \xi, \eta \rangle .$$

(3) The **right regular representation**

$$\rho : G \longrightarrow U(L^2(G)), \quad (\rho(g)\xi)(h) = \sqrt{\Delta(h)}\xi(gh).$$

Remark 3.2.3 *$\rho \cong \lambda$ by $U : L^2(G) \to L^2(G)$, $(U\xi)(g) := \sqrt{\Delta(g^{-1})}\xi(g^{-1})$, which is unitary:*

$$\langle U\xi, U\eta \rangle = \int_G \Delta(g^{-1})\xi(g^{-1})\overline{\eta(g^{-1})}dg = \int_G \xi(g)\overline{\eta(g)}dg = \langle \xi, \eta \rangle ,$$

and indeed an equivalence

$$\begin{aligned} (U\lambda(g)\xi)(h) &= \sqrt{\Delta(g^{-1})}(\lambda(g)\xi)(h^{-1}) = \sqrt{\Delta(g^{-1})}\xi(g^{-1}h^{-1}) \\ &= \sqrt{\Delta(g)}\sqrt{\Delta(hg^{-1})}\xi((hg)^{-1}) = \sqrt{\Delta(g)}(U\xi)(hg) = (\rho(g)U\xi)(h). \end{aligned}$$

Lemma 3.2.4 *If $\pi : G \to U(H_\pi)$ is a unitary representation and $E \subseteq H_\pi$ a closed $\pi(G)$-invariant subspace, then $E^\perp \subseteq H_\pi$ is also invariant.*

Proof: Let $\xi \in E^\perp, \eta \in E$, then $\pi(g)^* = \pi(g^{-1})$, and, since $\pi(g^{-1})\eta \in E$:

$$\langle \pi(g)\xi, \eta \rangle = \left\langle \xi, \pi(g^{-1})\eta \right\rangle = 0$$

□

Definition 3.2.5 **Irreducible Representation** *A unitary representation $\pi : G \to U(H_\pi)$ is called irreducible iff $\{0\}, H_\pi \subseteq H_\pi$ are the only closed invariant subspaces. The set of equivalent classes of irreducible representations of G is denoted $\hat{G}$.*

Lemma 3.2.6 **Schur** *Let $\pi : G \to U(H_\pi)$ be a unitary representation of G, then the following are equivalent*

(1) $\pi \in \hat{G}$, i.e. π is irreducible.

(2) Every vector $0 \neq \xi \in H_\pi$ is cyclic, i.e. $\overline{\mathrm{LH}\{\pi(g)\xi \,|\, g \in G\}} = H_\pi$.

(3) If $T \in L(H_\pi)$ with $T\pi(g) = \pi(g)T \ \forall g \in G$, then $T = \mathbb{1}$.

Remark 3.2.7 *Note that in part (3) we have $T = c\mathbb{1}$, with $c = 1$ due to unitarity of T.*

Lemma 3.2.8 *Let $f : G \to L(H)$ be a function such that*

(1) $g \mapsto \langle f(g)\xi, \eta\rangle$ is measurable $\forall \xi, \eta \in H$,

(2) $g \mapsto ||f(g)||$ is integrable.

Then there is exactly one operator $T_f \in L(H)$ such that $\forall \xi, \eta \in H$:

$$\boxed{\langle T_f\xi, \eta\rangle = \int_G \langle f(g)\xi, \eta\rangle \, dg, \qquad T_f =: \int_G f(g)dg \in L(H)}$$

For the integral it holds that

(a) $|| \int_G f(g)dg|| \leq \int_G ||f(g)||dg =: ||f||_1$,

(b) $\forall S \in L(H):\quad S\int_G f(g)dg = \int_G Sf(g)dg,\ \int_G f(g)Sdg = \int_G f(g)dgS$.

Proof: The map $(\xi, \eta) \mapsto \int_G \langle f(g)\xi, \eta\rangle \, dg$ is well defined and sesqui-linear. It holds, that

$$|\int_G \langle f(g)\xi, \eta\rangle \, dg| \leq \int_G |\langle f(g)\xi, \eta\rangle| dg \leq \int_G ||f(g)||dg \, ||\xi|| ||\eta||.$$

Thus there exists exactly one operator $T_f \in L(H)$, as stated above, with $||T_f|| \leq \int_G ||f(g)||dg$. Part (b) follows easily from identities like

$$\langle T_{Sf}\xi, \eta\rangle = \int_G \langle Sf(g)\xi, \eta\rangle \, dg = \int_G \langle f(g)\xi, S^*\eta\rangle \, dg = \langle T_f\xi, S^*\eta\rangle = \langle ST_f\xi, \eta\rangle \, .$$

□

In order to prove the main theorem of this section, we need one last

Lemma 3.2.9 *Let X be a set, $H, \tilde{H}$ Hilbert spaces and $\varphi, \tilde{\varphi} : X \to H, \tilde{H}$ maps with*

$$H = \overline{\mathrm{LH}\{\varphi(x) \,|\, x \in X\}}, \qquad \langle \varphi(x), \varphi(y)\rangle = \langle \tilde{\varphi}(x), \tilde{\varphi}(y)\rangle \, , \quad \forall x, y \in X.$$

Then there is exactly one linear isometry $U : H \to \tilde{H}$ with $U(\varphi(x)) = \tilde{\varphi}(x) \ \forall x \in X$. If additionally

$$\tilde{H} = \overline{\mathrm{LH}\{\tilde{\varphi}(x) \,|\, x \in X\}},$$

then U is unitary.

Proof: Let $H_0 := \mathrm{LH}\{\varphi(x) \,|\, x \in X\} \overset{\text{dense}}{\subset} H$. Define $U : H_0 \to \tilde{H}$ by

$$U(\sum_{i=1}^{m} \lambda_i \varphi(x_i)) = \sum_{i=1}^{m} \lambda_i \tilde{\varphi}(x_i).$$

Now we need to show that U is a well defined isometry. We have

$$\begin{aligned}\left\langle \sum_{i=1}^{m} \lambda_i \tilde{\varphi}(x_i), \sum_{i=1}^{m} \lambda_i \tilde{\varphi}(x_i) \right\rangle &= \sum_{i,j=1}^{m} \lambda_i \overline{\lambda}_j \left\langle \tilde{\varphi}(x_i), \tilde{\varphi}(x_i) \right\rangle = \sum_{i,j=1}^{m} \lambda_i \overline{\lambda}_j \left\langle \varphi(x_i), \varphi(x_i) \right\rangle \\ &= \left\langle \sum_{i=1}^{m} \lambda_i \varphi(x_i), \sum_{i=1}^{m} \lambda_i \varphi(x_i) \right\rangle.\end{aligned}$$

This shows that U is a well defined isometry, since

$$\sum_{i=1}^{m} \lambda_i \varphi(x_i) = \sum_{i=1}^{l} \mu_i \varphi(x_i), \qquad \Rightarrow \quad 0 = \sum_{i=1}^{m} \lambda_i \varphi(x_i) - \sum_{i=1}^{l} \mu_i \varphi(x_i)$$

and with this the above calculation shows, that

$$0 = \sum_{i=1}^{m} \lambda_i \tilde{\varphi}(x_i) - \sum_{i=1}^{l} \mu_i \tilde{\varphi}(x_i).$$

So the value of $U\xi$ does not depend on the specific representation of $\xi \in H_0$.
Functional analysis tells us that there is a unique isometric continuation $A : H = \overline{H}_0 \to \tilde{H}$, thus U is surjective and thus unitary. $\square$

We now get to the main theorem of this section.

Theorem 3.2.10 *Let G be a locally compact group. It then holds, that*

(1) If $\pi : G \to U(H_\pi)$ is a unitary representation of G, then

$$\tilde{\pi} : L^1(G) \longrightarrow L(H_\pi), \quad \tilde{\pi}(f) := \int_G f(g)\pi(g)dg$$

defines a nondegenerate $*$-representation of $L^1(G)$ on H_π.

(2) Vice versa: to every nondegenerate $*$-representation $\Pi : L^1(G) \to L(H)$ there corresponds exactly one unitary representation $\pi : G \to U(H)$, such that $\Pi = \tilde{\pi}$.

That is, the map $\pi \mapsto \tilde{\pi}$ is a bijection.

Proof: Let $f_1, f_2 \in L^1(G)$ and $\xi, \eta \in H_\pi$. We first prove (1):

- Multiplicativity of $f \mapsto \tilde{\pi}(f)$ is proved as follows:

$$\begin{aligned}\left\langle \tilde{\pi}(f_1 * f_2)\xi, \eta \right\rangle &= \int_G (f_1 * f_2)(g) \left\langle \pi(g)\xi, \eta \right\rangle dg = \int_G \int_G f_1(h) f_2(h^{-1}g) \left\langle \pi(g)\xi, \eta \right\rangle dhdg \\ &\overset{\text{Fubini}}{=} \int_G \int_G f_1(h) f_2(h^{-1}g) \left\langle \pi(g)\xi, \eta \right\rangle dgdh \\ &\overset{g \to gh}{=} \int_G \int_G f_1(h) f_2(g) \left\langle \pi(hg)\xi, \eta \right\rangle dgdh \\ &= \int_G \int_G f_1(h) f_2(g) \left\langle \pi(g)\xi, \pi(h^{-1})\eta \right\rangle dgdh \\ &= \int_G f_1(h) \int_G f_2(g) \left\langle \pi(g)\xi, \pi(h^{-1})\eta \right\rangle dgdh \\ &= \int_G f_1(h) \left\langle \tilde{\pi}(f_2)\xi, \pi(h^{-1})\eta \right\rangle dgdh \\ &= \left\langle \tilde{\pi}(f_1)\tilde{\pi}(f_2)\xi, \eta \right\rangle.\end{aligned}$$

- Compatability with the involution:

$$\begin{aligned}\langle\tilde{\pi}(f^*)\xi,\eta\rangle &= \int_G \Delta(g^{-1})\overline{f(g^{-1})}\,\langle\pi(g)\xi,\eta\rangle \\ &\overset{g\to g^{-1}}{=} \int_G \overline{f(g)}\,\langle\pi(g^{-1})\xi,\eta\rangle\,dg = \overline{\int_G f(g)\,\langle\pi(g^{-1})\eta,\xi\rangle\,dg} \\ &= \overline{\langle\tilde{\pi}(f)\eta,\xi\rangle} = \langle\xi,\tilde{\pi}(f)\eta\rangle = \langle\tilde{\pi}(f)^*\xi,\eta\rangle\,.\end{aligned}$$

- Nondegeneracy: Let $(\varphi_\lambda)_\lambda$ be an approximate unity as above and $\xi\in H_\pi$, then $\tilde{\pi}(\varphi_\lambda)\xi\to\xi$ since

$$||\tilde{\pi}(\varphi_\lambda)\xi-\xi|| = ||\int_G \varphi_\lambda(g)(\pi(g)\xi-\xi)dg|| \le \int_G \varphi_\lambda(g)(||\pi(g)\xi-\xi||dg \le \max_{g\in\mathrm{supp}\varphi_\lambda}||\pi(g)\xi-\xi||.$$

For an $\varepsilon>0$ choose a neighborhood V of $\mathbb{1}_G$ with $||\pi(g)\xi-\xi||<\varepsilon\ \forall g\in V$ and $\lambda_0\in\Lambda$ with $\mathrm{supp}(\varphi_\lambda)\subseteq V\ \forall\lambda\ge\lambda_0$. It then follows that $||\tilde{\pi}(\varphi_\lambda)\xi-\xi||\le\varepsilon\ \forall\lambda\ge\lambda_0$.

We now prove (2):

- First show that $({}_g\varphi)^* * {}_g f = \varphi^* * f\ \forall\varphi$ and $\forall g\in G$:

$$\begin{aligned}(({}_g\varphi)^* * {}_g f)(h) &= \int_G ({}_g\varphi)^*(l)_g f(l^{-1}h)dl = \int_G \Delta(l^{-1})\overline{{}_g\varphi(l^{-1})}_g f(l^{-1}h)dl \\ &= \int_G \Delta(l^{-1})\overline{\varphi(g^{-1}l^{-1})}f(g^{-1}l^{-1}h)dl \\ &\overset{l\to lg^{-1}}{=} \Delta(g^{-1})\int_G \Delta(gl^{-1})\overline{\varphi(l^{-1})}f(l^{-1}h)dl \\ &= \int_G \Delta(l^{-1})\overline{\varphi(l^{-1})}f(l^{-1}h)dl = (\varphi^* * f)(h).\end{aligned}$$

- Let now $\Pi : L^1(G)\to L(H)$ be a nondegenerate $*$-representation of $L^1(G)$. Set $X := L^1(G)\times H$ and define $\varphi,\tilde{\varphi} : X\to H$ by

$$\varphi(f,\xi) := \Pi(f)\xi, \qquad \tilde{\varphi}(f,\xi) := \Pi({}_g f)\xi.$$

Then for all $f,\varphi\in L^1(G)$ and $\xi,\eta\in H$, we have

$$\langle\Pi({}_g f)\xi,\Pi({}_g\varphi)\eta\rangle = \langle\Pi(({}_g\varphi)^* * {}_g f)\xi,\eta\rangle = \langle\Pi(\varphi^* * f)\xi,\eta\rangle = \langle\Pi(f)\xi,\Pi(\varphi)\eta\rangle\,.$$

Now, due to the last lemma, there exists exactly one unitary operator $\pi(g) : H\to H$ with

$$\pi(g)(\Pi(f)\xi) = \Pi({}_g f)\xi, \quad \forall f\in L^1(G),\xi\in H.$$

Because of ${}_{\mathbb{1}_G}f = f$ and ${}_{gh}f = {}_g({}_h f)$, we have $\pi(\mathbb{1}_G)=\mathbb{1}_H$ and $\pi(gh)=\pi(g)\pi(h)\ \forall g,h\in G$. So we have that $\pi : G\to U(H)$ is a homomorphism of groups. Since $\Pi : L^1(G)\to L(H)$ is norm decreasing and $g\mapsto {}_g f$ is continuous, it follows that also $g\mapsto\pi(g)(\Pi(f)\xi)=\Pi({}_g f)\xi$ is continuous $\forall\eta\in H_0$ and after a $\varepsilon/3$ argument, continuity holds for all $\eta\in H$.
We thus have that $\pi : G\to U(H)$ is a strongly continuous unitary representation of G.

- It remains to show that $\Pi=\tilde{\pi}$, for this it suffices to show that $\forall f,\varphi\in L^1(G),\xi,\eta\in H$:

$$\langle\tilde{\pi}(f)\Pi(\varphi)\xi,\eta\rangle = \langle\Pi(f*\varphi)\xi,\eta\rangle = \langle\Pi(f)\Pi(\varphi)\xi,\eta\rangle$$

that is, since with $\overline{\Pi(L^1(G))H}=H$ we get $\tilde{\pi}(f)=\Pi(f)$.
For fixed $\xi,\eta\in H$ the linear functional $\varphi\mapsto\langle\Pi(\varphi)\xi,\eta\rangle$ is continuous on $L^1(G)$. Because of $L^1(G)'=L^\infty(G)$, there exists a $\phi\in L^\infty(G)$, such that

$$\langle\Pi(\varphi)\xi,\eta\rangle = \int_G \varphi(l)\phi(l)dl, \quad \forall\varphi\in L^1(G).$$

With this we conclude that $\forall f, \varphi \in L^1(G)$ it holds that

$$\begin{aligned}
\langle \tilde{\pi}(f)\Pi(\varphi)\xi, \eta\rangle &= \int_G f(g)\,\langle \Pi(_g\varphi)\xi,\eta\rangle\, dg = \int_G f(g) \int_G {}_g\varphi(l)\phi(l) dl dg \\
&\overset{\text{Fubini}}{=} \int_G f(g) \int_G \varphi(g^{-1}l) dg\, \phi(l) dl = \int_G (f*\varphi)(l)\phi(l) dl \\
&= \langle \Pi(f*\varphi)\xi,\eta\rangle\,.
\end{aligned}$$

□

Remark 3.2.11 *It is easy to see, that the bijection $\pi \mapsto \tilde{\pi}$ preserves all important characteristics, like e.g. unitary equivalence, irreducibility, direct sums, etc. In particular we have the bijection*

$$\boxed{\hat{G} \longleftrightarrow \widehat{L^1(G)}}$$

Theorem 3.2.12 *Let $\lambda : G \to U(L^2(G))$ be the left regular representation of G, i.e. $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$. It then holds that*

$$\tilde{\lambda}(f)\xi = f * \xi \quad \forall f \in L^1(G),\ \xi \in C_c(G) \subseteq L^2(G)$$

and λ is faithful on $L^1(G)$, i.e. $\lambda(f) = 0 \;\Rightarrow\; f = 0$.

Proof: For all $\xi, \eta \in C_c(G) \subseteq L^2(G)$ we have

$$\begin{aligned}
\langle \lambda(f)\xi,\eta\rangle &= \int_G f(g)\,\langle \lambda(g)\xi,\eta\rangle\, dg = \int_G f(g)\int_G \xi(g^{-1}l)\overline{\eta(l)} dl dg \\
&\overset{\text{Fubini}}{=} \int_G (\int_G f(g)\xi(g^{-1}l)dg)\overline{\eta(l)} dl = \langle f*\xi,\eta\rangle\,.
\end{aligned}$$

Since $C_c(G)$ is dense in $L^2(G)$, it follows that $\lambda(f)\xi = f * \xi$. If now $f \in L^1(G)$ with $\lambda(f) = 0$, then $f * \xi = 0\ \forall \xi \in C_c(G)$, since $f * \xi$ is continuous. If then $(\varphi_\lambda)_\lambda$ is an approximate unity as above, it then follows that $f = \lim_\lambda f * \varphi_\lambda = 0 \in L^1(G)$. □

Theorem 3.2.13 **The maximal group algebra** *Let G be a locally compact group. The following is a norm on $L^1(G)$*

$$\boxed{||f||_{C^*} := \{||\pi(f)||\,|\,\pi \text{ is nondegenerate } *-\text{representation of } L^1(G)\}}$$

With this norm the following (called maximal group algebra) is a C^*-algebra

$$\boxed{C^*(G) := \overline{L^1(G)}^{||\cdot||_{C^*}}}$$

*Furthermore the restriction of a *-representation $\pi : C^*(G) \to L(H)$ to the dense subalgebra $L^1(G)$:*

$$\pi \longmapsto \pi|_{L^1(G)}$$

is a bijection and conserves unitary equivalence, direct sums, etc. In particular

$$\boxed{\widehat{C^*(G)} \cong \widehat{L^1(G)} \cong \hat{G}}$$

Proof: We know that for every *-representation $\pi : L^1(H) \to L(H)$ we have $||\pi(f)|| \le ||f||_1$. Thus there exists a $||f||_{C^*} = \sup_\pi ||\pi(f)|| \le ||f||_1$. Since $f \mapsto ||\pi(f)||$ is a semi norm for all π, we also get that $||\cdot||_{C^*}$ is a semi norm, and since

$$||f||_{C^*} \ge ||\lambda(f)|| \quad \text{and} \quad \lambda(f) = 0 \;\Rightarrow\; f = 0$$

we have that $||\cdot||_{C^*}$ is a norm. For $f \in L^1(G)$ we have

$$||f^* * f||_{C^*} = \sup_\pi ||\pi(f^* * f)|| = \sup_\pi ||\pi(f^*)\pi(f)|| = \sup_\pi ||\pi(f)||^2 = ||f||^2.$$

Thus $C^*(G) := \overline{L^1(G)}^{||\cdot||_{C^*}}$ is indeed a C^*-algebra.
If $\pi : L^1(H) \to L(H_\pi)$ is an arbitrary *-representation of $L^1(G)$, then $||\pi(f)|| \leq ||f||_{C^*}$ and thus π is continuous w.r.t. $||\cdot||_{C^*}$. Then there is a unique contiuation $\pi : C^*(G) \to L(H_\pi)$. This proves, that the restriction map $\pi \mapsto \pi|_{L^1(G)}$ is bijective. The rest is easily checked. $\square$

Definition 3.2.14 **Reduced group algebra** *The reduced group algebra is defined using the faithful left regular representation* $\lambda : L^1(G) \to L(L^1(G))$. *Setting*

$$\boxed{||f||_r := ||\lambda(f)||}$$

we define the reduced group algebra to be

$$\boxed{C_r^*(G) := \overline{L^1(G)}^{||\cdot||_r}}$$

which is also a C^**-algebra.*

Lemma 3.2.15 *The continuation of* λ *to* $C^*(G)$ *gives a surjective* $*$*-homomorphism*

$$\lambda : C^*(G) \longrightarrow C_r^*(G) \subseteq L(L^1(G)).$$

Definition 3.2.16 **Amenable group** *A group* G *is called amenable, iff the above surjection is also an injection, i.e. iff*

$$C^*(G) \xrightarrow{\cong} C_r^*(G).$$

Remark 3.2.17

- *All abelian and all compact groups are amenable.*
- $GL(n,\mathbb{R})$ *is not amenable.*
- *Since* $C_r^*(G) \cong C^*(G)/\ker(\lambda)$ *is a quotient, we can identify* C_r^* *with a closed subset*

$$\widehat{G}_r \subseteq \widehat{G} \cong \widehat{C^*(G)}.$$

- *One can choose the topology on* $\widehat{G}$ *such that the bijection* $\widehat{G} \cong \widehat{C^*(G)}$ *becomes a homeomorphism.*

3.3 Duality Theorem of Abelian Groups

The duality we shall establish in this section is

$$\boxed{G \text{ abelian locally compact} \quad \Rightarrow \quad \widehat{\widehat{G}} \cong G}$$

Lemma 3.3.1 *Let* G *be an abelian locally compact group and* $\pi : G \to L(H_\pi)$ *an irreducible unitary representation, then*

$$\widehat{G} = \{\chi : G \to S^1 \mid \chi \text{ continuous homomorphism}\}.$$

Proof: Since G is abelian, we have $\pi(g)\pi(h) = \pi(h)\pi(g)$ and thus with Schur $\pi(g) = \chi(g)\mathbb{1}_{H_\pi}$ for some $\chi(g) \in S^1$, since π is unitary. But then $\forall g$ every subspace of H_π is $\pi(g)$ invariant and we have $\dim(H_\pi) = 1$. $\square$

Definition 3.3.2 **Dual Group** *Let G be an abelian locally compact group, then*

$$\widehat{G} = \{\chi : G \to S^1 \,|\, \chi \text{ continuous homomorphism}\}, \qquad (\chi \cdot \mu)(g) := \chi(g)\mu(g) \quad \forall \chi, \mu \in \widehat{G}$$

is called the dual group, which is a group with neutral element and inverse:

$$\mathbb{1}_{\widehat{G}} : G \to \{1\}, \quad \chi^{-1} := \overline{\chi}.$$

Remark 3.3.3 **Fourier Transformation** *In the last section, we established the isomorphism*

$$\begin{array}{rcl} \widehat{G} = \{\chi : G \to S^1\} & \xrightarrow{\cong} & \widehat{L^1(G)} \cong \widehat{C^*(G)} \\ \chi & \longmapsto & \tilde{\chi} \end{array} \qquad \tilde{\chi}(f) := \int_G f(g)\chi(g)dg.$$

Where the second isomorphism is given by unique extension. The Gelfand-transformation

$$\begin{array}{rcl} L^1(G) \overset{\text{dense}}{\subset} C^*(G) & \xrightarrow{\cong} & C_0(\widehat{C^*(G)}) \cong C_0(\widehat{G}) \\ f & \longmapsto & \hat{f} \end{array}$$

gives the following isomorphism, called the Fourier transformation:

$$\begin{array}{rcl} C^*(G) & \xrightarrow{\cong} & C_0(\widehat{G}) \\ f & \longmapsto & \hat{f} \end{array} \qquad \hat{f}(\chi) = \tilde{\chi}(f) = \int_G f(g) \underbrace{\chi(g)}_{\in S^1} dg.$$

It holds that $\widehat{f_1 * f_2}(\chi) = \chi(f_1 * f_2) = \chi(f_1)\chi(f_2) = \hat{f}_1(\chi)\hat{f}_2(\chi)$ *and the following commutes*

$$\begin{tikzcd}
L^1(G) \arrow[r, "\wedge"] \arrow[dr] & C_0(\widehat{G}) \\
& C^*(G) \arrow[u, "\wedge"']
\end{tikzcd}$$

So we can identify $C^*(G)$ with $C_0(\widehat{G})$, where $\widehat{G}$ is given the topology of $\widehat{C^*(G)}$, i.e. the topology of pointwise convergence.

Remark 3.3.4 **Problem** *We need to check if with the above topology $\widehat{G}$ is a locally compact group. In particular, we need to assure that multiplication and the inverse map are continuous.*

Lemma 3.3.5 *The map $\widehat{C^*(G)} \to \widehat{L^1(G)}$ $\chi \mapsto \chi|_{L^1(G)}$ is a homeomorphism, where both spaces are taken to carry the weak-$*$-topology.*

Proof: It suffices to show that if $(\chi_\lambda)_{\lambda \in \Lambda}$ is a net in $\widehat{G}$ and $\chi \in \widehat{G}$ with $\chi_\lambda(f) \to \chi(f)$ $\forall f \in L^1(G)$, then also

$$\chi_\lambda(x) \to \chi(x) \quad \forall x \in C^*(G).$$

But if $x \in C^*(G)$ and $\varepsilon > 0$, then there exists a $f \in L^1(G)$ with $||f - x||_{C^*} < \frac{\varepsilon}{3}$. If then $\lambda_0 \in \Lambda$ with $||\chi_\lambda(f) - \chi(f)|| < \frac{\varepsilon}{3}$ $\forall \lambda \geq \lambda_0$, then $\forall \lambda \geq \lambda_0$:

$$||\chi_\lambda(x) - \chi(x)|| \leq ||\chi_\lambda(x) - \chi_\lambda(f)|| + ||\chi_\lambda(f) - \chi(f)|| + ||\chi(f) - \chi(x)|| \leq ||x - f||_{C^*} + \frac{\varepsilon}{3} + ||f - x||_{C^*} < \varepsilon.$$

□

We shall now give an alternative description of the topology on $\widehat{G}$:

Lemma 3.3.6 *Let G be an abelian locally compact group, $(\chi_\lambda)_{\lambda\in\Lambda}$ a net in $\widehat{G}$ and $\chi \in \widehat{G}$, then the following are equivalent:*

(1) $\chi_\lambda \to \chi_0$ *in* $\widehat{L^1(G)} \cong \widehat{C^*(G)}$.

(2) $\forall\varepsilon > 0$ *and* $\forall K \subseteq G$ *compact there is a* $\lambda \in \Lambda$ *with* $|\chi_\lambda(g) - \chi_0(g)| < \varepsilon\ \forall g \in K$ *(i.e.* $\chi_\lambda \to \chi_0$ *uniformally convergent on compact subsets of* G*).*

Proof:

(2) ⇒ (1) If (2) holds and $0 \neq \varphi \in C_c(G)$, then choose a $\varepsilon > 0$ and a $\lambda_0 \in \Lambda$ with $|\chi_\lambda(g) - \chi(g)| < \frac{\varepsilon}{\mu(K)||\varphi||_\infty}$ for all $g \in K := \mathrm{supp}(\varphi)$ and $\lambda \geq \lambda_0$. Then for all $\lambda \geq \lambda_0$ it follows, that

$$\begin{aligned}|\chi_\lambda(\varphi) - \chi_0(\varphi)| &= |\int_G \varphi(g)(\chi_\lambda(g) - \chi_0(g))dg| \leq \int_K |\varphi(g)||\chi_\lambda(g) - \chi_0(g)|dg \\ &\leq \mu(K)||\varphi||_\infty \frac{\varepsilon}{\mu(K)||\varphi||_\infty} = \varepsilon.\end{aligned}$$

Since $C_c(G)$ is dense in $L^1(G)$, we also have an analogous statement for $f \in L^1(G)$.

(1) ⇒ (2) We now have $\chi_\lambda \to \chi_0$ in $\widehat{L^1(G)} \cong \widehat{C^*(G)}$, then $\chi_\lambda\overline{\chi}_0 \to \chi_0\overline{\chi}_0 = \mathbb{1} \in \widehat{L^1(G)}$, since for $f \in L^1(G)$:

$$\chi_\lambda\overline{\chi}_0(f) = \int_G f(g)\chi_\lambda(g)\overline{\chi}_0(g)dg = \chi_\lambda(f\overline{\chi}_0) \to \chi_0(f\overline{\chi}_0) = \chi_0\overline{\chi}_0(f) = 1(f).$$

And due to $\chi_\lambda(g) - \chi_0(g) = |\chi_\lambda(g)\overline{\chi}_0(g) - 1|$ it holds that $\chi_\lambda \to \chi_0$ in the sense of (2) ⇔ $\chi_\lambda(g)\overline{\chi}_0 \to 1$ in the sense of (2). By using the net $(\chi_\lambda\overline{\chi}_0)_\lambda \in \widehat{G}$ w.l.o.g. we can assume that $\chi_0 = 1$. We thus take $\chi_\lambda \to 1$ in $\widehat{L^1(G)}$. Let $K \subseteq G$ be compact and $\varepsilon > 0$.
Select $\varphi \in C_c(G)$ with $1(\varphi) = \int_G \varphi(g)dg = 1$, then there exists a neighborhood W of $\mathbb{1}_G$ with $||_g\varphi - {}_{g_0}\varphi||_1 < \frac{\varepsilon}{3}$ for all $g, g_0 \in G$ with $g \in g_0W$. If then $\chi \in \widehat{G}$ with

$$|\chi(\varphi) - \underbrace{1(\varphi)}_{=1}| < \frac{\varepsilon}{3},\ |\chi({}_{g_0}\varphi) - \underbrace{1({}_{g_0}\varphi)}_{=1}| < \frac{\varepsilon}{3}.$$

So for all $g \in g_0W$ it follows that

$$\begin{aligned}|\chi(g) - 1| &= |\chi(g) - \chi(g)\chi(\varphi)| + |\chi(g)\chi(\varphi) - \chi(g_0)\chi(\varphi)| + |\chi(g_0)\chi(\varphi) - 1| \\ &= |\chi(g)||1 - \chi(\varphi)| + |\chi({}_g\varphi) - \chi({}_{g_0}\varphi)| + |\chi({}_{g_0}\varphi) - 1| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Where $\chi(g)\chi(\varphi) = \chi({}_g\varphi)\ \forall g \in G$ follows from inserting and left invariance of the Haar-integral.
Since K is compact, there are $g_1, \ldots, g_l \in G$ such that $K = \cup_{i=1}^l g_iW$. Define

$$f_0, \ldots, f_l \in L^1(G), \quad f_0 := \varphi 1,\ f_i := {}_{g_i}\varphi.$$

If then $\lambda_0 \in \Lambda$ such that $|\chi(f_i) - 1(f_i)| < \frac{\varepsilon}{3}\ \forall 0 \leq i \leq l$ and $\lambda \geq \lambda_0$, then with $|\chi(\varphi) - 1| < \varepsilon$, we have that $|\chi_\lambda(\varphi) - 1| < \varepsilon\ \forall g \in K$ and $\lambda \geq \lambda_0$.

□

Corollary 3.3.7 *If G is a locally compact abelian group, then its dual group $\widehat{G}$ is also locally compact abelian.*

Proof: Let $\chi_\lambda \to \chi$ $\mu_\lambda \to \mu$ be in $\widehat{G}$ and $K \subseteq G$ compact and $\varepsilon > 0$, then there exists $\lambda_0 \in \Lambda$ such that $|\chi_\lambda - \chi| < \frac{\varepsilon}{2}$ and $|\mu_\lambda - \mu| < \frac{\varepsilon}{2}$ $\forall g \in K, \lambda \geq \lambda_0$. For $g \in K$ and $\lambda \geq \lambda_0$ it then holds, that

$$|\chi_\lambda\mu_\lambda(g) - \chi\mu(g)| \leq |\chi_\lambda(g)||\mu_\lambda(g) - \mu(g)| + |\mu(g)||\chi_\lambda(g) - \chi(g)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that the multiplication is continuous. And due to $|\overline{\chi}_\lambda(g) - \overline{\chi}(g)| = |\chi_\lambda\mu_\lambda(g) - \chi\mu(g)|$, we also have the continuity of the map $\chi \mapsto \overline{\chi}$. □

Theorem 3.3.8 *Let G be a locally compact abelian group, then*

(1) G compact $\Rightarrow$ $\widehat{G}$ discrete.

(2) G discrete $\Rightarrow$ $\widehat{G}$ compact.

Proof:

(1) If G is compact, then w.l.o.g. $\int_G 1dg = 1$ and thus $\int_G \chi(g)dg = 0$ $\forall 1 \neq \chi \in \widehat{G}$, since if $g_0 \in G$ with $\chi(g_0) \neq 1$, then

$$\int_G \chi(g)dg = \int_G \chi(g_0 g)dg = \chi(g_0)\int_G \chi(g)dg \quad \overset{\chi(g_0)\neq 1}{\Rightarrow} \quad \int_G \chi(g)dg = 0.$$

Let now $(\chi_\lambda)_\lambda \to 1$ is a net in $\widehat{G}$. We know that $(\chi_\lambda)_\lambda \to 1$ converges uniformly and thus $\chi_\lambda = 1$ for all $\lambda \geq \lambda_0$, since $\int_G \chi_\lambda(g)dg \to \int_G 1dg = 1$. Thus $\{1\} \subset \widehat{G}$ is open, and since $\widehat{G}$ is a topological group, we also have $\{\chi\} \subset \widehat{G}$ is open for all $\chi \in \widehat{G}$.

(2) Let now G be discrete. Then $\delta_1 * f = f * \delta_1 = f$ $\forall f \in L^1(G)$, so $L^1(G)$ is unital, but then $\widehat{G} \cong \widehat{L^1(G)}$ is compact.

□

Lemma 3.3.9 *Let $G = G_1 \times \cdots \times G_n$ be a locally compact abelian group, then*

$$\begin{array}{ccc} \widehat{G}_1 \times \cdots \times \widehat{G}_n & \overset{\cong}{\longrightarrow} & \widehat{G}, \\ (\chi_1, \ldots, \chi_n) & \longmapsto & \chi_1 \cdots \chi_n \end{array}$$

with $(\chi_1 \cdots \chi_n)(g) := \chi_1(g) \cdots \chi_n(g)$.

Lemma 3.3.10 *Let $H \subseteq G$ be a closed subgroup of the locally compact abelian group G. Then*

$$\widehat{G/H} \cong \{\chi \in \widehat{G} \,|\, \chi|_H = 1\}.$$

Proof: Fundamental theorem on homomorphisms. □

Lemma 3.3.11 *Let G be compact abelian. If $A \subseteq \widehat{G}$ separates the points of G, then $\langle A\rangle = \widehat{G}$.*

Proof: Let w.l.o.g. $A = \langle A\rangle$ and let $D \subseteq C(\widehat{G})$ the subalgebra of $C(\widehat{G})$ generated by A. Since $1 \in A$ and if $\chi \in A$, then also $\overline{\chi} = \chi^{-1} \in A$. Since A (and thus also D) separates the points of G, we have that $D \subseteq C(\widehat{G})$ is dense w.r.t. $||\cdot||_\infty$ by Stone-Weierstraß.
We assume that $\exists \chi \in \widehat{G}/A$. Then $\mu \cdot \overline{\chi} \neq 1$ $\forall \mu \in A$. We have already shown that

$$\int_G \mu \cdot \overline{\chi}(g)dg = 0 \quad \forall \mu \in A.$$

Every element in D is of the form $\sum_{i=1}^m \alpha_i\chi_i$ with $\alpha_1 \in \mathbb{C}$, $\chi_i \in A$ and thus

$$\int_G \mu(g)\varphi(g)dg = 0 \quad \forall \varphi \in D.$$

Since $D \subseteq C(\widehat{G})$ is dense w.r.t. $||\cdot||_\infty$, we also have

$$\int_G \mu(g)\varphi(g)dg = 0 \quad \forall \varphi \in C(\widehat{G}).$$

In particular is follows that $1 = \int_G 1dg = \in_G \mu(g)\overline{\mu}(g)dg = 0$, which is a contradiction.

$\square$

Example 3.3.12

(1) It holds that

$$\boxed{S^1 \xrightarrow{\cong} \widehat{\mathbb{Z}}, \quad z \mapsto \chi_z, \quad \chi_z(n) := z^n}$$

If $\chi(n) = \chi(1)^n = z^n$, i.e. $\chi = \chi_z$ and thus $\widehat{\mathbb{Z}} \to S^1$ $\chi \mapsto \chi(1)$ is a continuous bijection and since $\widehat{\mathbb{Z}}, S^1$ are compact, it is a homeomorphism.

(2) It holds that

$$\boxed{\mathbb{Z} \xrightarrow{\cong} \widehat{S^1}, \quad n \mapsto \chi_n, \quad \chi_n(z) := z^n}$$

χ_1 separates the points of S^1, and since S^1 is compact, it follows that $\widehat{S^1} = \langle \chi_1 \rangle = \{\chi_n \,|\, n \in \mathbb{Z}\}$.

(3) It holds that

$$\boxed{\mathbb{R} \xrightarrow{\cong} \widehat{\mathbb{R}}, \quad s \mapsto \chi_s, \quad \chi_s(t) := e^{-1st}}$$

Let $\chi \in \widehat{\mathbb{R}}$ be arbitrary. Then $A := \ker \chi$ is a closed subgroup of $\mathbb{R}$, thus $A \in \{\{0\}, \mathbb{R}, b\mathbb{Z}\}$.

- *if $A = \{0\}$, then $1 = \chi(0) \neq \chi(1) := z$ and since χ is continuous, $\chi([0,1])$ is connected and contains a true arch in S^1, which spans from 1 to z, so $\chi(na) = 1$ with $na \neq 0$, which is in contradiction to $A = \{0\}$.*
- *If $A = \mathbb{R}$, then $\chi = 1$, i.e. $\chi = \chi_0$.*
- *If $A = b\mathbb{Z}$, $b > 0$, then $\tilde{\chi}(t + A) = \chi(t)$ defines a character on $\mathbb{R}/A$ and $\mathbb{R}/A \cong S^1$ via $t + A \mapsto e^{-it2\pi/b}$. So we get a $\tilde{\chi} \in \widehat{S^1}$ by setting $\tilde{\chi}(^{-it2\pi/b}) = (^{-it2\pi/b})^n =^{-it2\pi n/b}= \chi_s(t)$ with $s = \frac{2\pi n}{b}$.*

(4) It holds that

$$\boxed{\mathbb{Z}_m \cong C_m := \{\omega \in S^1 \,|\, \omega^m = 1\} \xrightarrow{\cong} \widehat{\mathbb{Z}}_m, \quad \omega \mapsto \chi_\omega, \quad \chi_\omega([n]) := \omega^n}$$

Theorem 3.3.13 **Structure Theorem for finitely generated Abelian Groups** *If G is finitely generated abelian, then there exist $l, m_1, \ldots, m_r \in \mathbb{N}$ such that*

$$\boxed{G \cong \mathbb{Z}^l \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r} \quad \Rightarrow \quad \widehat{G} \cong (S^1)^l \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}}$$

Definition 3.3.14 **Elementary Group** *A locally compact abelian group such that*

$$G \cong \mathbb{R}^n \times (S^1)^l \times \mathbb{Z}^r \times F$$

with some finite group F is called elementary.

Lemma 3.3.15 *Elementary groups $G \cong \mathbb{R}^n \times (S^1)^l \times \mathbb{Z}^r \times F$ are reflexive:*

$$\widehat{G} \cong \mathbb{R}^n \times \mathbb{Z}^l \times (S^1)^r \times F, \quad \Rightarrow \quad \widehat{\widehat{G}} \cong G$$

Theorem 3.3.16 **Pontrjagin + Plancherel 1940** *Let G be a locally compact abelian group, then the following hold*

(1) (Pontrjagin Theorem) $\boxed{\varphi : G \xrightarrow{\cong} \widehat{\widehat{G}}}$ with $\varphi(g)(\chi) := \chi(g)$, $\chi \in \widehat{G}$ is a topological isomorphism of groups.

(2) *(Inversion frormula) The Haar integral on $\widehat{G}$ can be normalized such that for $\hat{f} \in L^1(\widehat{G})$ we have*

$$f(g) = \int \widehat{G}\hat{f}(\chi)\overline{\chi(g)}d\chi \quad \text{for almost all } g \in G.$$

(3) *(Plancherel Theorem) Let $f \in L^1(G) \cap L^2(G)$, then $\hat{f} \in L^2(\widehat{G})$ with $||\hat{f}||_2 = ||f||_2$. This defines an isometry*

$$\mathcal{F} : L^1(G) \cap L^2(G) \longrightarrow L^2(\widehat{G}), \quad \mathcal{F}(f) := \hat{f},$$

which has a unique continuation $F : L^2(G) \to L^2(\widehat{G})$. If

$$\hat{F} : L^2(\widehat{G}) \longrightarrow L^2(\widehat{\widehat{G}}) \cong L^2(G)$$

is the corresponding Fourier transform for $\widehat{G}$, then

$$\widehat{F} \circ F(f) = f(g) = f(g^{-1}) \quad \forall f \in L^2(G).$$

Corollary 3.3.17 *Let G be a locally compact abelian group, then the regular representation*

$$\lambda : C^*(G) \xrightarrow{\cong} C^*_r(G)$$

is an isomorphism.

Proof: We want to use the Pointrjagin-Plancherel theorem to prove the assertion. We consider the following diagram

$$\begin{tikzcd} C^*(G) \arrow[r, "\lambda"] \arrow[d, "\wedge"'] & C^*_r(G) \subseteq L(L^2(G)) \arrow[d, "F\circ(\cdot)\circ F^{-1}"] \\ C_0(\widehat{G}) \arrow[r, "M"'] & L(L^2(\widehat{G})) \end{tikzcd}$$

where $M : C_0(\widehat{G}) \to L(L^2(\widehat{G}))$ is the multiplicative representation $(M\varphi)\xi = \varphi \cdot \xi$. Of all arrows, but λ, we know that they are isomorphisms, thus we only need to prove commutativity of the diagram.

For all $\xi \in C_c(G) \overset{\text{dense}}{\subseteq} L^2(G)$ and $f \in C_c(G)$, we have

$$\lambda(f)\xi = f * \xi.$$

And thus

$$F(\lambda(f)\xi) = \widehat{f * \xi} = \hat{f}\hat{\xi} = M(\hat{f})F(\hat{\xi}).$$

This gives $\lambda(f) = F^{-1}M(\hat{f})F$, i.e. $M(\hat{f}) = F\lambda(f)F^{-1}$. This gives $C_c(G) \overset{\text{dense}}{\subset} C^*(G)$ and everything is continuous the claim follows.

3.4 Universal C^*-Algebras and the noncommutative 2-Torus

One of the main lessons of this sections is, that the noncommutative 2-torus A_θ has a very different structure, depending of wether $\theta \in \mathbb{Q}$ or $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

There are many mathematical objects, to which one can assign C^*-algebra in a natural way, such that the algebra mirrors the properties of these objects:

- (C^*)-dynamical systems
- Foliations of manifolds

- Semigroups
- Groupoids
- Rings (see Cuntz and Xin Li)

Many of these constructions are of universal nature. E.g. for a discrete group G, the group algebra $C^*(G)$ is the universal C^*-algebra generated by G.

Definition 3.4.1 **Representation of a set X with relations** *Let X be a set and $\mathcal{R}$ a set of relations on X. A map*

$$\varphi : (X, \mathcal{R}) \longrightarrow B$$

is called a representation of $(X, \mathcal{R})$ if φ preserves all relations $\mathcal{R}$ of X in B.

Example 3.4.2 $X = \{u, v\}$, $\mathcal{R} = \{u^* = u^{-1},\ v^* = v^{-1},\ uv = vu\}$

Remark 3.4.3 *Not all pairs $(X, \mathcal{R})$ are representable. Not all relations make sense in a C^*-algebra, e.g. they could be contradictory to the defining relations of a C^*-algebra.*

Definition 3.4.4 **Universal $\mathbf{C*-}$algebra** *A C^*-algebra is called universal for $(X, \mathcal{R})$, if the following hold:*

(1) **Existence:** *There exists a representation $i_X : (X, \mathcal{R}) \to A$.*

(2) **Minimality:** *$A = C^*(i_X(X))$, i.e. there is no true C^*-subalgebra of A, which includes $i_X(X)$.*

(3) **Universality:** *If $\varphi : (X, \mathcal{R}) \to B$ is an arbitrary other representation, then there exists exactly one $*$-homomorphism $\phi : A \to B$ such that the following commutes:*

$$\begin{tikzcd} (X, \mathcal{R}) \arrow[r, "i_X"] \arrow[dr, "\varphi"'] & A \arrow[d, dotted, "\exists! \phi"] \\ & B \end{tikzcd}$$

We denote the universal algebra of $(X, \mathcal{R})$ by $\boxed{A := C^*(X, \mathcal{R})}$

Remark 3.4.5 **Existence** *It is not always clear if for a given pair $(X, \mathcal{R})$, there exists corresponding a universal C^*-algebra. However, a necessary condition is that there exists some representation $\varphi : (X, \mathcal{R}) \to B$.*

Lemma 3.4.6 **Uniqueness** *If A, B are both universal for $(X, \mathcal{R})$, then $A \cong B$.*

Proof: Let $i_X : X \to A$, $j_X : X \to B$ be representations, as in (1). According to the universality of A (3), there exists a unique $*$-homomorphism $\phi : A \to B$ such that $\phi(i_X(x)) = j_X(x)\ \forall x \in X$. By the universality of B there is also a unique $*$-homomorphism $\psi : B \to A$ such that $\phi(j_X(x)) = i_X(x)\ \forall x \in X$. Thus $\psi \circ \phi : A \to A$ is a $*$-homomorphism with $\psi \circ \phi(i_X(x)) = i_X(x)$ and due to uniqueness in (3) we get $\psi \circ \phi = \mathbb{1}_A$. Analogously $\phi \circ \psi = \mathbb{1}_B$ $\square$

Example 3.4.7

(i) **Group algebras** *Let G be a discrete group. We give G the relations coming from the group operation and $g^* = g^{-1}$. A representation $\varphi : (G, \mathcal{R}) \to B$ is then a homomorphism*

$$\varphi : G \longrightarrow U(B) := \{u \in B \,|\, u \text{ unitary}\}.$$

It then follows that

$$C^*(G, \mathcal{R}) = C^*(G).$$

That is since

(1) $i_G : G \to C^*(G)$ $i_G(g) = \delta_g$.

(2) Since $\mathrm{LH}\{\delta_g \,|\, g \in G\} \overset{\text{dense}}{\subset} l^1(G)$, *we also have* $\mathrm{LH}\{\delta_g \,|\, g \in G\} \overset{\text{dense}}{\subset} C^*(G)$.

(3) If $\varphi : G \to U(B)$ is an arbitrary representation of $(G, \mathcal{R})$, then, with Gelfand-Naimark, we can interprete $B \subseteq L(H)$ for some Hilbert space H, so

$$\varphi : G \longrightarrow B \subseteq L(H)$$

as a unitary representation of G. And then there is exactly one $*$-representation

$$\phi : C^*(G) \longrightarrow L(H), \quad \text{such that} \quad \phi(\delta_g) = \varphi(g).$$

Since $\phi(\mathrm{LH}\{\delta_g \,|\, g \in G\}) \subseteq B$, it also holds that $\phi(C^*(G)) \subseteq B$.

(ii) $X := \{u\}$, $\mathcal{R} := \{u^*u = uu^* = 1\}$, *then*

$$C^*(\{u\}, \mathcal{R}) \cong C(S^1).$$

(1) Let $v \in C(S^1)$ given by $v(z) = z$, then $i_X(u) = v$ is a representation of $(X, \mathcal{R})$.

(2) With Stone-Weierstraß, we have that $C(S^1) = C^*(v)$.

(3) If B is an arbitrary C^*-algebra and $w \in B$ unitary (i.e. fulfills $\mathcal{R}$), then $\sigma(w) \subseteq S^1$. Consider the $*$-homomorphism $\phi : C(S^1) \to B$ given by the composition

$$\begin{array}{rcl} C(S^1) \longrightarrow & C(\sigma(w)) & \xrightarrow{\cong} C^*(w) \subseteq B. \\ f \longmapsto & f|_{\sigma(w)} & \longmapsto f(w) \end{array}$$

Due to $v = \mathbb{1}_{S^1}$ it holds that $\phi(v) = \mathbb{1}(w) = w$.

(iii) Alternative to (ii) *If $w \in B$ is unitary, then so is w^n $\forall n \in \mathbb{Z}$. Every unitary element $w \in B$ then generates a unitary representation $\varphi : \mathbb{Z} \to B$, $\varphi(n) = w^n$ and it thus follows, that*

$$C^*(\{u\}, \mathcal{R}) \cong C^*(\mathbb{Z}) \cong C_0(\widehat{\mathbb{Z}}) = C(S^1).$$

(iv) $X := \{u, v\}$, $\mathcal{R} := \{u^*u = uu^* = 1 = v^*v = vv^*, uv = vu\}$, *then*

$$C^*(\{u, v\}, \mathcal{R}) \cong C^*(\mathbb{Z}^2) \cong C_0(\widehat{\mathbb{Z}}^2) = C(S^1 \times S^1).$$

That is since every pair $\{u, v\}$ with the relations $\mathcal{R}$ induces a representation $\varphi : \mathbb{Z}^2 \to B$ $\varphi(n, m) = u^n v^m$ and vice versa: every representation φ gives the unitary elements $u = \varphi(\delta_{(1,0)})$, $v = \varphi(\delta_{(0,1)})$.

Definition 3.4.8 **Noncommutative 2 – Torus** *is the universal algebra*

$$\boxed{A_\theta := C^*(X_\theta, \mathcal{R}_\theta)}$$

for the set and relations

$$X_\theta := \{u, v\}, \quad \mathcal{R}_\theta := \{u^*u = uu^* = 1 = v^*v = vv^*, uv = e^{2\pi i\theta} vu\}.$$

Remark 3.4.9 *Observe that we have already treaded the case A_0 above:*

$$A_0 := C^*(\{u,v\},\mathcal{R}_0) \cong C(S^1 \times S^1).$$

This is just the commutative algebra of the continuous functions on the 2-torus $S^1 \times S^1$, we thus understand why A_θ is referred to as the noncommutative 2-torus.

Remark 3.4.10 *The noncommutative 2-torus is a standard example in noncommutative geometry.*

Remark 3.4.11 **Constructive Definition** *Let $f \in C_c(\mathbb{Z}^2) = \{f : \mathbb{Z}^2 \to \mathbb{C} \,|\, \mathrm{supp}(f) = \text{finite}\}$, then take the formal sums*

$$\mathcal{A}_\theta := \{\sum_{\mathbb{Z}^2} f(n,m)u^n v^m \,|\, f \in C_c(\mathbb{Z}^2)\}.$$

The above relations $\mathcal{R}_\theta$ hold, so

$$u^n v^m u^l v^k = e^{-2\pi i\theta ml} u^{n+l} v^{m+k},$$

thus $\mathcal{A}_\theta$ is an algebra with

$$(\sum_{\mathbb{Z}^2} f(n,m)u^n v^m)(\sum_{\mathbb{Z}^2} g(n,m)u^n v^m) = \sum_{\mathbb{Z}^2}(f *_\theta g)(n,m)u^n v^m$$

and the product

$$(f *_\theta g)(n,m) := \sum_{(k,l)\in\mathbb{Z}^2} f(k,l)g(n-k,m-l)e^{-2\pi i\theta l(n-k)}.$$

Because of

$$(\sum_{\mathbb{Z}^2} f(n,m)u^n v^m)^* = \sum_{\mathbb{Z}^2} \overline{f(n,m)} v^{-m}u^{-n} = \sum_{\mathbb{Z}^2} \overline{f(n,m)} e^{-2\pi i\theta mn} u^{-n} v^{-m},$$

we get that $\mathcal{A}_\theta$ is a $*$-algebra. For an arbitrary representation $\varphi : \{u,v\} \to B$ of $(X_\theta, \mathcal{R}_\theta)$, we have that

$$\phi : \mathcal{A}_\theta \longrightarrow B, \quad \phi(\sum_{\mathbb{Z}^2} f(n,m)u^n v^m) := \sum_{\mathbb{Z}^2} f(n,m)\varphi(u)^n \varphi(v)^m$$

is a $*$-representation of $\mathcal{A}_\theta$ with

$$||\phi(\sum_{\mathbb{Z}^2} f(n,m)u^n v^m)|| \le \sum_{\mathbb{Z}^2} ||f(n,m)\varphi(u)^n\varphi(v)^m|| = \sum_{\mathbb{Z}^2} |f(n,m)| =: ||f||_1.$$

So we can define

$$||\sum_{\mathbb{Z}^2} f(n,m)u^n v^m||_{C^*} := \sup_\phi ||\phi(\sum_{\mathbb{Z}^2} f(n,m)u^n v^m)||,$$

which exists if there is such a ϕ. Finally we set

$$\boxed{A_\theta := \overline{\mathcal{A}_\theta}^{||\cdot||_{C^*}}}$$

From this constructive definition, it follows that A_θ together with the embedding $i_X : \{u,v\} \hookrightarrow A_\theta$ fulfills the conditions (1)-(3), if the relations $\mathcal{R}_\theta$ are realizable in at least one C^*-algebra, since otherwise $||\cdot||_{C^*}$ does not exist.

Example 3.4.12 *Let $B = L(l^2(\mathbb{Z}^2))$, define $U, V \in L(l^2(\mathbb{Z}^2))$ by*

$$\begin{aligned}(U\xi)(n,m) &:= \xi(n+1,m),\\ (V\xi)(n,m) &:= e^{2\pi i\theta n}\xi(n,m+1).\end{aligned}$$

Then U, V are unitary with

$$(UV\xi)(n,m) = (V\xi)(n+1,m) = e^{2\pi i\theta(n+1)}\xi(n+1,m+1) = e^{2\pi i\theta(n+1)}(U\xi)(n,m+1) = e^{2\pi i\theta}(VU\xi)(n,m).$$

Thus $UV = e^{2\pi i\theta}VU$ and we have found a representation

$$\varphi : (X_\theta, \mathcal{R}_\theta) \longrightarrow L(l^2(\mathbb{Z}^2)), \quad \varphi(u) := U,\ \varphi(v) := V.$$

Lemma 3.4.13

$$\boxed{\theta \in [0, \tfrac{1}{2}] \quad \Rightarrow \quad A_\theta \cong A_{1-\theta}}$$

.

Proof: Let $u, v \in A_\theta$ be the generators of A_θ and $\tilde{u}, \tilde{v} \in A_{1-\theta}$ the generators of $A_{1-\theta}$. Thus in particular

$$\tilde{u}\tilde{v} = e^{2\pi i(1-\theta)}\tilde{v}\tilde{u} = e^{-2\pi i\theta}\tilde{v}\tilde{u} \quad \Rightarrow \quad \tilde{v}\tilde{u} = e^{2\pi i\theta}\tilde{u}\tilde{v}.$$

There is exactly one $*$-homomorphism $\varphi : A_\theta \to A_{1-\theta}$ such that $\varphi(u) = \tilde{v}$, $\varphi(v) = \tilde{u}$. Analogously, due to

$$vu = e^{-2\pi i\theta}uv = e^{(2\pi i(1-\theta)}uv$$

there is exactly one $*$-homomorphism $\psi : A_{1-\theta} \to A_\theta$ such that $\psi(\tilde{u}) = v$, $\psi(\tilde{v}) = u$. It follows that

$$\psi \circ \varphi(u) = u,\ \psi \circ \varphi(v) = v$$

and thus $\psi \circ \varphi = \mathbb{1}_{A_\theta}$. Analogously it follows that $\varphi \circ \psi = \mathbb{1}_{A_{1-\theta}}$. $\square$

Remark 3.4.14 **Finite dimensional representations** *From the construction of A_θ it follows, that*

$$\dim(A_\theta) = \infty.$$

However, for $\theta = \frac{p}{q}$ with $\gcd(p,q) = 1$, we can define a family of finite dimensional representations

$$\pi_{(z,\omega)} : A_\theta \longrightarrow M_q(\mathbb{C}), \quad (z,\omega) \in S^1 \times S^1.$$

Let $\lambda = e^{2\pi i\theta}$ and let

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda^{q-1} \end{pmatrix}, \quad V := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Then U, V are unitary with

$$UV = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & \lambda^{q-2} \\ \lambda^{q-1} & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad VU = \begin{pmatrix} 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & \lambda^{q-1} \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

That is since $\lambda^q = e^{2\pi i\theta q} = e^{2\pi ip} = 1$. So there is exactly one $*$-homomorphism $\pi : A_\theta \to M_q(\mathbb{C})$ with $\pi(u) = U$, $\pi(v) = V$, one can prove that π is irreducible.

If now $(z,\omega) \in S^1 \times S^1$*, then* $zU, \omega V$ *are unitary, with*

$$(zU)(\omega V) = z\omega(UV) = z\omega e^{2\pi i\theta}(VU) = e^{2\pi i\theta}(\omega V)(zU)$$

so there is a representation

$$\pi_{(z,\omega)} : A_\theta \longrightarrow M_q(\mathbb{C}), \quad \pi_{(z,\omega)}(u) = zU,\, \pi_{(z,\omega)}(v) = \omega V.$$

One can then show, that all $\pi_{(z,\omega)}$ *are irreducible and*

$$\pi_{(z,\omega)} \cong \pi_{(z',\omega')} \quad \Leftrightarrow \quad z = \xi^k z',\, \omega = \xi^l \omega',\; \xi = e^{2\pi i \frac{1}{q}},\, k, l \in \mathbb{Z}.$$

And it holds that every irreducible representation of A_θ *is equivalent to some* $\pi_{(z,\omega)}$*.*
The moral of the story is, that for $\theta \in \mathbb{Q}$ *we have that* A_θ *is a CCR and it holds that*

$$\widehat{A}_\theta \cong \mathrm{Prim}(A_\theta) \cong (S^1 \times S^1)/\sim .$$

Where $\sim$ *is the above relation used in* $\pi_{(z,\omega)} \cong \pi_{(z',\omega')}$*. In particular it follows that* A_θ *has a lot of different ideals.*

We shall now study the case $\theta \in \mathbb{R} \setminus \mathbb{Q}$:

Lemma 3.4.15 *Let* $\theta \in \mathbb{R}$*, then for every* $(z,\omega) \in S^1 \times S^1$ *there is exactly one* $*$*-automorphism*

$$\beta_{(z,\omega)} : A_\theta \longrightarrow A_\theta, \quad \beta_{(z,\omega)}(u) := zu,\, \beta_{(z,\omega)}(v) := \omega v.$$

Proof: It holds that $(zu)(\omega v) = e^{2\pi i\theta}(\omega v)(zu)$ and thus there exists exactly one $*$-homomorphism $\beta_{(z,\omega)} : A_\theta \to A_\theta$ as in the lemma. Due to

$$\beta_{(\overline{z},\overline{\omega})} \circ \beta_{(z,\omega)}(u) = \beta_{(\overline{z},\overline{\omega})}(zu) = (\overline{z}zu) = u$$

and the analogous statement for v, we have $\beta_{(\overline{z},\overline{\omega})} = \beta_{(z,\omega)}^{-1}$. □

Remark 3.4.16 *The map*

$$\beta : S^1 \times S^1 \longrightarrow \mathrm{Aut}(A_\theta), \quad (z,\omega) \longmapsto \beta_{(z,\omega)}$$

is a homomorphism of groups.

Definition 3.4.17 **Inner Automorphism** *Let* A *be a unital* C^**-algebra and* $u \in A$ *unitary, then*

$$\boxed{u \longmapsto \mathrm{Ad}(u)(a) := uau^*}$$

is a $*$*-automorphism, called inner automorphism.*

Lemma 3.4.18 *Let* $\theta \in \mathbb{R}$ *and* $\lambda = e^{2\pi i\theta}$*, then*

$$\beta_{(\lambda^n,\lambda^m)} = \mathrm{Ad}(v^{-n}u^m), \quad \forall n, m \in \mathbb{Z}$$

In particular $\beta_{(\lambda^n,\lambda^m)}$ *is an inner automorphism* $\forall n, m \in \mathbb{Z}$*.*

Proof: Since $A_\theta = C^*(u,v)$, it suffices to show that $\beta_{(\lambda^n,\lambda^m)}(a) = (v^{-n}u^m)a(v^{-n}u^m)^*$ for $a = u, v$. We shall only prove the case $a = u$, the other is done in the exact same way.

$$(v^{-n}u^m)u(v^{-n}u^m)^* = v^{-n}u^m u u^{-m} v^n = v^{-n}uv^n = e^{2\pi 1\theta n}u = \lambda^n u = \beta_{(\lambda^n,\lambda^m)}(u)$$

□

Theorem 3.4.19 *If $H \subseteq \mathbb{R}$ is a closed subgroup, then $H \in \{\{0\}, \mathbb{R}, a\mathbb{Z}\}$ for some $a > 0$.*

Proof: Let $H \neq \{0\}$ and $a := \inf\{b \in H \mid b > 0\}$.

$\underline{a = 0}$ We will show that if $a = 0$, then $H = \mathbb{R}$. Let $x \in \mathbb{R}$ arbitrary and $c_x := \sup\{b \in H \mid b < x\}$, then $c_x \in H$, since H closed. Assume $c_x < x$, then $x - c_x > 0$ and there is a $b \in H$ with $0 < b < x - c_x$. But then $c_x < b + c_x < x$ and $b + c_x \in H$, since $b, c_x \in H$, which is a contradiction to $c_x = \sup\{b \in H \mid b < x\}$.

$\underline{a > 0}$ If $0 < a \in H$ we have $a\mathbb{Z} \subseteq H$. Assume that there is a $b \in H \setminus (a\mathbb{Z})$, then there is a $n \in \mathbb{Z}$ with $an < b < a(n+1)$, i.e. $0 < b - an < a$. But since $b - an \in H$ we have a contradiction to the definition of a.

$\square$

Corollary 3.4.20 *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda = e^{2\pi i\theta}$, then the following holds*

(1) $\mathbb{Z} + \theta\mathbb{Z}$ is a dense subgroup of $\mathbb{R}$.

(2) $\{\lambda^n \mid n \in \mathbb{Z}\}$ is a dense subgroup of S^1.

(3) $\{(\lambda^n, \lambda^m) \mid (n, m) \in \mathbb{Z}^2\}$ is a dense subgroup of $\mathbb{Z}^2$.

Proof:

(1) $\overline{\mathbb{Z} + \theta\mathbb{Z}}$ is a closed subgroup of $\mathbb{R}$. Assume $\exists a > 0 : \overline{\mathbb{Z} + \theta\mathbb{Z}} = a\mathbb{Z}$. Then there are $n, m \in \mathbb{Z}$ with $an = 1$, $am = \theta$. Thus $\frac{1}{n} = \frac{\theta}{m}$ and $\frac{m}{n} = \theta$ which is in contradiction to $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

(2) Consider $\varphi : \mathbb{R} \to S^1$ $\varphi(x) = e^{2\pi i x}$, then $\{e^{2\pi i\theta n} \mid n \in \mathbb{Z}\} = \varphi(\mathbb{Z} + \theta\mathbb{Z})$ which is dense in $\varphi(\mathbb{R}) = S^1$.

(3) If $D \subset X$ dense, then $D \times D \subset X \times X$ dense.

$\square$

Lemma 3.4.21 *For all $a \in A_\theta$ is holds that the following map is continuous:*

$$S^1 \times S^1 \longrightarrow A_\theta \quad (z, \omega) \longmapsto \beta_{(z,\omega)}.$$

Proof: Due to $\beta_{(z,\omega)}(u^n v^m) = (zu)^n(\omega v)^m = z^n\omega^m(u^n v^m)$ continuity follows for $a = u^n v^m$ and thus also for all

$$b = \sum_{n,m} f(n, m)u^n v^m, \quad f \in C_c(\mathbb{Z}^2).$$

If now $a \in A_\theta$ arbitrary and $(z_n, \omega_n) \to (z, \omega) \in S^1 \times S^1$, then there is a $\varepsilon > 0$ and a $b = \sum_{n,m} f(n, m)u^n v^m$ with $||b - a|| < \frac{\varepsilon}{3}$. Choose $N \in \mathbb{N}$ with $||\beta_{(z_n,\omega_n)}b - \beta_{(z,\omega)}b|| < \frac{\varepsilon}{3}$, then

$$||\beta_{(z_n,\omega_n)}a - \beta_{(z,\omega)}a|| \le ||\beta_{(z_n,\omega_n)}(a - b)|| + ||\beta_{(z_n,\omega_n)}b - \beta_{(z,\omega)}b|| + ||\beta_{(z,\omega)}(b - a)|| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

Definition 3.4.22 *With Gelfand-Naimark there exists a faithful representation $A_\theta \subset L(H)$ for some Hilbert space H. Then $(z, \omega) \mapsto \beta_{(z,\omega)}(a)$ is a continuous, operator valued function and we can define*

$$\boxed{E_\theta(a) := \int_{S^1 \times S^1} \beta_{(z,\omega)}(a) d(z, \omega) = \int_0^1 \int_0^1 \beta_{e^{2\pi i s}, e^{2\pi i t}} ds dt}$$

Approximating the integral by Riemannian sums, shows that $E_\theta(a) \in A_\theta$ $\forall a \in A_\theta$.

Theorem 3.4.23 *Let $E_\theta : A_\theta \to A_\theta$ as in the definition, then there is a state $\tau_\theta : A_\theta \to \mathbb{C}$ such that*

(1) $E_\theta(a) = \tau_\theta(a)\mathbb{1}\ \forall a \in A_\theta$.

(2) $\tau_\theta(ab) = \tau_\theta(ba)\ \forall a, b \in A_\theta$.

(3) $\tau_\theta(a^*a) > 0\ \forall 0 \neq a \in A_\theta$.

Proof:

(1) For $a = u^n v^m$ it holds that

$$\begin{aligned} E_\theta(u^n v^m) &= \int_{S^1\times S^1} \beta_{(z,\omega)}(u^n v^m) d(z,\omega) = \int_{S^1\times S^1} (zu)^n(\omega v)^m d(z,\omega) \\ &= \left(\int_{S^1\times S^1} z^n \omega^m d(z,\omega)\right) u^n v^m = \left(\int_0^1 \int_0^1 e^{2\pi isn} e^{2\pi itm} ds dt\right) u^n v^m \\ &= \begin{cases} 1, & \text{if } (n,m) \neq (0,0) \\ 0, & \text{if } (n,m) = (0,0). \end{cases} \end{aligned}$$

Thus $E_\theta(\sum_{n,m} f(n,m)u^n v^m) = f(0,0)\mathbb{1}$. $a \mapsto E_\theta(a)$ is continuous since $A_\theta \to C(S^1 \times S^1, A_\theta)$; $a \mapsto [(z,\omega) \mapsto]\beta_{(z\omega)}(a)$ is continuous as a *-homomorphism and $\int_{S^1\times S^1} : C(S^1 \times S^1, A_\theta) \to A_\theta$ is continuous. The continuity of $a \mapsto E_\theta(a)$ and the fact that $\mathbb{C}\mathbb{1}$ is closed in A_θ give $E_\theta(a) \in \mathbb{C}\mathbb{1}\ \forall a \in A_\theta$.
So we have $E_\theta(a) = \tau_\theta(a)\mathbb{1}\ \forall a \in A_\theta$, but we still need to show that τ_θ is a positive functional. Let $a = \sum_{n,m} f(n,m)u^n v^m$, $f \in C_c(Z^2)$, then

$$a^*a = (\sum_{n,m} f(n,m)u^n v^m)^*(\sum_{n,m} f(n,m)u^n v^m) = \sum_{n,m,k,l} \overline{f(n,m)} f(k,l) e^{2\pi i\theta(k-n)} u^{k-n} v^{l-m}.$$

It thus follows that $\tau_\theta(a^*a) = \sum \overline{f(n,m)} f(n,m) > 0$ if $f \neq 0$. Since τ_θ is continuous (E_θ continuous) we have $\tau_\theta(a^*a) \geq 0\ \forall a \in A_\theta$.

(2) If $a = \sum_{n,m} f(n,m)u^n v^m$, $b = \sum_{n,m} g(n,m)u^n v^m$, then

$$ab = \sum_{n,m} (f *_\theta g)(n,m)u^n v^m.$$

So it holds that

$$\begin{aligned} \tau_\theta(ab) &= (f *_\theta g)(0,0) = \sum f(k,l)g(-k,-l)e^{2\pi i\theta lk} \overset{(k,l)\to(-k,-l)}{=} \sum f(-k,-l)g(k,l)e^{2\pi i\theta lk} \\ &= (g *_\theta f)(0,0) = \tau_\theta(ba). \end{aligned}$$

And since τ_θ is continuous, it follows that $\tau_\theta(ab) = \tau_\theta(ba)\ \forall a, b \in A_\theta$.

(3) Let $0 \neq a \in A_\theta$ and let $\varphi : A_\theta \to \mathbb{C}$ is a linear functional with $\varphi(a^*a) > 0$ (which exists by the GNS construction), then

$$\varphi(E_\theta(a^*a)) = \varphi\left(\int_{S^1\times S^1} \beta_{(z,\omega)}(a^*a) d(z,\omega)\right) \overset{\varphi \text{ continuous}}{=} \int_{S^1\times S^1} \varphi(\beta_{(z,\omega)}(a^*a)) d(z,\omega) > 0$$

since $\beta_{(z,\omega)}(a^*a) = (\beta_{(z,\omega)}(a))^*\beta_{(z,\omega)}(a) \geq 0\ \forall (z,\omega) \in S^1 \times S^1$, and thus $\varphi(\beta_{(z,\omega)}(a^*a)) \geq 0$ $\forall (z,\omega) \in S^1 \times S^1$ and $\beta_{(1,1)}(a^*a) = \varphi(a^*a) \geq 0$. It follows that $E_\theta(a^*a) \neq 0$ and thus also $\tau_\theta(a^*a) \neq 0$.

$\square$

Remark 3.4.24 <u>**Traces**</u>

- *A positive functional* $\tau : A \to \mathbb{C}$ *with* $\tau(ab) = \tau(ba)\ \forall a, b \in A$ *is called a* **trace** *on* A. *E.g.* $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ *is such a trace.*
- *A trace* $\tau : A \to \mathbb{C}$ *with* $\tau(\mathbb{1}) = 1$ *is called* **trace state** *or a* **normalized trace**.
- *If* $\tau(a^*a) = 0 \Rightarrow a = 0$ *then* τ *is called a* **faithful trace**.
- *The above trace* $\tau_\theta : A_\theta \to \mathbb{C}$ *is a sort of Lebesgue-integral on the noncommutative torus* A_θ. *For* $\theta = 0$ τ_0 *is the usual Lebesgue-integral on* $A_0 = C(S^1 \times S^1)$.

Theorem 3.4.25 <u>**Uniqueness**</u> *Let* $\theta \in (0, 1)$ *be irrational, then* $\tau_\theta : A_\theta \to \mathbb{C}$ *is the only normalized trace on* A_θ, *i.e. if* $\tau : A_\theta \to \mathbb{C}$ *is an arbitrary state with* $\tau(ab) = \tau(ba)\ \forall a, b \in A_\theta$, *then* $\tau = \tau_\theta$.

Proof: Let $\lambda = e^{2\pi i\theta}$ and $H_\theta = \{(\lambda^n, \lambda^m) \,|\, n, m \in \mathbb{Z}\} \subseteq S^1 \times S^1$, then H_θ is dense in $S^1 \times S^1$ and we have

$$\tau(\beta_{(\lambda^n,\lambda^m)}(a)) = \tau(\underbrace{(v^{-n}u^m)^*}_{=:b}\underbrace{a(v^{-n}u^m)^*}_{=:c}) = \tau(bc) = \tau(cb) = \tau(a(v^{-n}u^m)^*(v^{-n}u^m)^*) = \tau(a).$$

Since $(z, \omega) \mapsto \beta_{(z,\omega)}(a)$ and τ are continuous, we have $\tau(\beta_{(z,\omega)}(a)) = \tau(a)\ \forall (z, \omega) \in S^1 \times S^1$ and finally

$$\begin{aligned}\tau(a) &= \int_{S^1\times S^1} \tau(a)d(z, \omega) = \int_{S^1\times S^1} \tau(\beta_{(z,\omega)}(a))d(z, \omega) = \tau(\int_{S^1\times S^1} \beta_{(z,\omega)}(a)d(z, \omega)) \\ &= \tau(E_\theta(a)) = \tau(\tau_\theta(a)\mathbb{1}) = \tau_\theta(a)\tau(\mathbb{1}) = \tau_\theta(a).\end{aligned}$$

□

Theorem 3.4.26 *Let* $\theta \in (0, 1)$ *be irrational, then* A_θ *is simple, i.e.* $\{0\}$ *and* A_θ *are the only ideals in* A_θ.

Proof: Let $I \neq \{0\}$ be an ideal in A_θ und let $0 \neq a \in I$. Let $H_\theta = \{(\lambda^n, \lambda^m) \,|\, n, m \in \mathbb{Z}\} \subseteq S^1 \times S^1$, then

$$\beta_{(\lambda^n,\lambda^m)}(a^*a) = (v^{-n}u^m)a^*a(v^{-n}u^m)^* \in I$$

for all $(\lambda^n, \lambda^m) \in H_\theta$. Now since I closed and H_θ dense in $S^1 \times S^1$, we also have $\beta_{(z,\omega)}(a^*a)$, for all $(z, \omega) \in S^1 \times S^1$. It follows that

$$\tau_\theta(a^*a)\mathbb{1}_{A_\theta} = E_\theta(a^*a) = \int_{S^1\times S^1} \beta_{(z,\omega)}(a^*a)d(z, \omega) \in I.$$

Since τ_θ is faithful, we have $\tau_\theta(a^*a) \neq 0$, and it follows that $\mathbb{1}_{A_\theta} \in I$, thus $I = A_\theta$. □

Remark 3.4.27 *We have seen that that the structure of* A_θ *is very different, depending on wether* θ *is rational or irrational. I.e. in the irrational case there are no nontrivial ideals, whereas in the rational case, there are many. In particular*

$$\theta \in \mathbb{Q},\ \tilde\theta \notin \mathbb{Q}, \quad \Rightarrow \quad A_\theta \ncong A_{\tilde\theta}$$

We already know

$$A_\theta \cong A_{1-\theta}$$

The question:

$$\text{When is } A_\theta \cong A_{\tilde\theta}\ ?$$

in solved with K-theory.

A Results from Topology

Theorem A.0.28 **Tietze Extension Theorem** *if X is a normal topological space, $A \subset X$ closed and $f : A \subset X \to \mathbb{R}$ continuous, then there exists a continuous extension*

$$F : X \longrightarrow \mathbb{R}, \quad F|_A = f$$

Lemma A.0.29 **Urysohn** *Let X be a locally compact Hausdorff space. Let $K \subset X$ be compact and $A \subset X$ closed with $K \cap A = \varnothing$. Then the following hold*

(i) There exists a relatively compact open neighborhood U of K such that $K \subset U \subset \overline{U} \subset X \setminus A$.

(ii) There exists a continuous function of compact support $f : X \to [0,1]$ with $f|_K = 1$ and $f|_A = 0$.

Lemma A.0.30 *Let K, X be Hausdorff spaces, K be compact, $f : K \to X$ continuous and bijective. Then f^{-1} is continuous also. I.e. f is a homeomorphism.*

B Results from Functional Analysis

Definition B.0.31 **Separates Points**

- *$F \subseteq X^Y$ is said to* **separate points in** *X iff*

$$\forall x_1, x_2 \in X \text{ with } x_1 \neq x_2 \ \exists \varphi \in F : \quad \varphi(x_1) \neq \varphi(x_2)$$

- *$F \subseteq X^{\mathbb{K}}$ is said to* **strongly separate points in** *X iff it separates points in X and*

$$\forall x \in X \ \exists \varphi \in F : \quad \varphi(x) \neq 0$$

Remark B.0.32 *Note: if $\mathbb{1} \in F \subseteq X^X$, then F separates points in X.*

Theorem B.0.33 **Stone – Weierstraß** *Let X be a locally compact Hausdorff space and $F \subseteq C_0(X, \mathbb{C})$ separates points in X, then the unital $*$-algebra $\langle F \rangle \subseteq C_0(X, \mathbb{C})$ is dense.*

Theorem B.0.34 **Hahn – Banach Continuation Theorem** *Let E be a topological $\mathbb{K}$-vector space and $p : E \to [0, \infty)$ a halfnorm on it. $F \subseteq E$ a linear subspace and $g : F \to \mathbb{K}$ a linear functional on F such that $|g(x)| \leq p(x)$ $\forall x \in F$, then there exists a linear continuation $\tilde{g} : E \to \mathbb{K}$ such that*

$$\tilde{g}|_F = g, \quad |\tilde{g}(x)| \leq p(x) \quad \forall x \in E.$$

Theorem B.0.35 **Banach – Alaoglu Theorem** *Let $(E, ||\cdot||)$ be a normed space, then*

$$\overline{B_r(f)} \subset E' \text{ compact}, \quad \forall f \in E', \, \varepsilon > 0$$

in the weak $*$-topology.

Theorem B.0.36 **Banach – Steinhaus Theorem** *Let E be a Banach space and F be a normed space. $\varnothing \neq I$ an arbitrary index set and $T_i \in L(E, F)$ $\forall i \in I$. If for all $x \in E$ there exists a $c_X \geq 0$ such that*

$$||T_i x|| \leq c_x \quad \forall i \in I$$

then there is a $c \geq 0$ such that $||T_i|| \leq c$ $\forall i \in I$. I.e. $\{T_i \,|\, i \in I\}$ is point wise bounded, then it is also norm bounded.

Theorem B.0.37 **Open Mapping Theorem** *Let E, F be Banach spaces. $T \in L(E, F)$ surjective, then T is open, i.e. if $U \subseteq E$ open, then $T(U) \subseteq F$ open.*

C Examples of Normed Algebras

(1) $C_0(X)$, X locally compact

- Multiplication: $(f \cdot g)(x) := f(x) \cdot g(x)$
- Involution: $f^* := \overline{f}$
- Banach $*$-algebra: $||\overline{f}||_\infty = ||f||_\infty$
- C^*-algebra: $||\overline{f}f||_\infty = ||\,|f|^2\,||_\infty = ||f||_\infty^2$
- Commutative: obvious
- Gelfand-space $\widehat{A} \cong X$, with the homeomorphism

$$\delta. : X \xrightarrow{\cong} \{\delta_x : C_0(X) \to \mathbb{C} \,|\, x \in X,\, \delta_x(f) := f(x)\} \cong \widehat{C_0(X)}$$

- Spectrum for X compact: $\sigma_{C(X)}(f) = f(X)$ since

$$f \in \mathrm{Inv}(C(X)) \quad \Leftrightarrow \quad f(x) \neq 0 \ \forall x \in X, then f^{-1} = \tfrac{1}{f}$$

(2) Disc-algebra $A = A^{\mathrm{Disc}} := \{f \in C(D) \,|\, f \text{ holomorphic on } \mathrm{int}(D)\}$

- Multiplication: obvious
- Involution: $f^*(z) := \overline{f(\overline{z})}$
- Banach $*$-algebra:
- C^*-algebra: A^{Disc} is not a C^*-algebra, that is since every commutative C^*-algebra is symmetric and A^{Disc} is not symmetric:

$$\mathbb{1} \in A^{\mathrm{Disc}} \quad \Rightarrow \quad \mathbb{1}^*(x) = \overline{\mathbb{1}(\overline{z})} = \overline{\overline{z}} = z \quad \Rightarrow \quad \mathbb{1} = \mathbb{1}^*$$

  and $\sigma_A(\mathbb{1}) = \mathbb{1}(D) = D \not\subseteq \mathbb{R}$, but symmetric algebras fulfill $\sigma_A(a) \subset \mathbb{R}$.
- Commutative: obvious
- Gelfand-space:

$$\begin{array}{ccl} \widehat{A} & \xrightarrow{\cong} & D \\ \varphi & \longmapsto & \varphi(z) \in \sigma_A(z) = z(D) = D \end{array}$$

- Spectrum: $\sigma_A(f) = f(D)$, if $f|_{\mathrm{int}(D)}$ is holomorphic, then so is $\frac{1}{f}|_{\mathrm{int}(D)}$.

(3) Convolution algebra $(l^1(\mathbb{Z}), ||f||_1 := \sum_{n\in\mathbb{Z}} |f(n)|)$

- Multiplication = Convolution: $(f * g)(n) := \sum_{m\in\mathbb{Z}} f(m)g(n-m)$
- Involution: $f^*(n) := \overline{f(-n)}$
- Banach $*$-algebra:

$$\begin{aligned} (f*g)^* &= \overline{(f*g)(-n)} = \overline{\sum_m f(m)g(-n-m)} = \sum_m \overline{f(-m)g(-n+m)} \\ &= \sum_m f^*(m)g^*(n-m) = (f^* * g^*)(n) \overset{\text{commutativity}}{=} (g^* * f^*)(n) \end{aligned}$$

  $||f^*||_1 = \sum_n |\overline{f(-n)}| = \sum_n |f(n)| = ||f||_1$
- C^*-algebra: $l^1(\mathbb{Z})$ is not a C^*-algebra. There exist counter examples $f \in l^1(\mathbb{Z})$ with $||f^* * f||_1 \neq ||f||_1^2$.

– Commutative:

$$(f * g)(n) = \sum_m f(m)g(n-m) \overset{m \to n-m}{=} \sum_m g(m)f(n-m) = (g * f)(n)$$

– Gelfand-space:

$$\begin{array}{rcl} \varphi : S^1 & \xrightarrow{\cong} & \widehat{l^1(\mathbb{Z})} \\ z & \longmapsto & \varphi_z \qquad \varphi_z(f) := \hat{f}(z) := \sum_n f(n)z^n \end{array}$$

– Spectrum: $\sigma_{l^1(\mathbb{Z})}(f) = \hat{f}(S^1)$, since $f \in \mathrm{Inv}(l^1(\mathbb{Z}))$ iff $\hat{f}(z) \neq 0\ \forall z \in S^1$