CROSSED PRODUCTS, THE MACKEY-RIEFFEL-GREEN MACHINE AND APPLICATIONS

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Abstract. We give an introduction into the ideal structure and representation theory of crossed products by actions of locally compact groups on C*-algebras. In particular, we discuss the Mackey-Rieffel-Green theory of induced representations of crossed products and groups. Although we do not give complete proofs of all results, we try at least to explain the main ideas. For a much more detailed exposition of many of the results presented here we refer to the beautiful book [123] by Dana Williams (which has not been available when most of this was written).

1. introduction

If a locally compact group G acts continuously via $*$-automorphism on a C^*-algebra A, one can form the full and reduced crossed products $A \rtimes G$ and $A \rtimes_r G$ of A by G. The full crossed product should be thought of as a skew *maximal* tensor product of A with the full group C^*-algebra $C^*(G)$ of G and the reduced crossed product should be regarded as a skew *minimal* (or spacial) tensor product of A by the reduced group C^*-algebra $C^*_r(G)$ of G.

The crossed product construction provides a huge source of examples in C^*-algebra theory, and they play important rôles in many applications of C^*-algebras in other fields of mathematics, like group representation theory and topology (here in particular in connection with the Baum-Connes conjecture). It is the purpose of this article to present in a concise way some of the most important constructions and features of crossed products with emphasis on the Mackey-Green-Rieffel mashine as a basic technique to investigate the structure of crossed products. Note that the material covered in this article is almost perpendicular to the material covered in Pedersen's book [98]. Hence we recommend the interested reader to also have a look into [98] to obttain a more complete and balanced picture of the theory. Pedersen's book also provides a good introduction into the general theory of C^*-algebras. An incomplete list of other good references on the general theory of C^*-algebas is [22, 23, 92]. Some general notation: if X is a locally compact Hausdorff space and E is a normed linear space, then we denote be $C_b(X, E)$ the space of bounded continuous E-valued functions on X and by $C_c(X, E)$ and $C_0(X, E)$ those function in $C_b(X, E)$ which have compact supports or which vanish at infinity. If $E = \mathbb{C}$, then we simply write $C_b(X), C_c(X)$ and $C_0(X)$, respectively. If E and F are two linear spaces, then $E \odot F$ always denotes the algebraic tensor product of E and F and we reserve the sign "$\otimes$" for certain kinds of topological tensor products.

2. Some preliminaris

We shall assume throughout this article that the reader is familiar with the basic concepts of C^*-algebras as can be found in any of the standard text books mentioned above. However, in order to make this treatment more self-contained we try to recall some basic facts and notation on C^*-algebras which will play an important rôle in this article.

2.1. **C^*-algebras.** A (complex) C^*-algebra is a complex Banach-algebra A together with an involution $a \mapsto a^*$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. Note that we usually do not assume that A has a unit. Basic examples are given by the algebras $C_0(X)$ and $C_b(X)$ equipped with the supremum-norm and the involution $f \mapsto \bar{f}$. These algebras are clearly commutative and a classical theorem of Gelfand and Naimark asserts, that *all* commutative C^*-algebras are isomorphic to some $C_0(X)$ (see paragraph 2.3 below). Other examples are given by the algebras $\mathcal{B}(H)$ of bounded operators on a Hilbert space H with operator norm and involution given by taking the adjoint operators, and all closed $*$-subalgebras of $\mathcal{B}(H)$ (like the algebra $\mathcal{K}(H)$ of compact operators on H). Indeed, another classic result by Gelfand and Naimark shows that every C^*-algebra is isomorphic to a closed $*$-subalgebra of some $\mathcal{B}(H)$. If $S \subseteq A$ is any subset of a C^*-algebra A, we denote by $C^*(S)$ the smallest sub-C^*-algebra of A which contains S. A common way to construct C^*-algebras is by describing a certain set $S \subseteq \mathcal{B}(H)$ and forming the algebra $C^*(S) \subseteq \mathcal{B}(H)$. If $S = \{a_1, \dots, a_l\}$ is a finte set of elements of A, we shall also write $C^*(a_1, \dots, a_l)$ for $C^*(S)$. For example, if $U, V \in \mathcal{B}(H)$ are unitary operators such that $UV = e^{2\pi i\theta}VU$ for some irrational $\theta \in [0,1]$, then $A_\theta := C^*(U,V)$ is the famous *irrational rotation algebra* corresponding to θ, a standard example in C^*-algebra theory (one can show that the isomorphism class of $C^*(U,V)$ does not depend on the particular choice of U and V).

C^*-algebras are very rigid objects: If A is a C^*-algebra, then every closed (two-sided) ideal of A is automatically selfadjoint and A/I, equipped with the obvious operations and the quotient norm is again a C^*-algebra. If B is any Banach $*$-algebra (i.e., a Banach algebra with isometric involution, which does not necessarily satisfy the C^*-relation $\|b^*b\| = \|b\|^2$), and if A is a C^*-algebra, then any $*$-homomorphism $\Phi : B \to A$ is automatically continuous with $\|\Phi(b)\| \leq \|b\|$ for all $b \in B$. If B is also a C^*-algebra, then Φ factors through an isometric $*$-homomorphism $\tilde{\Phi} : B/(\ker \Phi) \to A$. In particular, if A and B are C^*-algebras and $\Phi : B \to A$ is an injective (resp. bijective) $*$-homomorphism, then Φ is automatically isometric (resp. an isometric isomorphism).

2.2. **Multiplier Algebras.** The *multiplier algebra* $M(A)$ of a C^*-algebra A is the largest C^*-algebra which contains A as an essential ideal (an ideal J of a C^*-algebra B is called *essential* if $bJ = \{0\}$ implies $b = 0$ for all $b \in B$). If A is represented faithfully and non-degenerately on a Hilbert space H (i.e. $A \subseteq \mathcal{B}(H)$ with $AH = H$), then $M(A)$ can be realized as the idealizer

$$M(A) = \{T \in \mathcal{B}(H) : TA \cup AT \subseteq A\}$$

of A in $\mathcal{B}(H)$. In particular we have $M(\mathcal{K}(H)) = \mathcal{B}(H)$, where $\mathcal{K}(H)$ denotes the algebra of compact operators on H.

The *strict topology* on $M(A)$ is the locally convex topology generated by the semi-norms $m \mapsto \|am\|, \|ma\|$ with $a \in A$. Notice that $M(A)$ is the strict completion of A. $M(A)$ is always unital and $M(A) = A$ if A is unital. If $A = C_0(X)$ for some locally compact space X, then $M(A) \cong C_b(X) \cong C(\beta(X))$, where $\beta(X)$ denotes the Stone-Čech compactification of X. Hence $M(A)$ should be viewed as a noncommutative analogue of the Stone-Čech compactification. If A is any C^*-algebra, then the Algebra $A_1 := C^*(A \cup \{1\}) \subseteq M(A)$ is called the *unitization* of A (notice that $A_1 = A$ if A is unital). If $A = C_0(X)$ for some non-compact X, then $A_1 \cong C(X_+)$, where X_+ denotes the one-point compactification of X.

A $*$-homomorphism $\pi : A \to M(B)$ is called *non-degenerate* if $\pi(A)B = B$, which by Cohen's factorization theorem is equivalent to the weaker condition that $\mathrm{span}\{\pi(a)b : a \in A, b \in B\}$ is dense in B (e.g. see [105, Proposition 2.33]). If H is a Hilbert space, then $\pi : A \to M(\mathcal{K}(H)) = \mathcal{B}(H)$ is non-degenerate in the above sense iff $\pi(A)H = H$. If $\pi : A \to M(B)$ is non-degenerate, there exists a unique $*$-homomorphism $\bar{\pi} : M(A) \to M(B)$ such that $\bar{\pi}|_A = \pi$. We shall usually make no notational difference between π and its extension $\bar{\pi}$.

2.3. **Commutative C^*-algebras and functional calculus.** If A is commutative, then we denote by $\Delta(A)$ the set of all non-zero algebra homomorphisms $\chi : A \to \mathbb{C}$ equipped with the weak-$*$ topology. Then $\Delta(A)$ is locally compact and it is compact if A is unital. If $a \in A$, then $\widehat{a} : \Delta(A) \to \mathbb{C}; \widehat{a}(\chi) := \chi(a)$ is an element of $C_0(\Delta(A))$, and the Gelfand-Naimark theorem asserts that $A \to C_0(\Delta(A)) : a \mapsto \widehat{a}$ is an (isometric) $*$-isomorphism.

If A is any C^*-algebra, then an element $a \in A$ is called *normal* if $a^*a = aa^*$. If $a \in A$ is normal, then $C^*(a, 1) \subseteq A_1$ is a commutative sub-C^*-algebra of A_1. Let $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A_1\}$ denote the *spectrum* of a, a nonempty compact subset of $\mathbb{C}$. If $\lambda \in \sigma(a)$, then $a - \lambda 1$ generates a unique maximal ideal M_λ of $C^*(a, 1)$ and the quotient map $C^*(a, 1) \to C^*(a, 1)/M_\lambda \cong \mathbb{C}$ determines an element $\chi_\lambda \in \Delta(C^*(a, 1))$. One then checks that $\lambda \mapsto \chi_\lambda$ is a homeomorphism between $\sigma(a)$ and $\Delta(C^*(a, 1))$. Thus, the Gelfand-Naimark theorem provides a $*$-isomorphism $\Phi : C(\sigma(a)) \to C^*(a, 1)$. If $p(z) = \sum_{i,j=0}^n \alpha_{ij} z^i \bar{z}^j$ is a plynomial in z and $\bar{z}$ (which by the Stone-Weierstraß theorem form a dense subalgebra of $C(\sigma(a))$), then $\Phi(p) = \sum_{i,j=0}^n \alpha_{ij} a^i (a^*)^j$. In particular, we have $\Phi(1) = 1$ and $\Phi(\mathrm{id}_{\sigma(a)}) = a$. In what follows, we always write $f(a)$ for $\Phi(f)$. Note that $\sigma(f(a)) = f(\sigma(a))$ and if $g \in C\big(\sigma(f(a))\big)$, then $g(f(a)) = (g \circ f)(a)$, i.e., the functional calculus is compatible with composition of functions. If A is not unital, then $0 \in \sigma(a)$ and it is clear that for any polynomial p in z and $\bar{z}$ we have $p(a) \in A$ if and only if $p(0) = 0$. Approximating functions by polynomials, it follows that $f(a) \in A$ if and only $f(0) = 0$ and we obtain an isomorphism $C_0(\sigma(a) \setminus \{0\}) \to C^*(a) \subseteq A; f \mapsto f(a)$.

Example 2.1. An element $a \in A$ is called *positive* if $a = b^*b$ for some $b \in A$. This is equivalent to say that $\sigma(a) \subseteq [0, \infty)$. If $a \geq 0$, then the functional calculus provides the element $\sqrt{a} \in A$, which is the unique positive element of A such that $(\sqrt{a})^2 = a$.

If $a \in A$ is selfadjoint (i.e., $a = a^*$), then $\sigma(a) \subseteq \mathbb{R}$ and the functional calculus allows a unique decomposition $a = a_+ - a_-$ with $a_+, a_- \geq 0$. Simply take $a_+ = f(a)$ with $f(t) = \max\{t, 0\}$. Since we can write any $b \in A$ as a linear combination of two selfadjoint elements via $b = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*)$, we see that every element of A can be written as a linear combination of four positive elements. Since every positive element is a square, it follows that $A = A^2 := \mathrm{LH}\{ab : a, b \in A\}$ (Cohen's factorization theorem even implies that $A = \{ab : a, b \in A\}$).

Every C^*-algebra has an *approximate unit*, i.e., $(a_i)_{i\in I}$ is a net in A such that $\|a_i a - a\|, \|a a_i - a\| \to 0$ for all $a \in A$. In fact $(a_i)_{i\in I}$ can be chosen so that $a_i \geq 0$ and $\|a_i\| = 1$ for all $i \in I$, i.e., $(a_i)_{i\in I}$. If A is separable (i.e., A contains a countable dense set), then one can find a sequence $(a_n)_{n\in\mathbb{N}}$ with theses properties.

If A is a unital C^*-algebra, then $u \in A$ is called a *unitary*, if $uu^* = u^*u = 1$. If u is unitary, then $\sigma(u) \subseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and hence $C^*(u) = C^*(u, 1)$ is isomorphic to a quotient of $C(\mathbb{T})$. Note that if $u, v \in A$ are two unitaries such that $uv = e^{2\pi i\theta} vu$ for some irrational $\theta \in [0, 1]$, then one can show that $\sigma(u) = \sigma(v) = \mathbb{T}$, so that $C^*(u) \cong C^*(v) \cong C(\mathbb{T})$. It follows that the irrational rotation algebra $A_\theta = C^*(u, v)$ should be regarded as (the algebra of functions on) a "noncommutative product" of two tori which results in the expression of a *noncommutative* 2-*torus*.

2.4. **Representation and Ideal spaces of C^*-algebras.** If A is a C^*-algebra, the *spectrum* $\widehat{A}$ is defined as the set of all unitary equivalence classes of irreducible representations $\pi : A \to \mathcal{B}(H)$ of A on Hilbert space[1]. We shall usually make no notational difference between an irreducible representation π and its equivalence class $[\pi] \in \widehat{A}$. The *primitive ideals* of A are the kernels of the irreducible representations of A, and we write $\mathrm{Prim}(A) := \{\ker \pi : \pi \in \widehat{A}\}$ for the set of all primitive ideals of A. Every closed two-sided ideal I of A is an intersection of primitive ideals. The spaces $\widehat{A}$ and $\mathrm{Prim}(A)$ are equipped with the *Jacobson topologies*, where the closure operations are given by $\pi \in \overline{R} :\Leftrightarrow \ker \pi \supseteq \cap\{\ker \rho : \rho \in R\}$ (resp. $P \in \overline{R} :\Leftrightarrow P \supseteq \cap\{Q : Q \in R\}$) for $R \subseteq \widehat{A}$ (resp. $R \subseteq \mathrm{Prim}(A)$). In general, the Jacobson topologies are far away from being Hausdorff. In fact, while $\mathrm{Prim}(A)$ is at least always a T_0-space (i.e. for any two different elements in $\mathrm{Prim}(A)$ at least one of them has an open neighborhood which does not contain the other), this very weak separation property often fails for the space $\widehat{A}$. If A is commutative, it follows from Schur's lemma that $\widehat{A} = \Delta(A)$ and the Jacobson topology coincides in this case with the weak-$*$ topology.

If I is a closed two-sided ideal of A, then $\widehat{A}$ can be identified with the disjoint union of $\widehat{I}$ with $\widehat{A/I}$, such that $\widehat{I}$ identifies with $\{\pi \in \widehat{A} : \pi(I) \neq \{0\}\} \subseteq \widehat{A}$ and $\widehat{A/I}$ identifies with $\{\pi \in \widehat{A} : \pi(I) = \{0\}\} \subseteq \widehat{A}$. It follows from the definition of the Jacobson topology that $\widehat{A/I}$ is closed and $\widehat{I}$ is open in $\widehat{A}$. The correspondence $I \leftrightarrow \widehat{I}$

[1] A selfadjoint subset $S \subseteq \mathcal{B}(H)$ is called *irreducible* if there exists no proper nontrivial closed subspace $L \subseteq H$ with $SL \subseteq L$. By Schur's lemma, this is equivalent to saying that the commutator of S in $\mathcal{B}(H)$ is equal to $\mathbb{C} \cdot 1$. A representation $\pi : A \to \mathcal{B}(H)$ is *irreducible* if $\pi(A)$ is irreducible. Two representations π, ρ of A on H_π and H_ρ, respectively, are called unitarily equivalent, if there exists a unitary $V : H_\pi \to H_\rho$ such that $V \circ \pi(a) = \rho(a) \circ V$ for all $a \in A$.

(resp $I \leftrightarrow \widehat{A/I}$) is a one-to-one correspondence between the closed two-sided ideals of A and the open (resp. closed) subsets of $\widehat{A}$. Similar statements hold for the open or closed subsets of $\mathrm{Prim}(A)$.

A C^*-algebra is called *simple* if $\{0\}$ is the only proper closed two-sided ideal of A. Of course, this is equivalent to saying that $\mathrm{Prim}(A)$ has only one element (the zero ideal). Simple C^*-algebras are thought of as the basic "building blocks" of more general $C*$-algebras. Examples of simple algebras are the algebras $\mathcal{K}(H)$ of compact operators on a Hilber space H and the irrational rotation algebras A_θ. Notice that while $\widehat{\mathcal{K}(H)}$ has also only one element (the equivalance class of its embedding into $\mathcal{B}(H)$), one can show that $\widehat{A}_\theta$ is an uncountable infinite set (this can actually be deduced from Proposition 11.1 below).

A C^*-algebra A is called *type I* (or *GCR* or *postliminal*) if for every irreducible representation $\pi : A \to \mathcal{B}(H)$ we have $\pi(A) \supseteq \mathcal{K}(H)$. We refer to [23, Chapter XII] for some important equivalent characterizations of type I algebras. A C^*-algebra A is called *CCR* (or *liminal*), if $\pi(A) = \mathcal{K}(H)$ for every irreducible representation $\pi \in \widehat{A}$. If A is type I, then the mapping $\widehat{A} \to \mathrm{Prim}(A) : \pi \mapsto \ker \pi$ is a homeomorphism, and the converse holds if A is separable (in the non-separable case this converse is still an open problem). Furthermore, if A is type I, then A is CCR if and only if $\widehat{A} \cong \mathrm{Prim}(A)$ is a T_1-space, i.e., points are closed.

A C^*-algebra is said to have *continuous trace* if there exists a dense ideal $\mathfrak{m} \subseteq A$ such that for all positve elements $a \in \mathfrak{m}$ the operator $\pi(a) \in \mathcal{B}(H_\pi)$ is trace-class and the resulting map $\widehat{A} \to [0, \infty); \pi \mapsto \mathrm{tr}(\pi(a))$ is continuous. Continuous trace algebras are all CCR with Hausdorff spectrum $\widehat{A}$. Note that every type I C^*-algebra A contains a non-zero closed two-sided ideal I such that I is a continuous-trace algebra (see [23, Chapter 4]).

2.5. **Tensor products.** The algebraic tensor product $A \odot B$ of two C^*-algebras A and B has a canonical structure as a $*$-algebra. To make it a C^*-algebra, we have to take completions with respect to suitable cross-norms $\|\cdot\|_\mu$ satisfying $\|a \otimes b\|_\mu = \|a\|\|b\|$. Among the possible choices of such norms there is a maximal cross-norm $\|\cdot\|_{\max}$ and a minimal cross-norm $\|\cdot\|_{\min}$ giving rise to the *maximal tensor product* $A \otimes_{\max} B$ and the *minimal tensor product* $A \otimes_{\min} B$ (which we shall always denote by $A \otimes B$).

The maximal tensor product is characterized by the universal property that any $*$-homomorphisms $\pi : A \odot B \to \mathcal{B}(H)$ extends to the completion $A \otimes_{\max} B$. The minimal (or spatial) tensor product $A \otimes B$ is the completion of $A \odot B$ with respect to

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\min} = \left\| \sum_{i=1}^n \rho(a_i) \otimes \sigma(b_i) \right\|,$$

where $\rho : A \to \mathcal{B}(H_\rho), \sigma : B \to \mathcal{B}(H_\sigma)$ are faithful representations of A and B and the norm on the right is taken in $\mathcal{B}(H_\rho \otimes H_\sigma)$. It is a non-trivial fact (due to Takesaki) that $\|\cdot\|_{\min}$ is the smallest cross-norm on $A \odot B$ and that it does not depend on the choice of ρ and σ (e.g. see [105, Theorem B.38]).

A C^*-algebra A is called *nuclear*, if $A \otimes_{\max} B = A \otimes B$ for all B. Every type I C^*-algebra is nuclear (e.g. see [105, Corollary B.49]) as well as the irrational rotation algebra A_θ (which will follow from Theorem 4.7 below). In particular, all commutative C^*-algebras are nuclear and we have $C_0(X) \otimes B \cong C_0(X, B)$ for any locally compact space X. One can show that $\mathcal{B}(H)$ is not nuclear, if H is an infinite dimensional Hilbert space.

If H is an infinite dimensional Hilbert space, then $\mathcal{K}(H) \otimes \mathcal{K}(H)$ is isomorphic to $\mathcal{K}(H)$ (which can be deduced from a unitary isomorphism $H \otimes H \cong H$). A C^*-algebra A is called *stable* if A is isomorphic to $A \otimes \mathcal{K}$, where we write $\mathcal{K} := \mathcal{K}(l^2(\mathbb{N}))$. It follows from the associativity of taking tensor products that $A \otimes \mathcal{K}$ is always stable and we call $A \otimes \mathcal{K}$ the *stabilization* of $\mathcal{K}$. Note that $A \otimes \mathcal{K}$ and A have isomorphic representation and ideal spaces. For example, the map $\pi \mapsto \pi \otimes \mathrm{id}_{\mathcal{K}}$ gives a homeomorphism between $\widehat{A} \to (A \otimes \mathcal{K})\widehat{\ }$. Moreover A is type I (or CCR or continuous-trace or nuclear) if and only if $A \otimes \mathcal{K}$ is.

3. Actions and their crossed products

3.1. **Haar measure and Vector-valued integration on groups.** If X is a locally compact space, we denote by $C_c(X)$ the set of all continuous functions with compact supports on X. A *positive integral* on $C_c(X)$ is a linear functional $\int : C_c(X) \to \mathbb{C}$ such that $\int_X f(x)\,dx := \int(f) \geq 0$ if $f \geq 0$. We refer to [111] for a good treatment of the Riesz representation theorem which provides a one-to-one connection between integrals on $C_c(X)$ and positive measures on X. If H is a Hilbert space and $f : X \to \mathcal{B}(H)$ is a weakly continuous function (i.e., $x \mapsto \langle f(x), \xi, \eta\rangle$ is continuous for all $\xi, \eta \in H$) with compact support, then there exists a unique operator $\int_X f(x)\,dx \in \mathcal{B}(H)$ such that

$$\left\langle \left(\int_X f(x)\,dx \right) \xi, \eta \right\rangle = \int_X \langle f(x)\xi, \eta\rangle\,dx \qquad \text{for all } \xi, \eta \in H.$$

If A is a C^*-algebra imbedded faithfully into some $\mathcal{B}(H)$, then approximating f uniformly with controlled supports by elements in the algebraic tensor product $C_c(X) \odot A$ shows that $\int_X f(x)\,dx \in A$. Moreover, if $f : X \to M(A)$ is a strictly continuous function with compact support and $M(A) \subseteq \mathcal{B}(H)$, then f is weakly continuous as a function into $\mathcal{B}(H)$, and since $\big(x \mapsto af(x), f(x)a\big) \in C_c(X, A)$ for all $a \in A$ it follows that $\int_X f(x)\,dx \in M(A)$.

If G is a locally compact group, then there exists a nonzero positive integral $\int : C_c(X) \to \mathbb{C}$, called *Haar integral* on $C_c(G)$, such that $\int_G f(gx)\,dx = \int_G f(x)\,dx$ for all $f \in C_c(G)$ and $g \in G$. The Haar integral is unique up to multiplication with a positive number, which implies that for each $g \in G$ there exists a positive number $\Delta(g)$ such that $\int_G f(x)\,dx = \Delta(g) \int_G f(xg)\,dx$ for all $f \in C_c(G)$ (since the right hand side of the equation defines a new Haar integral). One can show that $\Delta : G \to (0, \infty)$ is a continuous group homomorphism. A group G is called unimodular if $\Delta(g) = 1$ for all $g \in G$. All discrete, all compact and all abelian groups are unimodular, however, the $ax+b$-group, which is the semidirect product $\mathbb{R} \rtimes \mathbb{R}^*$ via the action of the multiplicative group $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ on the additive group $\mathbb{R}$ by dilation, is not unimodular.

3.2. **C^*-dynamical systems and their crossed products.** An *action* of a locally compact group G on a C^*-algebra A is a homomorphism $\alpha : G \to \mathrm{Aut}(A); s \mapsto \alpha_s$ of G into the group $\mathrm{Aut}(A)$ of $*$-automorphisms of A such that $s \mapsto \alpha_s(a)$ is continuous for all $a \in A$ (i.e., α is strongly continuous). The triple (A, G, α) is then called a *C^*-dynamical system* (or *covariant system*). We also say that A is a *G-algebra*, when A is equipped with a given G-action α.

Example 3.1 (Transformation groups)**.** If $G \times X \to X; (s, x) \mapsto s \cdot x$ is a continuous action of G on a locally compact Hausdorff space X, then G acts on $C_0(X)$ by $\big(\alpha_s(f)\big)(x) := f(s^{-1} \cdot x)$, and it is not difficult to see that every action on $C_0(X)$ arises in this way. Thus, general G-algebras are non-commutative analogues of locally compact G-spaces.

If A is a G-algebra, then $C_c(G, A)$ becomes a $*$-algebra with respect to convolution and involution defined by

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))\, dt \quad \text{and} \quad f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1}))^*. \tag{3.1}$$

A *covariant homomorphisms* of (A, G, α) into the multiplier algebra $M(D)$ of some C^*-algebra D is a pair (π, U), where $\pi : A \to M(D)$ is a $*$-homomorphism and $U : G \to UM(D)$ is a strictly continuous homomorphism into the group $UM(D)$ of unitaries in $M(D)$ satisfying

$$\pi(\alpha_s(a)) = U_s\pi(a)U_{s^{-1}} \quad \text{for all } s \in G.$$

We say that (π, U) is *non-degenerate* if π is non-degenerate. A *covariant representation* of (A, G, α) on a Hilbert space H is a covariant homomorphism into $M(\mathcal{K}(H)) = \mathcal{B}(H)$. If (π, U) is a covariant homomorphism into $M(D)$, its *integrated form* $\pi \times U : C_c(G, A) \to M(D)$ is defined by

$$(\pi \times U)(f) := \int_G \pi(f(s))U_s\, ds \in M(D). \tag{3.2}$$

It is straightforward to check that $\pi \times U$ is a $*$-homomorphism.

Covariant homomorphisms do exist. Indeed, if $\rho : A \to M(D)$ is any $*$-homomorphism, then we can construct the induced covariant homomorphism $\mathrm{Ind}\,\rho := (\tilde{\rho}, 1 \otimes \lambda)$ of (A, G, α) into $M\big(D \otimes \mathcal{K}(L^2(G))\big)$ as follows: Let $\lambda : G \to U(L^2(G))$ denote the *left regular representation* of G given by $(\lambda_s\xi)(t) = \lambda(s^{-1}t)$, and define $\tilde{\rho}$ as the composition

$$A \xrightarrow{\tilde{\alpha}} M\big(A \otimes C_0(G)\big) \xrightarrow{\rho\otimes M} M\big(D \otimes \mathcal{K}(L^2(G))\big),$$

where the $*$-homomorphism $\tilde{\alpha} : A \to C_b(G, A) \subseteq M\big(A \otimes C_0(G)\big)$ [2] is defined by $\tilde{\alpha}(a)(s) = \alpha_{s^{-1}}(a)$, and where $M : C_0(G) \to \mathcal{B}(L^2(G)) = M(\mathcal{K}(L^2(G)))$ denotes the represention by multiplication operators. We call $\mathrm{Ind}\,\rho$ the *covariant homomorphism*

[2] $C_b(G, A)$ is regarded as a subset of $M\big(A \otimes C_0(G)\big)$ via the identification $A \otimes C_0(G) \cong C_0(G, A)$ and taking pointwise products of functions.

induced from ρ, and we shall make no notational difference between $\operatorname{Ind}\rho$ and its integrated form $\tilde{\rho}\times(1\otimes\lambda)$. $\operatorname{Ind}\rho$ is faithful on $C_c(G,A)$ whenever ρ is faithful on A. If $\rho=\mathrm{id}_A$, the identity on A, then we say that

$$\Lambda_A^G := \operatorname{Ind}(\mathrm{id}_A) : C_c(G,A)\to M\big(A\otimes\mathcal{K}(L^2(G))\big)$$

is *the regular representation* of (A,G,α). Notice that

$$\operatorname{Ind}\rho = (\rho\otimes\mathrm{id}_{\mathcal{K}})\circ\Lambda_A^G \tag{3.3}$$

for all $*$-homomorphisms $\rho : A\to M(D)$. [3]

Remark 3.2. If we start with a representation $\rho : A\to\mathcal{B}(H)=M(\mathcal{K}(H))$ of A on a Hilbert space H, then $\operatorname{Ind}\rho=(\tilde{\rho},1\otimes\lambda)$ is the representation of (A,G,α) into $\mathcal{B}(H\otimes L^2(G))$ (which equals $M(\mathcal{K}(H)\otimes\mathcal{K}(L^2(G)))$) given by the formulas

$$\big(\tilde{\rho}(a)\xi\big)(t)=\rho(\alpha_{t^{-1}}(a))(\xi(t))\quad\text{and}\quad\big(1\otimes\lambda)(s)\xi\big)(t)=\xi(s^{-1}t),$$

for $a\in A, s\in G$ and $\xi\in L^2(G,H)\cong H\otimes L^2(G)$. Its integrated form is given by the convolution formula

$$f*\xi(t):=\big(\operatorname{Ind}\rho(f)\xi\big)(t)=\int_G\rho\big(\alpha_{t^{-1}}(f(s))\big)\xi(s^{-1}t)\,ds$$

for $f\in C_c(G,A)$ and $\xi\in L^2(G,H)$.

In the literature the regular representation is often defined as such concrete representation on Hilbert space, but it has the disadvantage that it depends on a choice of an embedding of A into some $\mathcal{B}(H)$.

Definition 3.3. Let (A,G,α) be a C^*-dynamical system.

(i) The *full crossed produt* $A\rtimes_\alpha G$ (or just $A\rtimes G$ if α is understood) is the completion of $C_c(G,A)$ with respect to

$$\|f\|_{\max}:=\sup\{\|(\pi\times U)(f)\| : (\pi,U)\text{ is a covariant representation of }(A,G,\alpha)\}.$$

(ii) The *reduced crossed product* $A\rtimes_{\alpha,r}G$ (or just $A\rtimes_r G$) is defined as

$$\overline{\Lambda_A^G\big(C_c(G,A)\big)}\subseteq M\big(A\otimes\mathcal{K}(L^2(G))\big).$$

Remark 3.4. (1) It follows directly from the above definition that every integrated form $\pi\times U : C_c(G,A)\to M(D)$ of a covariant homomorphism (π,U) extends to a $*$-homomorphism of $A\rtimes_\alpha G$ into $M(D)$. Conversely, every **non-degenerate** $*$-homomorphism $\Phi : A\rtimes_\alpha G\to M(D)$ is of the form $\Phi=\pi\times U$ for some non-degenerate covariant homomorphism (π,U). To see this consider the canonical covariant homomorphism (i_A,i_G) of (A,G,α) into $M(A\rtimes_\alpha G)$ given by the formulas

$$(i_A(a)f)(s)=af(s)\qquad (i_G(t)f)(s)=\alpha_t(f(t^{-1}s))$$
$$(fi_A(a))(s)=f(s)\alpha_s(a)\qquad (fi_G(t))(s)=\Delta(t^{-1})\alpha_{s^{-1}}\big(f(st^{-1})\big),$$

[3] This equation even makes sense if ρ is degenerate since $\rho\otimes\mathrm{id}_{\mathcal{K}}$ is well defined on the image of $C_b(G,A)$ in $M(A\otimes\mathcal{K}(L^2(G)))$.

$f \in C_c(G, A)$ (the given formulas extend to left and right multiplications of $i_A(a)$ and $i_G(s)$ with elements in $A \rtimes G$). It is then relatively easy to check that $\Phi = \pi \times U$ with

$$\pi = \Phi \circ i_A \quad \text{and} \quad U = \Phi \circ i_G.$$

Nondegeneracy of Φ is needed to have the compositions $\Phi \circ i_A$ and $\Phi \circ i_G$ well defined. In the definition of $\|\cdot\|_{\max}$ one could restrict to non-degenerate or even (topologically) irreducible representations of (A, G, α) on Hilbert space. However, it is extremely useful to consider more general covariant homomorphisms into multiplier algebras.

(2) The above described correspondence between non-degenerate representations of (A, G, α) and $A \rtimes G$ induces a bijection between the set $(A, G, \alpha)\widehat{\ }$ of unitary equivalence classes of irreducible covariant Hilbert-space representations of (A, G, α) and $(A \rtimes G)\widehat{\ }$. We topologize $(A, G, \alpha)\widehat{\ }$ such that this bijection becomes a homeomorphisms.

(3) The reduced crossed product $A \rtimes_r G$ does not enjoy the above described universal properties, and therefore it is often more difficult to handle. However, it follows from (3.3) that whenever $\rho : A \to M(D)$ is a $*$-homomorphism, then $\operatorname{Ind}\rho$ factors through a representation of $A \rtimes_r G$ to $M(D)$ which is faithful iff ρ is faithful. In particular, if $\rho : A \to \mathcal{B}(H)$ is a faithful representation of A on Hilbert space, then $\operatorname{Ind}\rho$ is a faithful representation of $A \rtimes_r G$ into $\mathcal{B}(H \otimes L^2(G))$.

(4) It follows from the definition of $A \rtimes_r G$, that the canonical inclusion $i_{A,r} : A \to M(A \rtimes_r G) \subseteq M(A \otimes \mathcal{K}(L^2(G))$ maps a to $a \otimes 1$, and hence it is isometric. Since $i_{A,r} = \Lambda_A^G \circ i_A$, where i_A denotes the embedding of A into $M(A \rtimes G)$, we see that i_A is injective, too.

(5) If G is discrete, then A embeds into $A \rtimes_{(r)} G$ via $a \mapsto \delta_e \otimes a \in C_c(G, A) \subseteq A \rtimes_{(r)} G$. If, in addition, A is unital, then G also embeds into $A \rtimes_{(r)} G$ via $g \mapsto \delta_g \otimes 1$. If we identify $a \in A$ and $g \in G$ with their images in $A \rtimes_{(r)} G$, we obtain the relations $ga = \alpha_g(a)g$ for all $a \in A$ and $g \in G$. The full crossed product is then the universal algebra generated by A and G (viewed as a group of unitaries) subject to the relation $ga = \alpha_g(a)g$.

(6) In case $A = \mathbb{C}$ the maximal crossed product $C^*(G) := \mathbb{C} \rtimes G$ is called the *full group C^*-algebra* of G (note that $\mathbb{C}$ has only the trivial $*$-automorphism). The universal properties of $C^*(G)$ translate into a one-to-one correspondence between the unitary representations of G and the non-degenerate $*$-representations of $C^*(G)$ which induces a bijection between the set $\widehat{G}$ of equivalence classes of irreducible unitary Hilbert-space representations of G and $\widehat{C^*(G)}$. Again, we topologize $\widehat{G}$ so that this bijection becomes a homeomorphism.

The *reduced group C^*-algebra* $C_r^*(G) := \mathbb{C} \rtimes_r G$ is realized as the closure $\overline{\lambda(C_c(G))} \subseteq \mathcal{B}(L^2(G))$, where λ denotes the regular representation of G.

(7) If G is compact, then every irreducible representation is finite dimensional and the Jacobson topology on $\widehat{G} = \widehat{C^*(G)}$ is the discrete topology. Moreover, it follows from the Peter-Weyl theorem(e.g. see [48]) that $C^*(G)$ and $C_r^*(G)$ are isomorphic to the C^*-direct sum $\bigoplus_{U \in \widehat{G}} M_{\dim U}(\mathbb{C})$. In particular, we have $C^*(G) = C_r^*(G)$ if G

is compact.

(8) The convolution algebra $C_c(G)$, and hence also its completion $C^*(G)$, is commutative if and only if G is abelian. In that case $\widehat{G}$ coincides the set of continuous homomorphisms from G to the circle group $\mathbb{T}$, called *characters of* G, equipped with the compact-open topology. The Gelfand-Naimark theorem for commutative C^*-algebras then implies that $C^*(G) \cong C_0(\widehat{G})$ (which also coincides with $C_r^*(G)$ in this case). Note that $\widehat{G}$, equipped with the pointwise multiplication of characters, is again a locally compact abelian group and the Pontrjagin duality theorem asserts that $\widehat{\widehat{G}}$ is isomorphic to G via $g \mapsto \widehat{g} \in \widehat{\widehat{G}}$ defined by $\widehat{g}(\chi) = \chi(g)$. Notice that the Gelfand isomorphism $C^*(G) \cong C_0(\widehat{G})$ extends the *Fourier transform*

$$\mathcal{F} : C_c(G) \to C_0(\widehat{G}); \mathcal{F}(f)(\chi) = \chi(f) = \int_G f(x)\chi(x)\,dx.$$

For the circle group $\mathbb{T}$ we have $\mathbb{Z} \cong \widehat{\mathbb{T}}$ via $n \mapsto \chi_n$ with $\chi_n(z) = z^n$, and one checks that the above Fourier transform coincides with the classical Fourier transform on $C(\mathbb{T})$. Similarly, if $G = \mathbb{R}$, then $\mathbb{R} \cong \widehat{\mathbb{R}}$ via $s \mapsto \chi_s$ with $\chi_s(t) = e^{2\pi ist}$ and we recover the classical Fourier transform on $\mathbb{R}$ (see [48] for more details).

Example 3.5 (Transformation group algebras)**.** If (X, G) is a topological dynamical system, then we can form the crossed products $C_0(X) \rtimes G$ and $C_0(X) \rtimes_r G$ with respect to the corresponding action of G on $C_0(X)$. These algebras are often called the (full and reduced) *transformation group algebras* of the dynamical system (X, G). Many important C^*-algebras are of this type. For instance if $X = \mathbb{T}$ is the circle group and $\mathbb{Z}$ acts on $\mathbb{T}$ via $n \cdot z = e^{i2\pi\theta n}z$, $\theta \in [0, 1]$, then $A_\theta = C(\mathbb{T}) \rtimes \mathbb{Z}$ is the (rational or irrational) rotation algebra corresponding to θ (compare with §2.1 above). Indeed, since $\mathbb{Z}$ is discrete and $C(\mathbb{T})$ is unital, we have canonical embeddings of $C(\mathbb{T})$ and $\mathbb{Z}$ into $C(\mathbb{T}) \rtimes \mathbb{Z}$. If we denote by v the image of $\mathrm{id}_{\mathbb{T}} \in C(\mathbb{T})$ and by u the image of $1 \in \mathbb{Z}$ under these embeddings, then the relations given in part (5) of the above remark show that u, v are unitaries which satisfy the basic commutation relation $uv = e^{2\pi\theta i}vu$. It is this realization as a crossed product of A_θ which motivates the notion "rotation algebra".

There is quite some interesting and deep work on crossed products by actions of $\mathbb{Z}$ (or $\mathbb{Z}^d$) on compact spaces, which we cannot cover in this article. We refer the interested reader to the article [58] for a survey and for further references to this work.

Example 3.6 (Decomposition action)**.** Assume that $G = N \rtimes H$ is the semi-direct product ot two locally compact groups. If A is a G-algebra, then H acts canonically on $A \rtimes N$ (resp. $A \rtimes_r N$) via the extension of the action γ of H on $C_c(N, A)$ given by

$$\big(\gamma_h(f)\big)(n) = \delta(h)\alpha_h\big(f(h^{-1} \cdot n)\big),$$

where $\delta : H \to \mathbb{R}^+$ is determined by the equation $\int_N f(h \cdot n)\,dn = \delta(h)\int_N f(n)\,dn$ for all $f \in C_c(N)$. The inclusion $C_c(N, A) \subseteq A \rtimes_{(r)} N$ determines an inclusion $C_c(N \times K, A) \subseteq C_c(K, A \rtimes_{(r)} N)$ which extends to isomorphisms $A \rtimes (N \rtimes H) \cong$

$(A \rtimes N) \rtimes H$ and $A \rtimes_r (N \rtimes H) \cong (A \rtimes_r N) \rtimes_r H$. In particular, if $A = \mathbb{C}$, we obtain canonical isomorphisms $C^*(N \rtimes H) \cong C^*(N) \rtimes H$ and $C_r^*(N \rtimes H) \cong C_r^*(N) \rtimes_r H$.

We shall later extend the notion of crossed products to allow also the decompostion of crossed products by group extensions which are not topologically split.

Remark 3.7. When working with crossed products, it is often useful to use the following concrete realization of an approximate unit in $A \rtimes G$ (resp. $A \rtimes_r G$) in terms of a given approximate unit $(a_i)_{i\in I}$ in A: Let $\mathcal{U}$ be any neighborhood basis of the identity e in G, and for each $U \in \mathcal{U}$ let $\varphi_U \in C_c(G)^+$ such that $\int_G \varphi_U(t)\,dt = 1$. Let $\Lambda = I \times \mathcal{U}$ with $(i_1, U_1) \geq (i_2, U_2)$ if $i_1 \geq i_2$ and $U_1 \subseteq U_2$. Then a straightforward computation in the dense subalgebra $C_c(G, A)$ shows that $(\varphi_U \otimes a_i)_{(i,U)\in\Lambda}$ is an approximate unit of $A \rtimes G$ (resp. $A \rtimes_r G$), where we write $\varphi \otimes a$ for the function $(t \mapsto \varphi(t)a) \in C_c(G, A)$ if $\varphi \in C_c(G)$ and $a \in A$.

4. Crossed products versus tensor products

The following lemma indicates the conceptual similarity of full crossed products with maximal tensor products and of reduced crossed products with minimal tensor products of C^*-algebras.

Lemma 4.1. *Let (A, G, α) be a C^*-dynamical system and let B be a C^*-algebra. Let $\mathrm{id} \otimes_{\max} \alpha : G \to \mathrm{Aut}(B \otimes_{\max} A)$ be the diagonal action of G on $B \otimes_{\max} A$ (i.e., G acts trivially on B), and let $\mathrm{id} \otimes \alpha : G \to \mathrm{Aut}(B \otimes A)$ denote the diagonal action on $B \otimes A$. Then the obvious map $B \odot C_c(G, A) \to C_c(G, B \odot A)$ induces isomorphisms*

$$B \otimes_{\max} (A \rtimes_\alpha G) \cong (B \otimes_{\max} A) \rtimes_{\mathrm{id} \otimes \alpha} G \quad \textit{and} \quad B \otimes (A \rtimes_{\alpha,r} G) \cong (B \otimes A) \rtimes_{\mathrm{id} \otimes \alpha, r} G.$$

Sketch of proof. For the full crossed products check that both sides have the same non-degenerate representations and use the universal properties of full crossed products and maximal tensor product. For the reduced crossed products observe that the map $B \odot C_c(G, A) \to C_c(G, B \odot A)$ identifies $\mathrm{id}_B \otimes \Lambda_A^G$ with $\Lambda_{B\otimes A}^G$. □

Remark 4.2. As a special case of the above lemma (with $A = \mathbb{C}$) we see in particular that

$$B \rtimes_{\mathrm{id}} G \cong B \otimes_{\max} C^*(G) \quad \text{and} \quad B \rtimes_{\mathrm{id},r} G \cong B \otimes C_r^*(G).$$

We now want to study an important condition on G which implies that full and reduced crossed products by G always coincide.

Definition 4.3. Let $1_G : G \to \{1\} \subseteq \mathbb{C}$ denote the trivial representation of G. Then G is called *amenable* if $\ker 1_G \supseteq \ker \lambda$ in $C^*(G)$, i.e., if the integrated form of 1_G factors through a homomomorphism $1_G^r : C_r^*(G) \to \mathbb{C}$. [4]

Remark 4.4. The above definition is not the standard definition of amenability of groups, but it is one of the many equivalent formulations for amenability (e.g. see [54, 23]), and it is best suited for our purposes. It is not hard to check (even using the above C^*-theoretic definition) that abelian groups and compact groups

[4] In particular, it follows that $1_G^r(\lambda_s) = 1_G(s) = 1$ for all $s \in G$!

are amenable. Moreover, extensions, quotients, and closed subgroups of amenable groups are again amenable. In particular, all solvable groups are amenable.

On the other side, one can show that the non-abelian free group F_2 on two generators, and hence any group which contains F_2 as a closed subgroup, is not amenable. This shows that non-compact semi-simple Lie groups are never amenable. For extensive studies of amenability of groups (and groupoids) we refer to [54, 97, 2].

If (π, U) is is covariant representation of (A, G, α) on some Hilbert space H, then the covariant representation $(\pi \otimes 1, U \otimes \lambda)$ of (A, G, α) on $H \otimes L^2(G) \cong L^2(G, H)$ is unitarily equivalent to $\operatorname{Ind}\pi$ via the unitary $W \in U(L^2(G, H))$ defined by $(W\xi)(s) = U_s\xi(s)$. Thus, if π is faithful on A, then $(\pi \otimes 1) \times (U \otimes \lambda)$ factors through a faithful representation of $A \rtimes_r G$. As an important application we get

Proposition 4.5. *If G is amenable, then $\Lambda_A^G : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is an isomorphism.*

Proof. Choose any faithful representation $\pi \times U$ of $A \rtimes_\alpha G$ on some Hilbert space H. Regarding $(\pi \otimes 1, U \otimes \lambda)$ as a representation of (A, G, α) into $M(\mathcal{K}(H) \otimes C_r^*(G))$, we obtain the equation

$$(\mathrm{id} \otimes 1_G^r) \circ \big((\pi \otimes 1) \times (U \otimes \lambda)\big) = \pi \times U.$$

Since π is faithful, it follows that

$$\ker \Lambda_A^G = \ker(\operatorname{Ind}\pi) = \ker\big((\pi \otimes 1) \times (U \otimes \lambda)\big) \subseteq \ker(\pi \times U) = \{0\}.$$

□

The special case $A = \mathbb{C}$ gives

Corollary 4.6. *G is amenable if and only if $\lambda : C^*(G) \to C_r^*(G)$ is an isomorphism.*

A combination of Lemma 4.1 with Proposition 4.5 gives the following important result:

Theorem 4.7. *Let A be a nuclear G-algebra with G amenable. The $A \rtimes_\alpha G$ is nuclear.*

Proof. Using Lemma 4.1 and Proposition 4.5 we get

$$\begin{aligned} B \otimes_{\max} (A \rtimes_\alpha G) &\cong (B \otimes_{\max} A) \times_{\mathrm{id} \otimes \alpha} G \cong (B \otimes A) \times_{\mathrm{id} \otimes \alpha} G \\ &\cong (B \otimes A) \times_{\mathrm{id} \otimes \alpha, r} G \cong B \otimes (A \rtimes_{\alpha,r} G) \cong B \otimes (A \rtimes_\alpha G). \end{aligned}$$

□

If (A, G, α) and (B, G, β) are two systems, then a G-equivariant homomorphism $\phi : A \to M(B)$ [5] induces a $*$-homomorphism

$$\phi \rtimes G := (i_B \circ \phi) \times i_G : A \rtimes_\alpha G \to M(B \times_\beta G)$$

[5] where we uniquely extend β to an action of $M(B)$, which may fail to be strongly continuous

where (i_B, i_G) denote the canonical embeddings of (B, G) into $M(B \rtimes_\beta G)$, and a similar $*$-homomorphism

$$\phi \rtimes_r G := \operatorname{Ind}\phi : A \rtimes_{\alpha,r} G \to M(B \rtimes_{\beta,r} G) \subseteq M\big(B \otimes \mathcal{K}(L^2(G))\big).$$

Both maps are given on the level of functions by

$$\phi \rtimes_{(r)} G(f)(s) = \phi(f(s)), \quad f \in C_c(G, A).$$

If $\phi(A) \subseteq B$, then $\phi \rtimes G(A \rtimes_\alpha G) \subseteq B \rtimes_\beta G$ and similarly for the reduced crossed products. Moreover, $\phi \rtimes_r G = \operatorname{Ind}\phi$ is faithful if and only if ϕ is — a result which does **not** hold in general for $\phi \rtimes G$!

On the other hand, the following proposition shows that taking full crossed products gives an exact functor between the category of G-C^*-algebras and the category of C^*-algebras, which is not at all clear for the reduced crossed-product functer!

Proposition 4.8. *Assume that $\alpha : G \to \operatorname{Aut}(A)$ is an action and I is a G-invariant closed ideal in A. Let $j : I \to A$ denote the inclusion and let $q : A \to A/I$ denote the quotient map. Then the sequence*

$$0 \to I \rtimes_\alpha G \xrightarrow{j \rtimes G} A \rtimes_\alpha G \xrightarrow{q \rtimes G} (A/I) \rtimes_\alpha G \to 0$$

is exact.

Proof. If (π, U) is a non-degenerate representation of (I, G, α) into $M(D)$, then (π, U) has a canonical extension to a covariant homomorphism of (A, G, α) by defining $\pi(a)(\pi(b)d) = \pi(ab)d$ for $a \in A, b \in I$ and $d \in D$. By the definition of $\|\cdot\|_{\max}$, this implies that the inclusion $I \rtimes_\alpha G \to A \rtimes_\alpha G$ is isometric.

Assume now that $p : A \rtimes_\alpha G \to (A \rtimes_\alpha G)/(I \rtimes_\alpha G)$ is the quotient map. Then $p = \rho \times V$ for some covariant homomorphism (ρ, V) of (A, G, α) into $M\big((A \rtimes G)/(I \rtimes G)\big)$. Let $i_A : A \to M(A \rtimes G)$ denote the embedding. Then we have $i_A(I)C_c(G, A) = C_c(G, I) \subseteq I \rtimes G$ from which it follows that

$$\rho(I)(\rho \times V(C_c(G, A)) = \rho \times V\big(i_A(I)(A \rtimes G)\big) \subseteq \rho \times V(I \rtimes G) = \{0\}.$$

Since $\rho \times V(C_c(G, A))$ is dense in $A/I \rtimes G$, it follows that $\rho(I) = \{0\}$. Thus ρ factors through a representation of A/I and $p = \rho \times V$ factors through $A/I \rtimes_\alpha G$. This shows that the crossed product sequence is exact in the middle term. Since $C_c(G, A)$ clearly maps onto a dense subset in $A/I \rtimes_\alpha G$, $q \rtimes G$ is surjective and the result follows. $\square$

For quite some time it was an open question whether the analogue of Proposition 4.8 also holds for the reduced crossed products. This problem lead to

Definition 4.9 (Kirchberg – S. Wassermann)**.** A locally compact group G is called *C^*-exact* (or simply exact) if for any system (A, G, α) and any G-invariant ideal $I \subseteq A$ the sequence

$$0 \to I \rtimes_{\alpha,r} G \xrightarrow{j \rtimes_r G} A \rtimes_{\alpha,r} G \xrightarrow{q \rtimes_r G} A/I \rtimes_{\alpha,r} G \to 0$$

is exact.

Let us remark that only exactness in the middle term is the problem, since $q \rtimes_r G$ is clearly surjective, and $j \rtimes_r G = \operatorname{Ind} j$ is injective since j is. We shall later report on Kirchberg's and S. Wassermann's permanence results on exact groups, which imply that the class of exact groups is indeed very large. However, a recent construction of Gromov (see [56, 49]) implies that there do exist finitely generated discrete groups which are not exact!

5. The Morita categories

In this section we want to give some theoretical background for the discussion of imprimitivity theorems for crossed products and for the theory of induced representations. The basic notion for this is the notion of the Morita category in which the objects are C^*-algebras and the morphisms are unitary equivalence classes of Hilbert bimodules. Having this at hand, the theory of induced representations will reduce to taking compositions of morphisms in the Morita category. All this is based on the fundamental idea of Rieffel (see [106]) who first made systematic approach to the theory of induced representations of C^*-algebras in terms of (pre-) Hilbert modules, and who showed how the theory of induced group representations can be seen as part of this more general theory. However, it seems that a systematic categorical treatment of this theory was first given in [32] and, in parallel work by Landsman in [75].

5.1. **Hilbert modules.** If B is a C^*-algebra, then a (right) Hilbert B-module is a complex Banach space E equipped with a right B-module structure and a B-valued inner product $\langle\cdot,\cdot\rangle_B : E \times E \to B$, which is linear in the second and antilinear in the first variable and satisfies

$$(\langle\xi,\eta\rangle_B)^* = \langle\eta,\xi\rangle_B, \quad \langle\xi,\eta\rangle_B b = \langle\xi,\eta\cdot b\rangle_B, \quad \text{and} \quad \|\xi\|^2 = \|\langle\xi,\xi\rangle_B\|$$

for all $\xi,\eta \in E$ and $b \in B$. With the obvious modifications we can also define *left*-Hilbert B-modules. The Hilbert $\mathbb{C}$-modules are precisely the Hilbert spaces. Moreover, every C^*-algebra B becomes a Hilbert B-module by defining $\langle b,c\rangle_B := b^*c$. We say that E is a *full* Hilbert B-module, if

$$B = \langle E,E\rangle_B := \overline{\operatorname{span}}\{\langle\xi,\eta\rangle_B : \xi,\eta \in E\}.$$

In general $\langle E,E\rangle_B$ is a closed two-sided ideal of B.

If E is a Hilbert B-module, then a linear map $T : E \to E$ is called *adjointable* if there exists a map $T^* : E \to E$ such that $\langle T\xi,\eta\rangle_B = \langle\xi,T^*\eta\rangle_B$ for all $\xi,\eta \in E$. [6] Every adjointable operator on E is automatically bounded and B-linear. The set

$$\mathcal{L}_B(E) = \{T : E \to E : T \text{ is adjointable}\}$$

becomes a C^*-algebra with respect to the usual operator norm. Every pair of elements $\xi,\eta \in E$ determines an element $\Theta_{\xi,\eta} \in \mathcal{L}_B(E)$ given by

$$\Theta_{\xi,\eta}(\zeta) := \xi \cdot \langle\eta,\zeta\rangle_B \tag{5.1}$$

[6] Notice that, different from the operators on Hilbert space, a bounded B-linear operator $T : E \to E$ is **not** automatically adjointable.

with adjoint $\Theta_{\xi,\eta}^* = \Theta_{\eta,\xi}$. The closed linear span of all such operators forms the *ideal of compact operators* $\mathcal{K}_B(E)$ in $\mathcal{L}_B(E)$ Note that there is an obvious $*$-isomorphism between the multiplier algebra $M(\mathcal{K}_B(E))$ and $\mathcal{L}_B(E)$, which is given by extending the action of $\mathcal{K}_B(E)$ on E to all of $M(\mathcal{K}_B(E))$ in the canonical way.

Example 5.1. **(1)** If $B = \mathbb{C}$ H is a Hilbert space, then $\mathcal{L}_{\mathbb{C}}(H) = \mathcal{B}(H)$ and $\mathcal{K}_{\mathbb{C}}(H) = \mathcal{K}(H)$.
(2) Every C^*-algebra B can be viewed as a Hilbert B-module by defining $\langle b, c\rangle_B = b^*c$ and the obvious right module operation. It is then easy to check that $\mathcal{K}_B(B) = B$, where we B act on itself via left multiplication, and then we have $\mathcal{L}_B(B) = M(B)$.

It is important to notice that, in case $B \neq \mathbb{C}$, the notion of compact operators as given above does **not** coincide with the standard notion of compact operators on a Banach space (i.e., that the image of the unit ball has compact closure). For example, if B is unital, then $\mathcal{L}_B(B) = \mathcal{K}_B(B) = B$ and we see that the identity operator on B is a compact operator in the sense of the above definition. But if B is not finite dimensional, the identity operator is not a compact operator in the usual sense of Banach-space operators.

There is a one-to-one correspondence between right and left Hibert B-modules given by the operation $E \mapsto E^* := \{\xi^* : \xi \in E\}$, with left action of B on E^* given by $b \cdot \xi^* := (\xi \cdot b^*)^*$ and with inner product ${}_B\langle \xi^*, \eta^*\rangle := \langle \xi, \eta\rangle_B$ (notice that the inner product of a left Hilbert B-module is linear in the first and antilinear in the second variable). We call E^* the *adjoint module* of E. Of course, if F is a left Hilbert B-module, a similar construction yields the adjoint F^* — a right Hilbert B-module. Clearly, the notions of adjointable and compact operators also have their left analogues (thought of as acting on the right), and we have $\mathcal{L}_B(E) = \mathcal{L}_B(E^*)$ (resp. $\mathcal{K}_B(E) = \mathcal{K}_B(E^*)$) via $\xi^*T := (T^*\xi)^*$.

There are several important operations on Hilbert modules (like taking the direct sum $E_1 \bigoplus E_2$ of two Hilbert B-modules E_1 and E_2 in the obvious way). But for our considerations the construction of the interior tensor products is most important. For this assume that E is a (right) Hilbert A-module, F is a (right) Hilbert B-module, and $\Psi : A \to \mathcal{L}_B(F)$ is a $*$-homomorphism. Then the *interior tensor product* $E \otimes_A F$ is defined as the Hausdorff completion of $E \odot F$ with respect to the B-valued inner product

$$\langle \xi \otimes \eta, \xi' \otimes \eta'\rangle_B = \langle \Psi(\langle \xi, \xi'\rangle_A) \cdot \eta, \eta'\rangle_B,$$

where $\xi, \xi' \in E$ and $\eta, \eta' \in F$. With this inner product, $E \otimes_A F$ becomes a Hilbert B-module. Moreover, if C is a third C^*-algebra and if $\Phi : C \to \mathcal{L}_A(E)$ is a $*$-representation of C on $\mathcal{L}_A(E)$, then $\Phi \otimes 1 : C \to \mathcal{L}_B(E \otimes_A F)$ with $\Phi \otimes 1(c)(\xi \otimes \eta) = \Phi(c)\xi \otimes \eta$ becomes a $*$-representation of C on $E \otimes_A F$ (we refer to [74, 105] for more details). The construction of this representation is absolutely crucial in what follows below.

5.2. **Morita equivalences.** The notion of Morita equivalent C^*-algebras, which goes back to Rieffel [106] is one of the most important tools in the study of crossed products.

Definition 5.2 (Rieffel)**.** Let A and B be C^*-algebras. An A-B *imprimitivity bimodule*[7] X is a Banach space X which carries the structure of both, a right Hilbert B-module and a left Hilbert A-module with commuting actions of A and B such that

(i) ${}_A\langle X, X\rangle = A$ and $\langle X, X\rangle_B = B$ (i.e., both inner products on X are full);
(ii) ${}_A\langle \xi, \eta\rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_B$ for all $\xi, \eta, \zeta \in X$.

A and B are called *Morita equivalent* if such A-B bimodule X exists.

Remark 5.3. **(1)** It follows from the above definition together with (5.1) that, if X is an A-B imprimitivity bimodule, then A canonically identifies with $\mathcal{K}_B(X)$ and B canonically identifies with $\mathcal{K}_A(X)$. Conversely, if E is any Hilbert B-module, then ${}_{\mathcal{K}(E)}\langle \xi, \eta\rangle := \Theta_{\xi,\eta}$ (see (5.1)) defines a full $\mathcal{K}_B(E)$-valued inner product on E, and E becomes a $\mathcal{K}_B(E) - \langle B, B\rangle_B$ imprimitivity bimodule. In particular, if E is a full Hilbert B-module (i.e., $\langle E, E\rangle_B = B$), then B is Morita equivalent to $\mathcal{K}_B(E)$.
(2) As a very special case of (1) we see that $\mathbb{C}$ is Morita equivalent to $\mathcal{K}(H)$ for every Hilbert space H.
(3) It is easily checked that Morita equivalence is an equivalence relation: If A is any C^*-algebra, then A becomes an A-A imprimitivity bimodule with respect to ${}_A\langle a, b\rangle = ab^*$ and $\langle a, b\rangle_A = a^*b$ for $a, b \in A$. If X is an A-B imprimitivity bimodule and Y is a B-C imprimitivity bimodule, then $X \otimes_B Y$ is an $A - C$ equivalence bimodule. Finally, if X is an A-B-imprimitivity bimodule, then the adjoint module X^* is a $B - A$ imprimitivity bimodule.
(4) Recall that a C^*-algebra A is a *full corner* of the C^*-algebra C, if there exists a full projection $p \in M(C)$ (i.e., $\overline{CpC} = C$) such that $A = pCp$. Then pC equipped with canonical inner products and actions coming from multiplication and involution on C becomes an $A-C$ imprimitivity bimodule. Thus, if A and B can be represented as full corners of a C^*-algebra C, they are Morita equivalent. Conversely, let X be an A-B imprimitivity bimodule. Let $L(X) = \left(\begin{smallmatrix} A & X \\ X^* & B\end{smallmatrix}\right)$ with multiplication and involution defined by

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix}\begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + {}_A\langle \xi_1, \eta_2\rangle & a_1 \cdot \xi_2 + \xi_1 \cdot b_2 \\ \eta_1^* \cdot a_2 + b_1 \cdot \eta_2^* & \langle \eta_1, \xi_2\rangle_B + b_1b_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}.$$

Then $L(X)$ has a canonical embedding as a closed subalgebra of the adjointable operators on the Hilbert B-module $X \bigoplus B$ via

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}\begin{pmatrix} \zeta \\ d \end{pmatrix} = \begin{pmatrix} a\zeta + \xi d \\ \langle \eta, \zeta\rangle_B + bd \end{pmatrix}$$

which makes $L(X)$ a C^*-algebra. If $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right) \in M(L(X))$, then p and $q := 1 - p$ are full projections such that $A = pL(X)p$, $B = qL(X)q$ and $X = pL(X)q$. The algebra $L(X)$ is called the *linking algebra* of X. It often serves as a valuable tool for the study of imprimitivity bimodules.
(5) It follows from (4) that A is Morita equivalent to $A \otimes \mathcal{K}(H)$ for any Hilbert space H (since A is a full corner of $A \otimes \mathcal{K}(H)$). Indeed, a deep theorem of Brown, Green and Rieffel shows (see [9]) that if A and B are σ-unital[8], then A and B are Morita

[7] often called an A-B *equivalence* bimodule in the literature

[8] A C^*-algebra is called *σ-unital*, if it has a countable approximate unit. In particular, all separable and all unital C^*-algebras are σ-unital

equivalent if and only if they are *stably isomorphic*, i.e., there exists an isomorphism between $A \otimes \mathcal{K}(H)$ and $B \otimes \mathcal{K}(H)$ with $H = l^2(\mathbb{N})$. A similar result does not hold if the σ-unitality assumption is dropped (see [9]).

(6) The above results indicate that many important properties of C^*-algebras are preserved by Morita equivalences. Indeed, among these properties are: nuclearity, exactness, simplicity, the property of being a type I algebra (and many more). Moreover, Morita equivalent C^*-algebras have homeomorphic primitive ideal spaces and isomorphic K-groups. Most of these properties will be discussed later in more detail (e.g., see Propositions 5.4, 5.11 and 5.12 below).

A very important tool when working with imprimitivity bimodules is the Rieffel correspondence. To explain this suppose that X is an A-B imprimitivity bimodule and that I is a closed ideal of B. Then $X \cdot I$ is a closed A-B submodule of X and $\mathrm{Ind}^X I := {}_A\langle X \cdot I, X \cdot I\rangle$ (taking the closed span) is a closed ideal of A. The following proposition implies that Morita equivalent C^*-algebras have equivalent ideal structures:

Proposition 5.4 (Rieffel correspondence)**.** *Assume notation as above. Then*

(i) *The assignments $I \mapsto X \cdot I$, $I \mapsto \mathrm{Ind}^X I$ and $I \mapsto J_I := \begin{pmatrix} \mathrm{Ind}^X I & X\cdot I \\ I\cdot\widetilde{X} & I \end{pmatrix}$ provide inclusion preserving bijective correspondences between the closed two-sided ideals of B, the closed A-B-submodules of X, the closed two-sided ideals of A, and the closed two-sided ideals of the linking algebra $L(X)$, respectively.*

(ii) *$X \cdot I$ is an $\mathrm{Ind}^X I$-I imprimitivity bimodule and $X/(X \cdot I)$, equipped with the obvious inner products and bimodule actions, becomes an $A/(\mathrm{Ind}^X I)$-B/I imprimitivity bimodule. Moreover, we obviously have $J_I = L(X \cdot I)$ and $L(X)/J_I \cong L(X/X \cdot I)$.*

Remark 5.5. Assume that X is a A-B imprimitivity bimodule and Y is a C-D imprimitivity bimodule. An *imprimitivity bimodule homomorphism* from X to Y is then a triple (ϕ_A, ϕ_X, ϕ_B) such that $\phi_A : A \to C$ and $\Phi_B : B \to D$ are $*$-homomorphisms and $\phi_X : X \to Y$ is a linear map such that the triple (ϕ_A, ϕ_X, ϕ_B) satisfies the obvious compatibilty conditions with respect to the inner products and module actions on X and Y (e.g. $\langle\phi_X(\xi), \phi_X(\eta)\rangle_D = \phi_B(\langle\xi, \eta\rangle_B)$, $\phi_X(\xi b) = \phi_X(\xi)\phi_B(b)$, etc.).

If (ϕ_A, ϕ_X, ϕ_B) is such an imprimitivity bimodule homomorphism, then one can check that $\ker\phi_A$, $\ker\phi_X$ and $\ker\phi_B$ all correspond to each other under the Rieffel correspondence for X.

As a simple application of the Rieffel correspondence and the above remark we now show

Proposition 5.6. *Suppose that A and B are C^*-algebras. Then A is nuclear if and only if B is nuclear.*

Sketch of proof. Let X be an A-B imprimitivity bimodule. If C is any other C^*-algebra, we can equipp $X \odot C$ with $A \odot C$- and $B \odot C$-valued inner products and a $A \odot C$-$B \odot C$ module structure in the obvious way. Then $X \odot C$ completes to an

$A \otimes_{\max} C$-$B \otimes_{\max} C$ imprimitivity bimodule $X \otimes_{\max} C$ as well as to an $A \otimes C$-$B \otimes C$ imprimitivity bimodule $X \otimes C$. The identity map on $X \odot C$ then extends to a quotient map $X \otimes_{\max} C \to X \otimes C$ which together with the quotient maps $A \otimes_{\max} C \to A \otimes C$ and $B \otimes_{\max} C \to B \otimes C$ is an imprimitivity bimodule homomorphism. But then it follows from the above remark and the Rieffel correspondence that injectivity of any one of these quotient maps implies injectivity of all three of them. $\square$

5.3. **The Morita category.** We now come to the definition of the Morita categories. Suppose that A and B are C^*-algebras. A *(right) Hilbert A-B bimodule* is a pair (E, Φ) in which E is a Hilbert B-module and $\Phi : A \to \mathcal{L}_B(E)$ is a $*$-representation of A on E. We say that (E, Φ) is non-degenerate, if $\Phi(A)E = E$ (this is equivalent to $\Phi : A \to M(\mathcal{K}_B(E)) = \mathcal{L}_B(E)$ being non-degenerate in the usual sense). Two Hibert A-B bimodules (E_i, Φ_i), $i = 1, 2$ are called *unitarily equivalent* if there exists an isomorphism $U : E_1 \to E_2$ preserving the B-valued inner products such that $U\Phi_1(a) = \Phi_2(a)U$ for all $a \in A$. We denote by $[E, \Phi]$ the unitary equivalence class of (E, Φ).

Definition 5.7 (cf. [32, 33, 75])**.** The *Morita category* $\mathfrak{M}$ is the category whose objects are C^*-algebras and where the morphisms from A to B are given by unitary equivalence classes $[E, \Phi]$ of non-degenerate Hilbert A-B bimodules (E, Φ). The identity morphism from A to A is represented by the trivial A-A bimodule (A, id) and composition of two morphisms $[E, \Phi] \in \mathrm{Mor}(A, B)$ and $[F, \Psi] \in \mathrm{Mor}(B, C)$ is given by taking the interior tensor product $[E \otimes_B F, \Phi \otimes 1]$.

The *compact Morita category* $\mathfrak{M}_c$ is the subcategory of $\mathfrak{M}$ in which we additionally require $\Phi(A) \subseteq \mathcal{K}_B(E)$ for a morphism $[E, \Phi] \in \mathrm{Mor}_c(A, B)$.

If X is an A-B imprimitivity bimodule, then the adjoint module X^* satisfies $X \otimes_B X^* \cong A$ as A-A bimodule and $X^* \otimes_A X \cong B$ as B-B bimodule, so X^* is an inverse of X in the Morita categories. Indeed we have

Proposition 5.8 (cf [75, 32])**.** *The isomorphisms in the categories $\mathfrak{M}$ and $\mathfrak{M}_c$ are precisely the Morita equivalences.*

Remark 5.9. **(1)** Note that $\mathfrak{M}$ and $\mathfrak{M}_c$ are not categories in the strong sense, since the collections $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}_c(A, B)$ of morphisms from A to B do not form sets. However, if we restrict to separable C^*-algebras and bimodules, then $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}_c(A, B)$ do form sets and $\mathfrak{M}$ and $\mathfrak{M}_c$ will become real categories. Of course, one could similarly restrict to objects and modules with cardinality (of dense subsets) restricted by any other cardinal number to force $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}_c(A, B)$ to be sets. We shall mostly ignore this point below.
(2) It is necessary to require the bimodules (E, Φ) to be non-degenerate (i.e., $\Phi(A)E = E$) in order to have $[A, \mathrm{id}] \circ [E, \Phi] = [A \otimes_A E, \mathrm{id} \otimes 1] = [E, \Phi]$. However, if we are willing to identify an arbitrary A-B Hilbert bimodule (E, Φ) with $(\Phi(A)E, \Phi)$, we can include all bimodules into the picture.
(3) Note that every $*$-homomorphism $\Phi : A \to M(B)$ determines a morphism $[E, \Phi] \in \mathrm{Mor}(A, B)$ in $\mathfrak{M}$ (with $E = \Phi(A)B$), and $[E, \Phi]$ is a morphism in $\mathfrak{M}_c$ if and

only if $\Phi(A) \subseteq B$.
(4) Taking direct sums of bimodules allows to define sums of morphisms in the Morita categories (and hence a semi-group structure with neutral element given by the zero-module). It is easy to check that this operation is commutative and satisfies the distributive law with respect to composition.

5.4. **The equivariant Morita categories.** If G is a locally compact group, then the *G-equivariant Morita category* $\mathfrak{M}(G)$ is the category in which the objects are systems (A, G, α) and morphisms from (A, G, α) to (B, G, β) are the unitary equivalence classes of equivariant non-degenerate A-B Hilbert bimodules (E, Φ, u), i.e., E is equipped with a strongly continuous homomorphism $u : G \to \mathrm{Aut}(E)$ such that

$$\begin{aligned} \langle u_s(\xi), u_s(\eta)\rangle_B = \beta_s(\langle \xi, \eta\rangle_B), \quad u_s(\xi \cdot b) = u_s(\xi)\beta_s(b) \\ \text{and} \quad u_s(\Phi(a)\xi) = \Phi(\alpha_s(a))u_s(\xi). \end{aligned} \tag{5.2}$$

Again, composition of morphisms is given by taking interior tensor products equipped with the diagonal actions, and the equivalences in this category are just the equivariant Morita equivalences.

Notice that the crossed product constructions $A \rtimes_G$ and $A \rtimes_r G$ extend to descent functors

$$\rtimes_{(r)} : \mathfrak{M}(G) \to \mathfrak{M}.$$

In particular, Morita equivalent systems have Morita equivalent full (resp. reduced) crossed products. If $[E, \phi, u]$ is a morphism from (A, G, α) to (B, G, β), then the crossed product $[E \rtimes_{(r)} G, \Phi \rtimes_{(r)} G] \in \mathrm{Mor}(A \rtimes_{(r)} G, B \rtimes_{(r)} G)$ is given as the completion of $C_c(G, E)$ with respect to the $B \rtimes_{(r)} G$-valued inner product

$$\langle \xi, \eta\rangle_{B\rtimes_{(r)}G}(t) = \int_G \langle \xi(s), u_s(\eta(s^{-1}t))\rangle_B \, ds$$

(taking values in $C_c(G, B) \subseteq B \rtimes_{(r)} G$) and with left action of $C_c(G, A) \subseteq A \rtimes_{(r)} G$ on $E \rtimes_{(r)} G$ given by

$$\big(\Phi \rtimes_{(r)} G(f)\xi\big)(t) = \int_G \Phi(f(s))u_s(\xi(s^{-1}t)) \, ds.$$

The crossed product constructions for equivariant bimodules first appeared in Kasparov's famous Conspectus [65], which circulated as a preprint from the early eighties. A more detailed study in case of imprimitivity bimodules has been given in [18]. A very extensive study of the equivariant Morita categories for actions and coactions of groups together with their relations to duality theory are given in [33].

5.5. **Induced representations and ideals.** If B is a C^*-algebra we denote by $\mathrm{Rep}(B)$ the collection of all equivalence classes of non-degenerate $*$-representations of B on Hilbert space. In terms of the Morita category, $\mathrm{Rep}(B)$ coincides with the collection $\mathrm{Mor}(B, \mathbb{C})$ of morphisms from B to $\mathbb{C}$ in $\mathfrak{M}$. Thus, if A is any other C^*-algebra and if $[E, \Phi] \in \mathrm{Mor}(A, B)$, then composition with $[E, \Phi]$ determines a map

$$\mathrm{Ind}^{(E,\Phi)} : \mathrm{Rep}(B) \to \mathrm{Rep}(A); [H, \pi] \mapsto [E, \Phi] \circ [H, \pi] = [E \otimes_B H, \Phi \otimes 1].$$

If confusion seems unlikely, we will simply write π for the representation (H, π) and for its class $[H, \pi] \in \mathrm{Rep}(A)$ and we write $\mathrm{Ind}^E \pi$ for the representation $\Phi \otimes 1$ of A on $\mathrm{Ind}^E H := E \otimes_B H$. We call $\mathrm{Ind}^E \pi$ the *representation of A induced from π via E*.

Remark 5.10. (1) A special case of the above procedure is given in case when $\Phi : A \to M(B)$ is a non-degenerate $*$-homomorphism and $[B, \Phi] \in \mathrm{Mor}(A, B)$ is the corresponding morphism in $\mathfrak{M}$. Then the induction map $\mathrm{Ind}^B : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$ coincides with the obvious map

$$\Phi^* : \mathrm{Rep}(B) \to \mathrm{Rep}(A); \pi \mapsto \Phi^*(\pi) := \pi \circ \Phi.$$

(2) Induction in steps. If $[H, \pi] \in \mathrm{Rep}(B)$, $[E, \Phi] \in \mathrm{Mor}(A, B)$ and $[F, \Psi] \in \mathrm{Mor}(D, A)$ for some C^*-algebra D, then it follows directly from the associativity of composition in $\mathfrak{M}$ that (up to equivalence)

$$\mathrm{Ind}^F(\mathrm{Ind}^E \pi) = \mathrm{Ind}^{F \otimes_A E} \pi.$$

(3) If X is an A-B imprimitivity bimodule, then $\mathrm{Ind}^X : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$ gets inverted by $\mathrm{Ind}^{X^*} : \mathrm{Rep}(A) \to \mathrm{Rep}(B)$, where X^* denotes the adjoint of X (i.e., the inverse of $[X]$ in $\mathfrak{M}$). Since composition of morphisms in $\mathfrak{M}$ preserves direct sums, it follows from this that induction via X maps irreducible representations of B to irreducible representations of A and hence induces a bijection $\mathrm{Ind}^X : \widehat{B} \to \widehat{A}$ between the spectra.

It is useful to consider a similar induction map on the set $\mathcal{I}(B)$ of closed two sided ideals of the C^*-algebra B. If (E, Φ) is any Hilbert A-B bimodule, we define

(5.3) $\quad \mathrm{Ind}^E : \mathcal{I}(B) \to \mathcal{I}(A);\ \mathrm{Ind}^E I := \{a \in A : \langle \Phi(a)\xi, \eta \rangle_B \in I \text{ for all } \xi, \eta \in E\}.$ [9]

It is clear that induction preserves inclusion of ideals and with a little work one can check that

$$\mathrm{Ind}^E(\ker \pi) = \ker(\mathrm{Ind}^E \pi) \quad \text{for all } \pi \in \mathrm{Rep}(B). \tag{5.4}$$

Hence it follows from part (3) of Remark 5.10 that, if X is an A-B imprimitivity bimodule, then induction of ideals via X restricts to give a bijection $\mathrm{Ind}^X : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ between the primitive ideal spaces of B and A. Since induction preserves inclusion of ideals, the next proposition follows directly from the description of the closure operations in $\widehat{A}$ and $\mathrm{Prim}(A)$ (see §2.4).

Proposition 5.11 (Rieffel)**.** *Let X be an A-B imprimitivity bimodule. Then the bijections*

$$\mathrm{Ind}^X : \widehat{B} \to \widehat{A} \quad \textit{and} \quad \mathrm{Ind}^X : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$$

are homeomorphisms.

[9] If X is an A-B imprimitivity bimodule, the induced ideal $\mathrm{Ind}^X I$ defined here coincides with the induced ideal $\mathrm{Ind}^X I = {}_A\langle X \cdot I, X \cdot I \rangle$ of the Rieffel correspondence (see Proposition 5.4).

Notice that these homeomorphisms are compatible with the Rieffel-correspondence (see Proposition 5.4): If I is any closed ideal of B and if we identify $\widehat{B}$ with the disjoint union $\widehat{I} \cup \widehat{B/I}$ in the canonical way (see §2.4), then induction via X "decomposes" into induction via $Y := X \cdot I$ from $\widehat{I}$ to $(\operatorname{Ind}^X I)\widehat{\ }$ and induction via X/Y from $\widehat{B/I}$ to $(A/\operatorname{Ind}^X I)\widehat{\ }$. This helps to prove

Proposition 5.12. *Suppose that A and B are Morita equivalent C^*-algebras. Then*

(i) *A is type I if and only if B is type I.*
(ii) *A is CCR if and only if B is CCR.*
(iii) *A has continuous trace algebra if and only if B has contininuous trace.*

Proof. Recall from §2.4 that a C^*-algebra B is type I if and only if for each $\pi \in \widehat{B}$ the image $\pi(B) \subseteq \mathcal{B}(H_\pi)$ contains $\mathcal{K}(H_\pi)$. Furthermore, B is CCR if and only if B is type I and points are closed in $\widehat{B}$.

If X is an A-B imprimitivity bimodule and $\pi \in \widehat{B}$, we may pass to $B/\ker\pi$ and $A/\ker(\operatorname{Ind}^X \pi)$ via the Rieffel correspondence to assume that π and $\operatorname{Ind}^E \pi$ are injective, and hence that $B \subseteq \mathcal{B}(H_\pi)$ and $A \subseteq \mathcal{B}(X \otimes_B H_\pi)$. If B is type I, it follows that $\mathcal{K} := \mathcal{K}(H_\pi)$ is an ideal of B. Let $Z := X \cdot \mathcal{K}$. Then Z is an $\operatorname{Ind}^X \mathcal{K} - \mathcal{K}$ imprimitivity bimodule and $Z \otimes_{\mathcal{K}} H_\pi$, the composition of Z with the $\mathcal{K} - \mathbb{C}$ imprimitivity bimodule H_π, is an $\operatorname{Ind}^X \mathcal{K} - \mathbb{C}$ imprimitivity bimodule. It follows that $\operatorname{Ind}^X \mathcal{K} \cong \mathcal{K}(Z \otimes_K H_\pi)$. Since $Z \otimes_{\mathcal{K}} H_\pi \cong X \otimes_B H_\pi$ via the identity map on both factors, we conclude that $\operatorname{Ind}^X \pi(A)$ contains the compact operators $\mathcal{K}(X \otimes_B H_\pi)$. This proves (i). Now (ii) follows from (i) since $\widehat{B}$ is homeomorphic to $\widehat{A}$. The proof of (iii) needs a bit more room and we refer the interested reader to [122,]. $\square$

Of course, similar induction procedures as described above can be defined in the equivariant settings: If (A, G, α) is a system, then the morphisms from (A, G, α) to $(\mathbb{C}, G, \mathrm{id})$ in $\mathfrak{M}(G)$ are just the unitary equivalence classes of non-degenerate covariant representations of (A, G, α) on Hilbert space, which we shall denote by $\operatorname{Rep}(A, G)$ (supressing the given action α in our notation). Composition with a fixed equivariant morphism $[E, \Phi, u]$ between two systems (A, G, α) and (B, G, β) gives an induction map

$$\operatorname{Ind}^E : \operatorname{Rep}(B, G) \to \operatorname{Rep}(A, G); [H, (\pi, U)] \mapsto [E, \Phi, u] \circ [H, \pi, U].$$

As above, we shall write $\operatorname{Ind}^E H := E \otimes_A H$, $\operatorname{Ind}^E \pi := \Phi \otimes 1$, and $\operatorname{Ind}^E U := u \otimes U$, so that the composition $[E, \Phi, u] \circ [H, \pi, U]$ becomes the triple $[\operatorname{Ind}^E H, \operatorname{Ind}^E \pi, \operatorname{Ind}^E U]$. Taking integrated forms allows to identify $\operatorname{Rep}(A, G)$ with $\operatorname{Rep}(A \rtimes G)$ and hence we may topologize $\operatorname{Rep}(A, G)$ so that these identifications become homeomorphisms. A more or less straight-forward computation gives:

Proposition 5.13. *Assume that $[E, \Phi, u]$ is a morphism from (A, G, α) to (B, G, β) in $\mathfrak{M}(G)$ and let $[E \rtimes G, \Phi \rtimes G] \in \operatorname{Mor}(A \rtimes G, B \rtimes G)$ denote its crossed product. Then, for each $[H, (\pi, U)] \in \operatorname{Rep}(B, G)$ we have*

$$[\operatorname{Ind}^E H, \operatorname{Ind}^E \pi \times \operatorname{Ind}^E U] = [\operatorname{Ind}^{E \rtimes G} H, \operatorname{Ind}^E (\pi \times U)] \quad \text{in} \quad \operatorname{Rep}(A \rtimes G).$$

Hence induction from $\mathrm{Rep}(B,G)$ *to* $\mathrm{Rep}(A,G)$ *via* $[E,\Phi,u]$ *is equivalent to induction from* $\mathrm{Rep}(A\rtimes G)$ *to* $\mathrm{Rep}(B\rtimes G)$ *via* $[E\rtimes G,\Phi\rtimes G]$ *under the canonical identifications* $\mathrm{Rep}(A,G)\cong\mathrm{Rep}(A\rtimes G)$ *and* $\mathrm{Rep}(B,G)\cong\mathrm{Rep}(B\rtimes G)$.

Proof. Simply check that the map

$$W: C_c(G,E)\odot H\to E\otimes_A H;\quad W(\xi\otimes v)=\int_G \xi(s)\otimes U_s v\,ds$$

extends to a unitary from $(E\rtimes G)\otimes_{A\rtimes G} H$ to $E\otimes_A H$ which intertwines both representations (see [30] for mor details). $\square$

5.6. **The Fell-topologies and weak containment.** For later use and for completeness it is necessary to discuss some more topological notions on the spaces $\mathrm{Rep}(B)$ and $\mathcal{I}(B)$: For $I\in\mathcal{I}(B)$ let $U(I):=\{J\in\mathcal{I}(B): J\cap I\neq\emptyset\}$. Then $\{U(I): I\in\mathcal{I}(B)\}$ is a sub-basis for the *Fell topology* on $\mathcal{I}(B)$. The Fell topology on $\mathrm{Rep}(B)$ is then defined as the inverse image topology with respect to the map $\ker:\mathrm{Rep}(B)\to\mathcal{I}(B);\pi\mapsto\ker\pi$. [10] The Fell topologies restrict to the Jacobson topologies on $\mathrm{Prim}(B)$ and $\widehat{B}$, respectively. Convergence of nets in $\mathrm{Rep}(B)$ (and hence also in $\mathcal{I}(B)$) can conveniently be described in terms of *weak containment*: If $\pi\in\mathrm{Rep}(B)$ and R is a subset of $\mathrm{Rep}(B)$, then π is said to be *weakly contained* in R (denoted $\pi\prec R$) if $\ker\pi\supseteq\cap\{\ker\rho:\rho\in R\}$. Two subsets S,R of $\mathrm{Rep}(A)$ are said to be *weakly equivalent* ($S\sim R$) if $\sigma\prec R$ for all $\sigma\in S$ and $\rho\prec S$ for all $\rho\in R$.

Lemma 5.14 (Fell). *Let* $(\pi_j)_{j\in J}$ *be a net in* $\mathrm{Rep}(B)$ *and let* $\pi,\rho\in\mathrm{Rep}(B)$. *Then*

(i) $\pi_j\to\pi$ *if and only if* π *is weakly contained in every subnet of* $(\pi_j)_{j\in J}$.

(ii) *If* $\pi_j\to\pi$ *and if* $\rho\prec\pi$, *then* $\pi_j\to\rho$.

For the proof see [43, Propositions 1.2 and 1.3]. As a direct consequence of this and the fact that induction via bimodules preserves inclusion of ideals we get

Proposition 5.15. *Let* $[E,\Phi]\in\mathrm{Mor}(A,B)$. *Then induction via* E *preserves weak containment and the maps*

$$\mathrm{Ind}^E:\mathrm{Rep}(B)\to\mathrm{Rep}(A)\quad\textit{and}\quad\mathrm{Ind}^E:\mathcal{I}(B)\to\mathcal{I}(A)$$

are continuous with respect to the Fell topologies. Both maps are homeomorphisms if E *is an imprimitivity bimodule.*

Another important observation is the fact that tensoring representations and ideals of C^*-algebras is continuous:

Proposition 5.16. *Suppose that* A *and* B *are* C^**-algebras. For* $\pi\in\mathrm{Rep}(A)$ *and* $\rho\in\mathrm{Rep}(B)$ *let* $\pi\otimes\rho\in\mathrm{Rep}(A\otimes B)$ *denote the tensor product representation on the minimal tensor product* $A\otimes B$. *Moreover, if* $I\in\mathcal{I}(A)$ *and* $J\in\mathcal{I}(B)$, *define* $I\otimes J$ *as the closed two-sided ideal of* $A\otimes B$ *generated by* $I\otimes B+A\otimes J$. *Then the maps*

$$\mathrm{Rep}(A)\times\mathrm{Rep}(B)\to\mathrm{Rep}(A\otimes B);\ (\pi,\rho)\mapsto\pi\otimes\rho$$
$$\textit{and}\qquad \mathcal{I}(A)\times\mathcal{I}(B)\to\mathcal{I}(A\otimes B);\ (I,J)\mapsto I\otimes J$$

are continuous with respect to the Fell-topologies.

[10] Recall that $\mathrm{Rep}(B)$ is a set only if we restrict the cardinality of the Hilbert spaces.

Proof. Notice first that if $I = \ker\pi$ and $J = \ker\rho$, then $I \otimes J = \ker(\pi\otimes\rho)$. Since tensoring ideals clearly preserves inclusion of ideals, the map $(\pi,\rho) \mapsto \pi\otimes\rho$ preserves weak containment in both variables. Hence the result follows from Lemma 5.14. $\square$

It follows from deep work of Fell (e.g. see [42, 23]) that weak containment (and hence the topologies on $\mathcal{I}(B)$ and $\operatorname{Rep}(B)$) can be described completely in terms of matrix coefficients of representations. In particular, if G is a locally compact group and if we identify the collection $\operatorname{Rep}(G)$ of equivalence classes of unitary representations of G with $\operatorname{Rep}(C^*(G))$ via integration, then it is shown in [42, 23] that weak containment for representations of G can be described in terms of convergence of positive definite functions on G associated to the given representations.

6. Green's imprimitivity theorem

We are now presenting (a slight extension of) Phil Green's imprimitivity theorem as presented in [52]. For this we start with the construction of an induction functor

$$\operatorname{Ind}_H^G : \mathfrak{M}(H) \to \mathfrak{M}(G); (A, H, \alpha) \mapsto \big(\operatorname{Ind}_H^G(A,\alpha), G, \operatorname{Ind}\alpha\big),$$

if H is a closed subgroup of G and $\alpha : H \to \operatorname{Aut}(A)$ an action of H on the C^*-algebra A. The induced C^*-algebra $\operatorname{Ind}_H^G(A,\alpha)$ (or just $\operatorname{Ind} A$ if all data are understood) is defined as

$$\operatorname{Ind}_H^G(A,\alpha) := \left\{ f \in C^b(G,A) : \begin{matrix} f(sh) = \alpha_{h^{-1}}(f(s)) \ \text{ for all } s \in G, h \in H \\ \text{and } sH \mapsto \|f(s)\| \in C_0(G/H) \end{matrix} \right\},$$

equipped with the pointwise operations and the supremum norm. The induced action $\operatorname{Ind}\alpha : G \to \operatorname{Aut}(\operatorname{Ind} A)$ is given by

$$\big(\operatorname{Ind}\alpha_s(f)\big)(t) := f(s^{-1}t) \qquad \text{for all} \quad s,t \in G.$$

A similar construction works for morphisms in $\mathfrak{M}(H)$, i.e., if $[E,\Phi,u]$ is an H-equivariant morphism from A to B, then a fairly obvious extension of the above construction yields the induced morphism $[\operatorname{Ind}_H^G(E,u), \operatorname{Ind}\Phi, \operatorname{Ind} u]$ from $(\operatorname{Ind}_H^G(A,\alpha), G, \operatorname{Ind}\alpha)$ to $(\operatorname{Ind}_H^G(B,\beta), G, \operatorname{Ind}\beta)$. One then checks that induction preserves composition of morphisms, and hence gives a functor from $\mathfrak{M}(H)$ to $\mathfrak{M}(G)$ (see [32] for more details).

Remark 6.1. (1) If we start with an action $\alpha : G \to \operatorname{Aut}(A)$ and restrict this action to the closed subgroup H of G, then $\operatorname{Ind}_H^G(A,\alpha)$ is canonically G-isomorphic to $C_0(G/H, A) \cong C_0(G/H) \otimes A$ equipped with the diagonal action $l \otimes \alpha$, where l denotes the left-translation action of G on G/H. The isomorphism is given by

$$\Phi : \operatorname{Ind}_H^G(A,\alpha) \to C_0(G/H, A); \quad \Phi(f)(sH) = \alpha_s(f(s)).$$

(2) The construction of the induced algebra $\operatorname{Ind}_H^G(A,\alpha)$ is the C^*-analogue of the usual construction of the induced G-space $G \times_H Y$ of a topological H-space Y, which is defined as the quotient of $G \times Y$ by the H-action $h(g,y) = (gh^{-1}, hy)$ and which is equipped with the obvious G-action. Indeed, if Y is locally compact, then $\operatorname{Ind}_H^G C_0(Y) \cong C_0(G \times_H Y)$.

A useful characterization of induced systems is given by the following result:

Theorem 6.2 (cf [27, Theorem]). *Let (B,G,β) be a system and assume that H is a closed subgroup of G. Then (B,G,β) is isomorphic to an induced system $(\mathrm{Ind}_H^G(A,\alpha), G, \mathrm{Ind}\,\alpha)$ if and only if there exists a continuous G-equivariant map $\varphi : \mathrm{Prim}(B) \to G/H$, where G acts on $\mathrm{Prim}(B)$ via $s\cdot P := \beta_s(P)$.*

Indeed, we can always define a continuous G-map $\varphi : \mathrm{Prim}(\mathrm{Ind}\,A) \to G/H$ by sending a primitive ideal P to sH iff P contains the ideal $I_s := \{f \in \mathrm{Ind}\,A : f(s) = 0\}$. Conversely, if $\varphi : \mathrm{Prim}(B) \to G/H$ is given, define $A := B/I_e$ with $I_e := \cap\{P \in \mathrm{Prim}(B) : \phi(P) = eH\}$. Since I_e is H-invariant, the action $\beta|_H$ induces an action α of H on A and (B,G,β) is isomorphic to $(\mathrm{Ind}_H^G A, G, \mathrm{Ind}\,\alpha)$ via $b \mapsto f_b \in \mathrm{Ind}_H^G A$; $f_b(s) := \beta_{s^{-1}}(b) + I_e$. We should remark at this point that a much more general result has been shown by Le Gall in [78] in the setting of Morita equivalent groupoids. Applying Theorem 6.2 to commutative G-algebras, one gets:

Corollary 6.3. *Let X be a locally compact G-space and let H be a closed subgroup of G. Then X is G-homeomorphic to $G\times_H Y$ for some locally compact H-space Y if and only if there exists a continuous G-map $\varphi : X \to G/H$. If such a map is given, then Y can be chosen as $Y = \varphi^{-1}(\{eH\})$ and the homeomorphism $G\times_H Y \cong X$ is given by $[g,y] \mapsto gy$.*

In what follows let $B_0 = C_c(H,A)$ and $D_0 = C_c(G, \mathrm{Ind}\,A)$, viewed as dense subalgebras of the full (resp. reduced) crossed products $A \rtimes_{(r)} H$ and $\mathrm{Ind}\,A \rtimes_{(r)} G$, respectively. Let $X_0(A) = C_c(G,A)$. We define left and right module actions of D_0 and B_0 on $X_0(A)$, and D_0- and B_0-valued inner products on $X_0(A)$ by the formulas

$$\begin{aligned}
e\cdot x(s) &= \int_G e(t,s)x(t^{-1}s)\Delta_G(t)^{1/2}dt\\
x\cdot b(s) &= \int_H \alpha_h\big(x(sh)b(h^{-1})\big)\Delta_H(h)^{-1/2}dh\\
{}_{D_0}\langle x,y\rangle(s,t) &= \Delta_G(s)^{-1/2}\int_H \alpha_h\big(x(th)y(s^{-1}th)^*\big)\,dh\\
\langle x,y\rangle_{B_0}(h) &= \Delta_H(h)^{-1/2}\int_G x(t^{-1})^*\alpha_h(y(t^{-1}h))\,dt,
\end{aligned}\tag{6.1}$$

for $e \in D_0, x, y \in X_0(A)$, and $b \in B_0$. The $C_c(H,A)$-valued inner product on $X_0(A)$ provides $X_0(A)$ with two different norms: $\|\xi\|^2_{\max} := \|\langle\xi,\xi\rangle_{B_0}\|_{\max}$ and $\|\xi\|_r^2 := \|\langle\xi,\xi\rangle_{B_0}\|_r$, where $\|\cdot\|_{\max}$ and $\|\cdot\|_r$ denote the maximal and reduced normes on $C_c(G,A)$. Then Green's inprimitivity theorem reads as follows:

Theorem 6.4 (Green). *The actions and inner products on $X_0(A)$ extend to the completion $X_H^G(A)$ of $X_0(A)$ with respect to $\|\cdot\|_{\max}$ such that $X_H^G(A)$ becomes an $(\mathrm{Ind}_H^G A \rtimes G)$-$(A \rtimes H)$ imprimitivity bimodule.*

Similarly, the completion $X_H^G(A)_r$ of $X_0(A)$ with respect to $\|\cdot\|_r$ becomes an $(\mathrm{Ind}_H^G A \rtimes_r G)$-$(A \rtimes_r H)$ imprimitivity bimodule.

Remark 6.5. **(1)** Although the statement of Green's theorem looks quite straightforward, the proof requires a fair amount of work. The main problem is to show positivity of the inner products and continuiuty of the left and right actions of D_0

and B_0 on X_0 with respect to the appropriate norms. In [52] Green only considered full crossed products. The reduced versions were obtained later by Kasparov ([66]), by Quigg and Spielberg ([102]) and by Kirchberg and Wassermann ([70]).

The reduced module $X_H^G(A)_r$ can also be realized as the quotient of $X_H^G(A)$ by the submodule $Y := X_H^G(A) \cdot I$ with $I := \ker\big(A \rtimes H \to A \rtimes_r H\big)$. This follows from the fact that the ideal I corresponds to the ideal $J := \ker\big(\operatorname{Ind} A \rtimes G \to \operatorname{Ind} A \rtimes_r G\big)$ in $\operatorname{Ind} A \rtimes G$ via the Rieffel correspondence (see Proposition 5.4). We shall give an argument for this fact in Remark 9.14 below.

(2) In his original work [52], Green first considered the special case where the action of H on A restricts from an action of G on A. In this case one obtains a Morita equivalence between $A \rtimes_{(r)} H$ and $C_0(G/H, A) \rtimes_{(r)} G$ (compare with Remark 6.1 above). Green then deduced from this a more general result (see [52, Theorem 17]), which by Theorem 6.2 is equivalent to the above formulation for full crossed products.

(3) In [32] it is shown that the construction of the equivalence bimodule $X(A)$, viewed as an isomorphism in the Morita category $\mathfrak{M}$, provides a natural equivalence between the descent functor $\rtimes : \mathfrak{M}(H) \to \mathfrak{M}; (A, H, \alpha) \mapsto A \rtimes H$ with the composition $\rtimes \circ \operatorname{Ind}_H^G : \mathfrak{M}(H) \to \mathfrak{M}; (A, H, \alpha) \mapsto \operatorname{Ind} A \rtimes G$ (and similarly for the reduced descent functors $\rtimes_r$). This shows that the assignment $(A, H, \alpha) \mapsto X_H^G(A)$ is, in a very strong sense, natural in A.

Let us now present some basic examples:

Example 6.6. **(1)** Let H be a closed subgroup of G. Consider the trivial action of H on $\mathbb{C}$. Then $\operatorname{Ind}_H^G \mathbb{C} = C_0(G/H)$ and Green's theorem provides a Morita equivalence between $C^*(H)$ and $C_0(G/H) \rtimes G$, and similarly between $C_r^*(H)$ and $C_0(G/H) \rtimes_r G$. It follows then from Proposition 5.15 that induction via $X_H^G(\mathbb{C})$ identifies the representation spaces $\operatorname{Rep}(H)$ and $\operatorname{Rep}(C_0(G/H), G)$. This is a very strong version of Mackey's original imprimitivity theorem for groups (e.g., see [81, 82, 7]).

(2) If $H = \{e\}$ is the trivial subgroup of G, we obtain a Morita equivalence between A and $C_0(G, A) \rtimes G$, where G acts on itself by left translation. Indeed, in this case we obtain a unitary isomorphism between Green's bimodule $X_{\{e\}}^G(A)$ and the Hilbert A-module $L^2(G, A) \cong A \otimes L^2(G)$ via the transformation

$$U : X_{\{e\}}^G(A) \to L^2(G, A); \big(U(x)\big)(s) = \delta(s)^{-\frac{1}{2}} x(s).$$

It follows from this that $C_0(G, A) \rtimes G$ is isomorphic to $\mathcal{K}(A \otimes L^2(G)) \cong A \otimes \mathcal{K}(L^2(G))$. In particular, it follows that $C_0(G) \rtimes G$ is isomorphic to $\mathcal{K}(L^2(G))$ if G acts on itself by translation.

Since full and reduced crossed products by the trivial group coincide, it follows from part (2) of Remark 6.5 that $C_0(G, A) \rtimes_r G \cong C_0(G, A) \rtimes G$, and hence that $C_0(G, A) \rtimes_r G \cong A \otimes \mathcal{K}(L^2(G))$, too.

(3) Let H_3 denote the three-dimensional real Heisenberg group, i.e., $H_3 = \mathbb{R}^2 \rtimes \mathbb{R}$ with action of $\mathbb{R}$ on $\mathbb{R}^2$ given by $x \cdot (y, z) = (y, z + xy)$. We want to use Green's theorem to analyse the structure of $C^*(H_3) \cong C^*(\mathbb{R}^2) \rtimes \mathbb{R}$. We first identify $C^*(\mathbb{R}^2)$ with $C_0(\mathbb{R}^2)$ via Fourier transform. The transformed action of $\mathbb{R}$ on $\mathbb{R}^2$ is then given by $x \cdot (\eta, \zeta) = (\eta - x\zeta, \zeta)$. The short exact sequence

$$0 \to C_0\big(\mathbb{R} \times \mathbb{R}^*\big) \to C_0(\mathbb{R}^2) \to C_0(\mathbb{R} \times \{0\}) \to 0$$

determines a short exact sequence

$$0 \to C_0\big(\mathbb{R} \times \mathbb{R}^*\big) \rtimes \mathbb{R} \to C^*(H_3) \to C_0(\mathbb{R} \times \{0\}) \rtimes \mathbb{R} \to 0.$$

Since the action of $\mathbb{R}$ on the quotient $C_0(\mathbb{R}) \cong C_0(\mathbb{R} \times \{0\})$ is trivial, we see that $C_0(\mathbb{R} \times \{0\}) \rtimes \mathbb{R} \cong C_0(\mathbb{R}) \otimes C^*(\mathbb{R}) \cong C_0(\mathbb{R}^2)$. The homeomorphism $h : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R} \times \mathbb{R}^*; h(\eta, \zeta) = (-\frac{\eta}{\zeta}, \zeta)$ transforms the action of $\mathbb{R}$ on $C_0(\mathbb{R} \times \mathbb{R}^*) \cong C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^*)$ to the diagonal action $l \otimes \mathrm{id}$, where l denotes left translation. Thus, it follows from (2) and Lemma 4.1 that $C_0(\mathbb{R} \times \mathbb{R}^*) \rtimes \mathbb{R} \cong C_0(\mathbb{R}^*) \otimes \mathcal{K}(L^2(\mathbb{R}))$ and we obtain a short exact sequence

$$0 \to C_0(\mathbb{R}^*) \otimes \mathcal{K}(L^2(\mathbb{R})) \to C^*(H_3) \to C_0(\mathbb{R}^2) \to 0$$

for $C^*(H_3)$.

(4) Let $\mathbb{R}$ act on the two-torus $\mathbb{T}^2$ by an irrational flow, i.e. there exists an irrational number $\theta \in (0, 1)$ such that $t \cdot (z_1, z_2) = (e^{2\pi i t} z_1, e^{2\pi i \theta t} z_2)$. Then $\mathbb{T}^2$ is $\mathbb{R}$-homeomorphic to the induced space $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}$, where $\mathbb{Z}$ acts on $\mathbb{T}$ by irrational rotation given by θ (compare with Example 3.5). Hence, it follows from Green's theorem that $C(\mathbb{T}^2) \rtimes_\theta \mathbb{R}$ is Morita equivalent to the irrational rotation algebra $A_\theta = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.

7. The Takesaki-Takai duality theorem.

From the second of the above examples it is fairly easy to obtain the Takesaki-Takai duality theorem for crossed products by abelian groups. For this assume that (A, G, α) is a system with G abelian. The *dual action* $\widehat{\alpha} : \widehat{G} \to \mathrm{Aut}(A \rtimes G)$ of the dual group $\widehat{G}$ on the crossed product $A \rtimes G$ is defined by

$$\widehat{\alpha}_\chi(f)(s) := \chi(s) f(s) \qquad \text{for } \chi \in \widehat{G} \text{ and } f \in C_c(G, A) \subseteq A \rtimes G.$$

With a similar action of $\widehat{G}$ on crossed products $E \rtimes G$ for an equivariant bimodule (E, Φ, u) we obtain from this a descent functor

$$\rtimes : \mathfrak{M}(G) \to \mathfrak{M}(\widehat{G}).$$

The double dual crossed product $A \rtimes G \rtimes \widehat{G}$ is isomorphic to $C_0(G, A) \rtimes G$ with respect to the diagonal action $l \otimes \alpha$ of G on $C_0(G, A) \cong C_0(G) \otimes A$. Indeed, we have canonical (covariant) representations $(k_A, k_G, k_{\widehat{G}})$ of the triple $(A, G, \widehat{G})$ into $M\big(C_0(G, A) \rtimes G\big)$ given by the formulas

$$\begin{aligned} \big(k_A(a) \cdot f\big)(s, t) = & a\big(f(s, t)\big), \quad \big(k_G(r) \cdot f\big)(s, t) = \alpha_r\big(f(r^{-1}s, r^{-1}t)\big), \quad \text{and} \\ & \big(k_{\widehat{G}}(\chi) \cdot f\big)(s, t) = \chi(t) f(s, t), \end{aligned}$$

for f in the dense subalgebra $C_c(G, C_0(G, A))$ of $C_0(G, A) \rtimes G$. Making extensive use of the universal properties, one checks that the integrated form

$$(k_A \times k_G) \times k_{\widehat{G}} : (A \rtimes G) \rtimes \widehat{G} \to M(C_0(G, A) \rtimes G)$$

gives the desired isomorphism $A \rtimes G \rtimes \widehat{G} \cong C_0(G, A) \rtimes G$. Using the isomorphism $C_0(G, A) \rtimes G \cong A \otimes \mathcal{K}(L^2(G))$ of Example 6.6 (2) and checking what this isomorphism does on the double-dual action $\widehat{\widehat{\alpha}}$ we arrive at

Theorem 7.1 (Takesaki-Takai)**.** *Suppose that (A, G, α) is a system with G abelian. Then the double dual system $(A \rtimes G \rtimes \widehat{G}, G, \widehat{\widehat{\alpha}})$ is equivariantly isomorphic to the system $(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \operatorname{Ad} \rho)$, where $\rho : G \to U(L^2(G))$ denotes the right regular representation of G on $L^2(G)$.*

Recall that the *right regular representation* $\rho : G \to U(L^2(G))$ is defined by $(\rho_s \xi)(t) = \sqrt{\Delta(t)}\xi(st)$ for $\xi \in L^2(G)$. Note that the system $(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \operatorname{Ad} \rho)$ in the Takesaki-Takai theorem is Morita equivalent to the original system (A, G, α) via the equivariant imprimitivity bimodule $(A \otimes L^2(G), \alpha \otimes \rho)$. In fact, the assignment $(A, G, \alpha) \mapsto (A \otimes L^2(G), \alpha \otimes \rho)$ is easily seen to give a natural equivalence between the identity functor on $\mathfrak{M}(G)$ and the composition

$$\mathfrak{M}(G) \xrightarrow{\rtimes} \mathfrak{M}(\widehat{G}) \xrightarrow{\rtimes} \mathfrak{M}(G).$$

In general, if G is not abelian, one can obtain similar duality theorems by replacing the dual action of $\widehat{G}$ by a dual coaction of the group algebra $C^*(G)$ on $A \rtimes G$. A fairly complete account of that theory in the group case is given in the appendix of [33] — however a much more general duality theory for Hopf-C^*-algebras was developed by Baaj and Skandalis in [5].

8. Permanence properties of exact groups

As a further application of Green's imprimitivity theorem we now present some of Kirchberg's and Wassermann's permanence results for C^*-exact groups. Recall from Definition 4.9 that a group G is called C^*-exact (or just exact) if for every system (A, G, α) and for every G-invariant ideal $I \subseteq A$ the sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

is exact (which is equivalent to exactness of the sequence in the middle term). Recall from Proposition 4.8 that the corresponding sequence of full crossed products is always exact. Using Proposition 4.5, this implies that all amenable groups are exact.

In what follows we want to relate exactness of G with exactnesss of a closed subgroup H of G. For this we start with a system (A, H, α) and a closed H-invariant ideal I of A. Recall that Green's $\operatorname{Ind} A \rtimes_r G - A \rtimes_r H$ imprimitivity bimodule $X_H^G(A)_r$ is a completion of $C_c(G, A)$. Using the formulas for the actions and inner products as given in (6.1) one observes that $X_H^G(I)_r$ can be identified with the closure of $C_c(G, I) \subseteq C_c(G, A)$ in $X_H^G(A)_r$. It follows that the ideals $\operatorname{Ind} I \rtimes_r G$ and $I \rtimes_r H$ are linked via the Rieffel correspondence with respect to $X_H^G(A)_r$ (see Proposition 5.4). Similarly, the imprimitivity bimodule $X_H^G(A/I)_r$ is isomorphic to the quotient $X_H^G(A)_r/Y$ with $Y := X_H^G(A)_r \cdot \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big)$, which implies that the ideals

$$\ker\big(A \rtimes_r H \to A/I \rtimes_r H\big) \quad \text{and} \quad \ker\big(\operatorname{Ind} A \rtimes_r G \to \operatorname{Ind}(A/I) \rtimes_r G\big)$$

are also linked via the Rieffel correspondence. Since the Rieffel correspondence is one-to-one, we obtain

$$\begin{aligned} & I \rtimes_r H = \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big) \\ \iff & \operatorname{Ind} I \rtimes_r G = \ker\big(\operatorname{Ind} A \rtimes_r G \to \operatorname{Ind}(A/I) \rtimes_r G\big). \end{aligned} \tag{8.1}$$

Using this, we now give proofs of two of the main results of [70].

Theorem 8.1 (Kirchberg and Wassermann). *Let G be a locally compact group. Then the following are true:*

(i) *If G is exact and H is a closed subgroup of G, then H is exact.*

(ii) *Let H be a closed subgroup of G such that G/H is compact. Then H exact implies G exact.*

Proof. Suppose that I is an H-invariant ideal of the H-algebra A. If G is exact, then $\operatorname{Ind} I \rtimes_r G = \ker\big(\operatorname{Ind} A \rtimes_r G \to \operatorname{Ind}(A/I) \rtimes_r G\big)$ and hence $I \rtimes_r H = \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big)$ by (8.1). This proves (i).

To see (ii) we start with an arbitrary G-algebra A and a G-invariant ideal I of G. Since A, I, and A/I are G-algebras and G/H is compact, we have $\operatorname{Ind}_H^G A \cong C(G/H, A)$ and similar statements hold for I and A/I. Since H is exact, we see that $I \rtimes_r H = \ker\big(A \rtimes_r H \to A/I \rtimes_r H\big)$ and (8.1) implies that the lower row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \rtimes_r G & \longrightarrow & A \rtimes_r G & \longrightarrow & (A/I) \rtimes_r G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(G/H, I) \rtimes_r G & \longrightarrow & C(G/H, A) \rtimes_r G & \longrightarrow & C(G, A/I) \rtimes_r G & \longrightarrow & 0, \end{array}$$

is exact, where the vertical maps are induced by the canonical inclusions of I, A, and A/I into $C(G/H, I), C(G/H, A)$ and $C(G/H, A/I)$, respectively. Since these inclusions are injective, all vertical maps are injective, too (see the remarks preceeding Proposition 4.8). This and the exactness of the lower horizontal row imply that

$$\ker\big(A \rtimes_r G \to (A/I) \rtimes_r G\big) =: J = \big(A \rtimes_r G\big) \cap \big(C(G/H, I) \rtimes_r G\big).$$

Let $(x_i)_i$ be a bounded approximate unit of I and let $(\varphi_j)_j$ be an approximate unit of $C_c(G)$ (compare with Remark 3.7). Then $z_{i,j} := \varphi_j \otimes x_i \in C_c(G, A)$ serves as an approximate unit of $I \rtimes_r G$ and of $J := \big(A \rtimes_r G\big) \cap \big(C(G/H, I) \rtimes_r G\big)$. Thus if $y \in J$, then $z_{i,j} \cdot y \in I \rtimes_r G$ and $z_{i,j} \cdot y$ converges to y. Hence $J \subseteq I \rtimes_r G$. □

Corollary 8.2. *Every closed subgroup of an almost connected group is exact (in particular, every free group in countably many generators is exact). Also, every closed subgroup of* $\mathrm{GL}(n, \mathbb{Q}_p)$*, where $\mathbb{Q}_p$ denotes the field of p-adic rational numbers equipped with the Hausdorff topology is exact.*

Proof. Recall first that a locally compact group is called *almost connected* if the component G_0 of the identity in G is cocompact. By part (i) of Theorem 8.1 it is enough to show that every almost connected group G is exact and that $\mathrm{GL}(n, \mathbb{Q}_p)$ is exact for all $n \in \mathbb{N}$. But structure theory for those groups implies that in both cases one can find an amenable cocompact subgroup. Since amenable groups are exact (by Propositions 4.5 and 4.8), the result then follows from part (ii) of the theorem. □

Remark 8.3. We should mention that Kirchberg and Wassermann proved some further permanence results: If H is a closed subgroup of G such that G/H carries a finite invariant measure, then H exact implies G exact[11]. Another important result is the extension result: If N is a closed normal subgroup of G such that N and G/N are exact, then G is exact. The proof of this result needs the notion of twisted actions and twisted crossed products. We shall present that theory and the proof of the extension result for exact groups in §12 below. We should also mention that the proof of part (ii) of Theorem 8.1, and hence of Corollary 8.2 followed some ideas of Skandalis (see also the discussion at the end of [69]).

By work of Ozawa and others (e.g. see [94] for a general discussion), the class of discrete exact groups is known to be identical to the class of all discrete groups which can act amenably on some compact Hausdorff space X (we refer to [2] for a quite complete exposition of amenable actions). A similar result is not known, so far, for more general locally compact groups. If we apply the exactness condition of a group G to trivial actions, it follows from Remark 4.2 that $C_r^*(G)$ is an exact C^*-algebra if G is exact — the converse is known for discrete groups by [69] but is still open in the general case. As mentioned at the end of §4, it is now known that there are non-exact finitely generated discrete groups.

9. Induced representations of groups and crossed products

If (A, G, α) is a system and H is a closed subgroup of G, then Green's imprimitivity theorem provides an imprimitivity bimodule $X_H^G(A)$ between $C_0(G/H, A) \rtimes G \cong \operatorname{Ind}_H^G A \rtimes G$ and $A \rtimes H$. In particular, $C_0(G/H, A) \rtimes G$ identifies with the compact operators $\mathcal{K}(X_H^G(A))$ on $X_H^G(A)$. There is a canonical covariant homomorphism

$$(k_A, k_G) : (A, G) \to M(C_0(G, A) \rtimes G) \cong \mathcal{L}(X_H^G(A)),$$

where $k_A = i_{C_0(G,A)} \circ j_A$ denotes the composition of the inclusion $j_A : A \to M\big(C_0(G/H, A)\big)$ with the inclusion $i_{C_0(G/H,A)} : C_0(G/H, A) \to M\big(C_0(G/H, A) \rtimes G\big)$ and $k_G = i_G$ denotes the canonical inclusion of G into $M\big(C_0(G/H, A) \rtimes G\big)$. The integrated form

$$k_A \times k_G : A \rtimes G \to M\big(C_0(G/H, A) \rtimes G\big) \cong \mathcal{L}(X_H^G(A))$$

determines a left action of $A \rtimes G$ on $X_H^G(A)$ and we obtain a canonical element $[X_H^G(A), k_A \times k_G] \in \operatorname{Mor}(A \rtimes G, A \rtimes H)$ — a morphism from $A \rtimes G$ to $A \rtimes H$ in the Morita category. Using the techniques of §5.5, we can define induced representations of $A \rtimes G$ as follows:

Definition 9.1. For $\rho \times V \in \operatorname{Rep}(A \rtimes H)$ we define the induced representation $\operatorname{ind}_H^G(\rho \times V) \in \operatorname{Rep}(A \rtimes G)$ as the representation induced from $\rho \times V$ via $[X_H^G(A), k_A \times k_G] \in \operatorname{Mor}(A \rtimes G, A \rtimes H)$.

[11] Nachschauen!!!

Similarly, for $J \in \mathcal{I}(A \rtimes H)$, we define the induced ideal $\operatorname{ind}_H^G J \in \mathcal{I}(A \rtimes G)$ as the ideal induced from J via $[X_H^G(A), k_A \times k_G]$.

On the other hand, if we restrict the canonical embedding $i_G : G \to M(A \rtimes G)$ to H, we obtain a non-degenerate homomorphism $i_A \times i_G|_H : A \rtimes H \to M(A \rtimes G)$ which induces a morphism $[A \rtimes G, i_A \times i_G|_H] \in \operatorname{Mor}(A \rtimes H, A \rtimes G)$. This leads to

Definition 9.2. For $\pi \times U \in \operatorname{Rep}(A \rtimes G)$ we define the *restriction* $\operatorname{res}_H^G(\pi \times U) \in \operatorname{Rep}(A \rtimes H)$ as the representation induced from $\pi \times U$ via $[A \rtimes G, i_A \times i_G|_H] \in \operatorname{Mor}(A \rtimes H, A \rtimes G)$.

Similarly, for $I \in \mathcal{I}(A \rtimes G)$, we define the restricted ideal $\operatorname{res}_H^G I \in \mathcal{I}(A \rtimes H)$ as the ideal induced from I via $[A \rtimes G, i_A \times i_G|_H]$.

Remark 9.3. It is a good exercise to show that for any $\pi \times U \in \operatorname{Rep}(A \rtimes G)$ we have $\operatorname{res}_H^G(\pi \times U) = \pi \times U|_H$ — the integrated form of the restriction of (π, U) to (A, H, α).

As a consequence of the Definitions 9.1 and 9.2 and Proposition 5.15 we get

Proposition 9.4. *The maps* $\operatorname{ind}_H^G : \operatorname{Rep}(A \rtimes H) \to \operatorname{Rep}(A \rtimes G)$ *and* $\operatorname{ind}_H^G : \mathcal{I}(A \rtimes H) \to \mathcal{I}(A \rtimes G)$ *as well as the maps* $\operatorname{res}_H^G : \operatorname{Rep}(A \rtimes G) \to \operatorname{Rep}(A \rtimes H)$ *and* $\operatorname{res}_H^G : \mathcal{I}(A \rtimes G) \to \mathcal{I}(A \rtimes H)$ *are continuous with respect to the Fell topologies.*

Remark 9.5. (1) Note that the left action of $A \rtimes G$ on $X_H^G(A)$ can be described conveniently on the level of $C_c(G, A)$ via convolution: If $f \in C_c(G, A) \subseteq A \rtimes G$ and $\xi \in C_c(G, A) \subseteq X_H^G(A)$, then $k_A \times k_G(f)\xi = f * \xi$.

(2) For $A = \mathbb{C}$ we obtain, after identifying unitary representations of G (resp. H) with $*$-representations of $C^*(G)$ (resp. $C^*(H)$) an induction map $\operatorname{ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G)$. With a bit of work one can check that $\operatorname{ind}_H^G U$ for $U \in \operatorname{Rep}(H)$ coincides (up to equivalence) with the induced representations defined by Mackey in [81] or Blattner in [7]. Similarly, the induced representations for C^*-dynamical systems as defined above coincide up to equivalence with the induced representations as constructed by Takesaki in [116]. We will present some more details on these facts in Proposition 9.7 and Corollary 9.8 below.

(3) If ρ is a non-degenerate representation of A on a Hilbert space H_ρ, then $\operatorname{ind}_{\{e\}}^G \rho$ is equivalent to the regular representation $\operatorname{Ind}\rho$ of $A \rtimes G$ on $L^2(G, H_\rho)$ (see Remark 3.2). The intertwining unitary $V : X_H^G(A) \otimes_A H_\rho \to L^2(G, H_\rho)$ is given by $\big(V(\xi \otimes v)\big)(s) = \pi\big(\alpha_{s^{-1}}(\xi(s))\big)v$ for $\xi \in C_c(G, A) \subseteq X_H^G(A)$ and $v \in H_\rho$.

The construction of $[X_H^G(A), k_A \times k_G]$ shows that we have a decomposition

$$[X_H^G(A), k_A \times k_G] = [C_0(G/H, A) \rtimes G, k_A \times k_G] \circ [X_H^G(A)]$$

as morphisms in the Morita category. Hence the induction map $\operatorname{ind}_H^G : \operatorname{Rep}(A \rtimes H) \to \operatorname{Rep}(A \rtimes G)$ factors as the composition

$$\operatorname{Rep}(A \rtimes H) \xrightarrow[\cong]{\operatorname{Ind}^{X_H^G(A)}} \operatorname{Rep}\big(C_0(G/H, A) \rtimes G\big) \xrightarrow{(k_A \times k_G)^*} \operatorname{Rep}(A \rtimes G)$$

(see Remark 5.10 for the meaning of $(k_A \times k_G)^*$). The representations of $C_0(G/H, A) \rtimes G$ are of the form $(P \otimes \pi) \times U$, where P and π are commuting representations of $C_0(G/H)$ and A, respectively (we use the identification $C_0(G/H, A) \cong C_0(G) \otimes A$). The covariance condition for $(P \otimes \pi, U)$ is equivalent to (π, U) and (P, U) being covariant representations of (A, G, α) and $(C_0(G/H), G, l)$, respectively (where $l : G \to \mathrm{Aut}(C_0(G/H))$ is the left translation action). One then checks that

$$\big((P \otimes \pi) \times U\big) \circ (k_A \times k_G) = \pi \times U.$$

Since induction from $\mathrm{Rep}(A \rtimes H)$ to $\mathrm{Rep}\,\big(C_0(G/H, A) \rtimes G\big)$ via $X_H^G(A)$ is a bijection, we obtain the following general version of Mackey's classical imprimitivity theorem for group representation (see [81] and [116]):

Theorem 9.6 (Mackey-Takesaki-Rieffel-Green). *Suppose that (A, G, α) is a system and let H be a closed subgroup of G. Then:*

(i) *A representation $\pi \times U \in \mathrm{Rep}(A \rtimes G)$ on a Hilbert space H_π is induced from a representation $\sigma \times V \in \mathrm{Rep}(A \rtimes H)$ if and only if there exists a non-degenerate representation $P : C_0(G/H) \to \mathcal{B}(H_\pi)$ which commutes with π and such that (P, U) is a covariant representation of $(C_0(G/H), G, l)$.*

(ii) *If $\pi \times U \in \mathrm{Rep}(A \rtimes G)$ is induced from the irreducible representation $\sigma \times V \in \mathrm{Rep}(A \rtimes H)$, and if $P : C_0(G/H) \to \mathcal{B}(H_\pi)$ is the corresponding representation such that $(P \otimes \pi) \times U \cong \mathrm{Ind}^{X_H^G(A)}(\rho \times V)$, then $\pi \times U$ is irreducible if and only if every $W \in \mathcal{B}(H_\pi)$ which intertwines π and U (and hence $\pi \times U$) also intertwines P.*

Proof. The first assertion follows directly from the above discussions. The second statement follows from Schur's irreducibilty criterion (a representation is irreducible iff every intertwiner is a multiple of the identity) together with the fact that induction via imprimitivity bimodules preserves irreducibility of representations in both directions (see Proposition 5.11). □

In many situations it is convenient to have a more concrete realization of the induced representations. The following construction follows Blattner's construction of induced group representations (see [7, 48]). It is actually convenient to start with the more general situation of an induced system. So assume that H is a closed subgroup of G and that $\alpha : H \to \mathrm{Aut}(A)$ is an H-action. If $\rho \times V \in \mathrm{Rep}(A \rtimes H)$ is a representation on the Hilbert space H_ρ we put

$$\mathcal{F}_{\rho\times V} := \left\{ \xi : G \to H_\rho : \begin{array}{c} \xi(sh) = \sqrt{\Delta_H(h)/\Delta_G(h)}V_{h^{-1}}\xi(s) \text{ for all } s \in G, h \in H \\ \text{and } \xi \text{ is continuous with compact support modulo } H \end{array} \right\}.$$

Let $c : G \to [0, \infty)$ be a Bruhat section for H, i.e., c is continuous with $\mathrm{supp}\, c \cap C \cdot H$ compact for all compact $C \subseteq G$ and such that $\int_H c(sh)\, dh = 1$ for all $s \in G$ (for the existence of such c see [8]). Then

$$\langle \xi, \eta \rangle := \int_G c(s) \langle \xi(s), \eta(s) \rangle\, ds$$

determines a well defined inner product on $\mathcal{F}_{\rho\times V}$ and we let $H_{\mathrm{ind}(\rho\times V)}$ denote its Hilbert space completion. We can now define representations σ and U of $\mathrm{Ind}_H^G A$ and G on $H_{\mathrm{ind}(\rho\times V)}$, respectively, by

$$(\sigma(f)\xi)(s) := \rho(f(s))\xi(s) \quad \text{and} \quad (U_t\xi)(s) := \xi(t^{-1}s). \tag{9.1}$$

Then $\sigma \times U$ is a representation of $\mathrm{Ind}_H^G A \rtimes G$ on $H_{\mathrm{ind}(\rho\times V)}$ and a straightforward but lengthy computation gives:

Proposition 9.7. *Let $X := X_H^G(A)$ denote Green's $\mathrm{Ind}_H^G A\rtimes G - A\rtimes H$ imprimitivity bimodule and let $\rho\times V$ be a representation of $A\rtimes H$ on H_ρ. Then there is a unitary $W : X\otimes_{A\rtimes H} H_\rho \to H_{\mathrm{ind}(\rho\times V)}$, given on elementary tensors $x\otimes v\in X\odot H_\rho$ by*

$$W(x\otimes v)(s) = \Delta_G(s)^{-\frac{1}{2}}\int_H \Delta_H(h)^{-\frac{1}{2}} V_h\rho(x(sh))v\,dh,$$

which implements a unitary equivalence between $\mathrm{Ind}^X(\rho\times V)$ and the representation $\sigma\times U$ defined above.

In the special case where A is a G-algebra we identify $\mathrm{Ind}_H^G A$ with $C_0(G/H, A)$ via the isomorphism Φ of Remark 6.1 (1). It is then easy to check that the representation σ defined above corresponds to the representation $P\otimes\pi$ of $C_0(G/H,A)\cong C_0(G/H)\otimes A$ on $H_{\mathrm{ind}(\rho\times V)}$ given by the formula

$$(P(\varphi)\xi)(s) := \varphi(sH)\xi(s) \quad \text{and} \quad (\pi(a)\xi)(s) = \rho(\alpha_{s^{-1}}(a))\xi(s). \tag{9.2}$$

Hence, as a direct corollary of the above proposition we get:

Corollary 9.8. *Let (A, G, α) be a system and let $\rho\times V\in \mathrm{Rep}(A\rtimes H)$ for some closed subgroup H of G. Then $\mathrm{ind}_H^G(\rho\times V)$ is unitarily equivalent to the representation $\pi\times U$ of $A\rtimes G$ on $H_{\mathrm{ind}(\rho\times V)}$ with π and U as in Equations (9.2) and (9.1), respectively.*

Another corollary which we can easily obtain from Blattner's realization is the following useful observation: Assume that H is a closed subgroup of G and that A is an H-algebra. Let $\varepsilon_e : \mathrm{Ind}_H^G A\rtimes H \to A\rtimes H$ be the H-equivariant surjection defined by evaluation of functions $f\in \mathrm{Ind}_H^G A$ at the unit $e\in G$. If $\rho\times V\in\mathrm{Rep}(A\rtimes H)$ then $(\rho\circ\varepsilon_e)\times V$ is a representation of $\mathrm{Ind}_H^G A\rtimes H$. We then get

Corollary 9.9. *The induced representation $\mathrm{ind}_H^G\big((\rho\circ\varepsilon_e)\times V\big)$ (induction from H to G for the system $(\mathrm{Ind}_H^G A, G, \mathrm{Ind}\,\alpha)$) is unitarily equivalent to $\mathrm{ind}^{X_H^G(A)}(\rho\times V)$ (induction via Green's $\mathrm{Ind}_H^G A\rtimes G - A\rtimes H$ imprimitivity bimodule $X_H^G(A)$).*

Proof. By Proposition 9.7 and Corollary 9.8, both representations can be realized on the Hilbert space $H_{\mathrm{ind}(\rho\times V)}$ whose construction only depends on G and the unitary representation V of H. Applying the formula for π in (9.2) to the present situation, we see that the $\mathrm{Ind}_H^G A$-part of $\mathrm{ind}_H^G\big((\rho\circ\varepsilon_e)\times V\big)$ is given by the formula

$$(\pi(f)\xi)(s) = \rho\big(\mathrm{ind}\,\alpha_{s^{-1}}(f)(e)\big)\xi(s) = \rho(f(s))\xi(s) = (\sigma(f)\xi)(s)$$

with σ as in (9.1). $\square$

We now turn to some further properties of induced representations. To obtain those properties we shall pass from Green's to Blattner's realizations of the induced representations and back whenever it seems convenient. We start the discussion with the theorem of induction in steps. For this suppose that $L \subseteq H$ are closed subgroups of G. To avoid confusion, we write Φ_H^G for the left action of $A \rtimes G$ on $X_H^G(A)$ (i.e., $\Phi_H^G = k_A \times k_G$ in the notation used above) and we write Φ_L^G and Φ_L^H for the left actions of $A \rtimes G$ and $A \rtimes L$ on $X_L^G(A)$ and $X_L^H(A)$, respectively. Then the theorem of induction in steps reads as

Theorem 9.10 (Green)**.** *Let (A, G, α) and $L \subseteq H$ be as above. Then*

$$[X_H^G(A), \Phi_H^G] \circ [X_L^H(A), \Phi_L^H] = [X_L^G(A), \Phi_L^G]$$

as morphisms from $A \rtimes L$ to $A \rtimes G$ in the Morita category $\mathfrak{M}$. As a consequence, we have

$$\operatorname{ind}_H^G \big(\operatorname{ind}_L^H(\rho \times V)\big) = \operatorname{ind}_L^G(\rho \times V)$$

for all $\rho \times V \in \operatorname{Rep}(A \rtimes L)$.

Proof. For the proof one has to check that $X_H^G \otimes_{A \rtimes H} X_L^H \cong X_L^G(A)$ as Hilbert $A \rtimes G - A \rtimes L$ bimodule. Indeed, one can check that such isomorphism is given on the level functions by the pairing $C_c(G, A) \otimes C_c(H, A) \to C_c(G, A)$ as given by the second formula in (6.1). We refer to [52] and [123, Theorem 5.9] for more details. □

By an *automorphism* γ of a system (A, G, α) we understand a pair $\gamma = (\gamma_A, \gamma_G)$, where γ_A is a $*$-automorphism of A and $\gamma_G : G \to G$ is an automorphism of G such that $\alpha_{\gamma_G(t)} = \gamma_A \circ \alpha_t \circ \gamma_A^{-1}$ for all $t \in G$. An *inner automorphism* of (A, G, α) is an automorphism of the form (α_s, C_s), $s \in G$, with $C_s(t) = sts^{-1}$. If $\gamma = (\gamma_A, \gamma_G)$ is an automorphism of (A, G, α) and if H is a closed subgroup of G, then γ induces an isomorphism $\gamma_{A \rtimes H} : A \rtimes H \to A \rtimes H_\gamma$ with $H_\gamma := \gamma_G(H)$ via

$$\gamma_{A \rtimes H}(f)(h) := \gamma_A\big(f(\gamma_G^{-1}(h))\big) \quad \text{for } h \in H_\gamma \text{ and } f \in C_c(H, A),$$

where we adjust Haar measures on H and H_γ such that $\int_H f(\gamma_G(h))\, dh = \int_{H_\gamma} f(h')\, dh'$ for $f \in C_c(H_\gamma)$. Note that if $(\rho, V) \in \operatorname{Rep}(A, H_\gamma)$ then $(\rho \circ \gamma_A, V \circ \gamma_G) \in \operatorname{Rep}(A, H)$ and we have

$$(\rho \times V) \circ \gamma_{A \rtimes H} \cong (\rho \circ \gamma_A) \times (V \circ \gamma_G)$$

for their integrated forms.

Remark 9.11. If $H = N$ is normal in G and if $\gamma_s = (\alpha_s, C_s)$ is an inner automorphism of (A, G, α), then we will write α_s^N for the corresponding automorphism of $A \rtimes N$. Then $s \mapsto \alpha_s^N$ is an action of G on $A \rtimes N$. This action will serve as a starting point for the study of twisted actions in §12 below.

Proposition 9.12. *Suppose that $\gamma = (\gamma_A, \gamma_G)$ is an automorphism of (A, G, α) and let $H \subseteq L$ be two closed subgroups of G. Then*

$$\operatorname{ind}_H^L \big((\rho \times V) \circ \gamma_{A \rtimes H}\big) \cong \big(\operatorname{ind}_{H_\gamma}^{L_\gamma}(\rho \times V)\big) \circ \gamma_{A \rtimes L}$$

for all $\rho \times V \in \operatorname{Rep}(A \rtimes H_\gamma)$, *where "$\cong$" denotes unitary equivalence. In particular, if* $\rho \times V \in \operatorname{Rep}(A, H)$ *and* (α_s, C_s) *is an inner automorphism of* (A, G, α) *then*

$$\operatorname{ind}_H^G(\rho \times V) \cong \operatorname{ind}_{sHs^{-1}}^G \big(s \cdot (\rho \times V)\big),$$

where we put $s \cdot (\rho \times V) := (\rho \circ \alpha_{s^{-1}}) \times (V \circ C_{s^{-1}}) \in \operatorname{Rep}(A, sHs^{-1})$.

Proof. Simply check that the map $\gamma_L : C_c(L, A) \to C_c(L_\gamma, A)$ as defined above also extends to a bijection $\Phi_L : X_H^L(A) \to X_{H_\gamma}^{L_\gamma}(A)$ which is compatible with the isomorphisms $\gamma_L : A \rtimes L \to A \rtimes L_\gamma$ and $\gamma_H : A \rtimes H \to A \rtimes H_\gamma$ on the left and right. This implies that $\gamma_L^* \circ [X_{H_\gamma}^{L_\gamma}(A), k_A \times k_{L_\gamma}] = [X_H^G(A), k_A \times k_L] \circ \gamma_H^*$ in $\mathfrak{M}$ and the first statement follows. The second statement follows from the first applied to $L = G$ and $\gamma = (\alpha_s, C_s)$ together with the fact that for any $\pi \times U \in \operatorname{Rep}(A, G)$ the unitary $U_s \in \mathcal{U}(H_\pi)$ implements a unitary equivalence between $s \cdot (\pi \times U) = (\pi \circ \alpha_{s^{-1}}) \times (U \circ C_{s^{-1}})$ and $\pi \times U$. $\square$

As a direct consequence we get:

Corollary 9.13. *Let* (A, G, α) *be a system. For* $J \in \mathcal{I}(A)$ *let*

$$J^G := \cap\{\alpha_s(J) : s \in G\}.$$

Then $\operatorname{ind}_{\{e\}}^G J^G = \operatorname{ind}_{\{e\}}^G J$ *in* $A \rtimes G$*. As a consequence, if* $\rho \in \operatorname{Rep}(A)$ *such that* $\cap\{\ker(\rho \circ \alpha_s) : s \in G\} = \{0\}$, *then* $\operatorname{ind}_{\{e\}}^G \rho$ *factors through a faithful representation of the reduced crossed product* $A \rtimes_r G$.

Proof. Let $J = \ker \rho$ for some $\rho \in \operatorname{Rep}(A)$ and let $\rho^G := \bigoplus_{s \in G} \rho \circ \alpha_s$. Then $J^G = \ker \rho^G$. It follows from Proposition 9.12 that $\operatorname{ind}_{\{e\}}^G \rho \circ \alpha_s \cong \operatorname{ind}_{\{e\}}^G \rho$ for all $s \in G$. Since induction preserves direct sums, it follows that

$$\operatorname{ind}_{\{e\}}^G J = \ker(\operatorname{ind}_{\{e\}}^G \rho) = \ker(\operatorname{ind}_{\{e\}}^G \rho^G) = \operatorname{ind}_{\{e\}}^G J^G.$$

If $\cap\{\ker(\rho \circ \alpha_s) : s \in G\} = \{0\}$, then ρ^G is faithful and it follows from Remark 3.4 (2) and Remark 9.5 (3) that $\ker \Lambda_A^G = \ker(\operatorname{ind}_{\{e\}}^G \rho^G) = \ker(\operatorname{ind}_{\{e\}}^G \rho)$. $\square$

Remark 9.14. From the previous results it is now possible to obtain a fairly easy proof of the fact that Green's $\operatorname{Ind}_H^G A \rtimes G - A \rtimes H$ imprimitivity bimodule $X_H^G(A)$ factors to give a $\operatorname{Ind}_H^G A \rtimes_r G - A \rtimes_r H$ imprimitivity bimodule for the reduced crossed products (compare with Remark 6.5). Indeed, if ρ is any faithful representation of A, and if $\varepsilon_e : \operatorname{Ind} A \to A$ denotes evaluation at the unit e, it follows from Corollary 9.13, that $\ker(\Lambda_{\operatorname{Ind} A}^G) = \ker\big(\operatorname{ind}_{\{e\}}^G(\rho \circ \varepsilon_e)\big)$. The latter coincides with $\ker(\operatorname{ind}_H^G(\operatorname{ind}_{\{e\}}^H(\rho \circ \varepsilon_e)))$ by Theorem 9.10. If $\sigma \times V$ denotes the representation $\operatorname{ind}_{\{e\}}^H \rho \in \operatorname{Rep}(A \rtimes H)$, then $\ker(\sigma \times V) = \ker \Lambda_A^H$ since ρ is faithful on A and a short computation shows that $\operatorname{ind}_{\{e\}}^H(\rho \circ \varepsilon_e) = (\sigma \circ \varepsilon_e) \times V$, where on the left-hand side we use induction in the system $(\operatorname{Ind} A, H, \operatorname{Ind} \alpha)$. Putting all this together we get

$$\begin{aligned}
\ker(\Lambda_{\operatorname{Ind} A}^G) &= \ker\big(\operatorname{ind}_{\{e\}}^G(\rho \circ \varepsilon_e)\big) = \ker\big(\operatorname{ind}_H^G(\operatorname{ind}_{\{e\}}^H(\rho \circ \varepsilon_e))\big) \\
&= \ker\big(\operatorname{ind}_H^G\big((\sigma \circ \varepsilon_e) \times V\big)\big) \overset{*}{=} \ker\big(\operatorname{ind}^{X_H^G(A)}(\sigma \times V)\big) \\
&= \ker\big(\operatorname{ind}^{X_H^G(A)} \Lambda_A^H\big) \overset{**}{=} \operatorname{ind}^{X_H^G(A)}\big(\ker \Lambda_A^H\big),
\end{aligned}$$

where * follows from Corollary 9.9 and ** follows from Equation (5.4). The desired result then follows from the Rieffel correspondence (Proposition 5.4).

We now come to some important results concerning the relation between induction and restriction of representations and ideals (see Definition 9.2 for the definition of the restriction maps). We start with

Proposition 9.15. *Suppose that (A, G, α) is a system and let $N \subseteq H$ be closed subgroups of G such that N is normal in G. Let $\mathcal{F}_{\rho\times V}$ be the dense subspace of Blattner's induced Hilbert space $H_{\mathrm{ind}(\rho\times V)}$ as constructed above. Then*

$$\big(\mathrm{res}_N^G(\mathrm{ind}_H^G(\rho\times V))(f)\xi\big)(s) = \mathrm{res}_N^H(\rho\times V)(\alpha^N_{s^{-1}}(f))\xi(s) \tag{9.3}$$

for all $f \in A \rtimes N$, $\xi \in \mathcal{F}_{\rho\times V}$ and $s \in G$, where $\alpha^N : G \to \mathrm{Aut}(A \rtimes N)$ is the canonical action of G on $A \rtimes N$ (see §12). As a consequence, if $J \in \mathcal{I}(A \times H)$, we get

$$\mathrm{res}_N^G\left(\mathrm{ind}_H^G J\right) = \cap\{\alpha_s^N(\mathrm{res}_N^H(J)) : s \in G\}. \tag{9.4}$$

Proof. Define $\sigma : A \rtimes N \to \mathcal{B}(H_{\mathrm{ind}(\rho\times V)})$ by $(\sigma(f)\xi)(s) = \rho\times V|_N(\alpha^N_{s^{-1}}(f))\xi(s)$ for $f \in A \rtimes N$ and $\xi \in \mathcal{F}_{\rho\times V}$. Then σ is a non-degenerate $*$-representation and hence it suffices to check that the left-hand side of (9.3) coincides with $(\sigma(f)\xi)(s)$ for $f \in C_c(N, A)$. But using (9.2) together with the transformation $n \mapsto sns^{-1}$ and the equation $\xi(sn^{-1}) = V_n\xi(s)$ for $s \in G, n \in N$, the left-hand side becomes

$$\begin{aligned}\big(\pi\times U|_N(f)\xi\big)(s) &= \int_N \rho(\alpha_{s^{-1}}(f(n))\xi(n^{-1}s)\,dn = \delta(s^{-1})\int_N \rho(\alpha_{s^{-1}}(f(sns^{-1}))\xi(sn^{-1})\,dn\\ &= \int_N \rho(\alpha^N_{s^{-1}}(f)(n))V_n\xi(s)\,dn = (\sigma(f)\xi)(s).\end{aligned}$$

□

Remark 9.16. Suppose that (A, G, α) is a system, H is a closed subgroup of G, and $J \subseteq A$ is a G-invariant ideal of A. If $\rho \times V$ is a representation of $A \rtimes H$ and if we put $\pi \times U := \mathrm{ind}_H^G(\rho \times V)$, then it follows from the above proposition that

$$J \subseteq \ker\rho \quad\Longleftrightarrow\quad J \subseteq \ker\pi.$$

Hence, we see that the induction map for the system (A, G, α) determines a map from $\mathrm{Rep}(A/J \rtimes H)$ to $\mathrm{Rep}(A/J \rtimes G)$ if we identify representations of $A/J \rtimes H$ with the representations $\rho\times V$ of $A \rtimes H$ wich satisfy $J \subseteq \ker\rho$ (and similarly for G). It is easy to check (e.g., by using Blattner's construction of the induced representations) that this map coincides with the induction map for the system $(A/J, G, \alpha)$.

Also, if $\rho\times V$ is a representation of $A \rtimes H$ such that ρ restricts to a non-degenerate representation of J, then one can check that the restriction of $\mathrm{ind}_H^G(\rho\times V)$ to $J \rtimes G$ conicides with the induced representation $\mathrm{Ind}_H^G(\rho|_J \times V)$ where the latter representation is induced from $J \rtimes H$ to $J \rtimes G$ via $X_H^G(J)$.[12] We shall use these facts quite frequently below.

[12] Of course, these results are also consequences of the naturality of the assignment $A \mapsto X_H^G(A)$ as stated in Remark 6.5 (3).

We close this section with some useful results on tensor products of representations. If (π, U) is a covariant representation of the system (A, G, α) on H_π and if V is a unitary representation of G on H_V, then $(\pi \otimes 1_{H_V}, U \otimes V)$ is a covariant representation of (A, G, α) on $H_\pi \otimes H_V$ and we obtain a pairing

$$\begin{aligned} \otimes : \operatorname{Rep}(A \rtimes G) \times \operatorname{Rep}(G) &\to \operatorname{Rep}(A \rtimes G); \\ \big((\pi \times U), V\big) &\mapsto (\pi \times U) \otimes V := (\pi \otimes 1_{H_V}) \times (U \otimes V). \end{aligned}$$

Identifying $\operatorname{Rep}(G) \cong \operatorname{Rep}(C^*(G))$, this map can also be obtained via the composition

$$\operatorname{Rep}(A \rtimes G) \times \operatorname{Rep}(C^*(G)) \to \operatorname{Rep}\big((A \rtimes G) \otimes C^*(G)\big) \xrightarrow{D^*} \operatorname{Rep}(A \rtimes G),$$

where $D : A \rtimes G \to M\big((A \rtimes G) \otimes C^*(G)\big)$ denotes the integrated form of the tensor product $(i_A \otimes 1_{C^*(G)}, i_G \otimes i_G)$ of the canonical inclusions $(i_A, i_G) : (A, G) \to M(A \rtimes G)$ with the inclusion $i_G : G \to M(C^*(G))$. Thus, from Propositions 5.15 and 5.16 we get

Proposition 9.17. *The map* $\otimes : \operatorname{Rep}(A \rtimes G) \times \operatorname{Rep}(G) \to \operatorname{Rep}(A \rtimes G)$ *preserves weak containment in both variables and is jointly continuous with respect to the Fell topologies.*

Proposition 9.18. *Let* (A, G, α) *be a system and let* H *be a closed subgroup of* G*. Then*

(i) $\operatorname{ind}_H^G\big((\rho \times V) \otimes U|_H)\big) \cong \big(\operatorname{ind}_H^G(\rho \times V)\big) \otimes U$ *for all* $\rho \times V \in \operatorname{Rep}(A \rtimes H)$ *and* $U \in \operatorname{Rep}(G)$*;*
(ii) $\operatorname{ind}_H^G\big((\pi \times U|_H) \otimes V\big) \cong (\pi \times U) \otimes \operatorname{ind}_H^G V$ *for all* $V \in \operatorname{Rep}(H)$ *and* $\pi \times U \in \operatorname{Rep}(A \rtimes G)$.

In particular, if $\pi \times U \in \operatorname{Rep}(A \rtimes G)$ *and* N *is a normal subgroup of* G*, then*

$$\operatorname{ind}_N^G(\pi \times U|_N) \cong (\pi \times U) \otimes \lambda_{G/N},$$

where $\lambda_{G/N}$ *denotes the regular representation of* G/N*, viewed as a representation of* G*.*

Proof. This result can be most easily shown using Blattner's realization of the induced representations: In the first case define

$$W : \mathcal{F}_{\rho \times V} \otimes H_U \to \mathcal{F}_{(\rho \times V) \otimes U|_H};\ W(\xi \otimes v)(s) = \xi(s) \otimes U_{s^{-1}} v.$$

Then a short computation shows that W is a unitary intertwiner of $\big(\operatorname{ind}_H^G(\rho \times V)\big) \otimes U$ and $\operatorname{ind}_H^G\big((\rho \times V) \otimes U|_H)\big)$. A similar map works for the second equivalence. Since $\lambda_{G/N} = \operatorname{ind}_N^G 1_N$, the last assertion follows from (ii) in case $V = 1_N$. $\square$

Corollary 9.19. *Suppose that* (A, G, α) *is a system and that* N *is a normal subgroup of* G *such that* G/N *is amenable. Then* $\pi \times U$ *is weakly contained in* $\operatorname{ind}_N^G(\operatorname{res}_N^G(\pi \times U))$ *for all* $\pi \times U \in \operatorname{Rep}(A \rtimes G)$*. As a consequence,* $\operatorname{ind}_N^G(\operatorname{res}_N^G I) \subseteq I$ *for all* $I \in \mathcal{I}(A \rtimes G)$*.*

Proof. Since G/N is amenable if and only if $1_{G/N} \prec \lambda_{G/N}$ we obtain from Proposition 9.18

$$\pi \times U = (\pi \times U) \otimes 1_{G/N} \prec (\pi \times U) \otimes \lambda_{G/N} = \mathrm{ind}_N^G(\pi \times U|_N),$$

which proves the first statement. The second statement follows from the first by choosing $\pi \times U \in \mathrm{Rep}(A \rtimes G)$ such that $I = \ker(\pi \times U)$. □

10. The Mackey-Rieffel-Green machine for the ideal structure of crossed products

In this section we come to the main results on the Mackey-Rieffel-Green machine, namely the description of the spectrum $(A \rtimes G)\widehat{\ }$ and the primitive ideal space $\mathrm{Prim}(A \rtimes G)$ in terms of induced representations (resp. ideals) under some favorable circumstances. We start with some topological notations:

Definition 10.1. Let Y be a topological space.

(i) We say that Y is *almost Hausdorff* if every closed subset F of Y contains a nonempty relatively open Hausdorff subset U (which can then be chosen to be dense in F).

(ii) A subset $C \subseteq Y$ is called *locally closed* if C is relatively open in its closure $\overline{C}$.

It is important to note that if A is a type I algebra, then the spectrum $\widehat{A}$ (and then also $\mathrm{Prim}(A) \cong \widehat{A}$) is almost Hausdorff with respect to the Jacobson topology. This follows from the fact that every quotient of a type I algebra is type I and that every nonzero type I algebra contains a nonzero continuous-trace ideal, and hence its spectrum contains a nonempty Hausdorff subset U (see [23, Chapter 4] and §2.4). Notice also that if Y is almost Hausdoff, then the one-point sets $\{y\}$ are locally closed for all $y \in Y$.

If A is a C^*-algebra and if $J \subseteq I$ are two closed two-sided ideals of A, then we may view $\widehat{I/J}$ (resp. $\mathrm{Prim}(I/J)$) as a locally closed subset of $\widehat{A}$ (resp. $\mathrm{Prim}(A)$). Indeed, we first identify $\widehat{A/J}$ with the closed subset $\{\pi \in \widehat{A} : J \subseteq \ker \pi\}$ of $\widehat{A}$ and then we identify $\widehat{I/J}$ with the open subset $\{\pi \in \widehat{A/J} : \pi(I) \neq \{0\}\}$ (and similarly for $\mathrm{Prim}(I/J)$ — compare with §2.4)).

Conversely, if C is a locally closed subset of $\widehat{A}$, then C is canonical homeomorphic $\widehat{I_C/J_C}$ if we take $J_C := \ker(C)$ and $I_C := \ker\left(\overline{C} \smallsetminus C\right)$ (we write $\ker(E) := \cap\{\ker \pi : \pi \in E\}$ if $E \subseteq \widehat{A}$ and similarly $\ker(D) := \cap\{P : P \in D\}$ for $D \subseteq \mathrm{Prim}(A)$). If we apply this observation to commutative C^*-algebras, we recover the well known fact that the locally closed subsets of a locally compact Hausdorff space Y are precisely those subsets of Y which are locally compact in the relative topology.

Definition 10.2. Let A be a C^*-algebra and let C be a locally closed subset of $\widehat{A}$ (resp. $\mathrm{Prim}(A)$). Then $A_C := I_C/J_C$ with I_C, J_C as above is called the *restriction of A to C*. In the same way, we define the restriction A_D of A to D for a locally closed subset D of $\mathrm{Prim}(A)$.

In what follows, we shall use the following notations:

Notations 10.3. If (A, G, α) is a system, we consider $\mathrm{Prim}(A)$ as a G-space via the continuous action $G \times \mathrm{Prim}(A) \to \mathrm{Prim}(A)$; $(s, P) \mapsto s \cdot P := \alpha_s(P)$. We write

$$G_P := \{s \in G : s \cdot P = P\} \quad \text{and} \quad G(P) := \{s \cdot P : s \in G\}$$

for the for the *stabilizer* and the *G-orbit* of $P \in \mathrm{Prim}(A)$, respectively. Moreover, we put

$$P^G := \ker G(P) = \cap\{s \cdot P : s \in G\}.$$

Notice that the stabilizers G_P are closed subgroups of G for all $P \in \mathrm{Prim}(A)$.[13]

Remark 10.4. Similarly, we may consider the G-space $\widehat{A}$ with G-action $(s, \pi) \mapsto s \cdot \pi := \pi \circ \alpha_{s^{-1}}$ (identifying representations with their equivalence classes) and we then write G_π and $G(\pi)$ for the stabilizers and the G-orbits, respectively. However, the stabilizers G_π are not necessarily closed in G if A is not a type I algebra. If A is type I, then $\pi \mapsto \ker \pi$ is a G-equivariant homeomorphism from $\widehat{A}$ to $\mathrm{Prim}(A)$.

The following theorem is due to Glimm:

Theorem 10.5 (cf [50])**.** *Suppose that (A, G, α) is a separable type I system (i.e., A is a separable type I algebra and G is second countable). Then the following are equivalent:*

(i) *The quotient space $G\backslash\mathrm{Prim}(A)$ is almost Hausdorff.*
(ii) *$G\backslash\mathrm{Prim}(A)$ is a T_0-space.*
(iii) *All points in $G\backslash\mathrm{Prim}(A)$ are locally closed.*
(iv) *For all $P \in \mathrm{Prim}(A)$ the quotient G/G_P is homeomorphic to $G(P)$ via $s \cdot G_P \mapsto s \cdot P$.*
(v) *There exists an ordinal number μ and an increasing sequence $\{I_\nu\}_{\nu \le \mu}$ of G-invariant ideals of A such that $I_0 = \{0\}$, $I_\mu = A$ and $G\backslash\mathrm{Prim}(I_{\nu+1}/I_\nu)$ is Hausdorff for all $\nu < \mu$.*

Hence, if (A, G, α) is a separable type I system satisfying one of the equivalent conditions above, then (A, G, α) is smooth in the sense of:

Definition 10.6. The system (A, G, α) is called *smooth* if the following two conditions are satisfied:

(i) The map $G/G_P \to G(P)$; $s \cdot G_P \to s \cdot P$ is a homeomorphism for all $P \in \mathrm{Prim}(A)$.
(ii) The quotient $G\backslash\mathrm{Prim}(A)$ is almost Hausdorff, or A is separable and all orbits $G(P)$ are locally closed in $\mathrm{Prim}(A)$.

[13] The the fact that G_P is closed in G follows from the fact that $\mathrm{Prim}(A)$ is a T_0-space. Indeed, if $\{s_i\}$ is a net in G_P which converges to some $s \in G$, then $P = s_i \cdot P \to s \cdot P$, so $s \cdot P$ is in the closure of $\{P\}$. Conversely, we have $s \cdot P = ss_i^{-1} \cdot P \to P$, and hence $\{P\} \in \overline{\{sP\}}$. Since $\mathrm{Prim}(A)$ is a T_0-space it follows that $P = s \cdot P$.

If $G(P)$ is a locally closed orbit of $\mathrm{Prim}(A)$, then we may identify $G(P)$ with $\mathrm{Prim}(A_{G(P)})$, where $A_{G(P)} = I_{G(P)}/J_{G(P)}$ denotes the restriction of A to $G(P)$ as in Definition 10.2 (note that $J_{G(P)} = P^G$ in our notation). Since the ideals $I_{G(P)}$ and $J_{G(P)}$ are G-invariant, the action of G on A restricts to an action of G on $A_{G(P)}$. Using exactness of the full crossed-product functor, we get

$$A_{G(P)} \rtimes G \cong (I_{G(P)} \rtimes G)/(J_{G(P)} \rtimes G). \tag{10.1}$$

If G is exact, a similar statement holds for the reduced crossed products.

Proposition 10.7. *Suppose that (A, G, α) is a system such that*

(i) *$G\backslash\mathrm{Prim}(A)$ is almost Hausdorff, or*

(ii) *A is separable.*

Then, for each $\pi \times U \in (A \rtimes G)\widehat{\ }$, there exists an orbit $G(P) \subseteq \mathrm{Prim}(A)$ such that $\ker\pi = P^G$. If, in addition, all orbits in $\mathrm{Prim}(A)$ are locally closed (which is automatic in case of (i)), then $G(P)$ is uniquely determined by $\pi \times U$.

Proof. (Following ideas from [107,]) Let $J = \ker\pi$. By passing from A to A/J we may assume without loss of generality that π is faithful. We then have to show that there exists a $P \in \mathrm{Prim}(A)$ such that $G(P)$ is dense in $\mathrm{Prim}(A)$.

We first show that under these assumptions every open subset $W \subseteq G\backslash\mathrm{Prim}(A)$ is dense. Indeed, since π is faithful, it follows that $\pi \times U$ restricts to a non-zero, and hence irreducible representation of $I \rtimes G$, whenever I is a nonzero G-invariant ideal of A. In particular, $\pi(I)H_\pi = H_\pi$ for all such ideals I. Assume now that there are two nonempty G-invariant open sets $U_1, U_2 \subseteq \mathrm{Prim}(A)$ with $U_1 \cap U_2 = \emptyset$. Put $I_i = \ker(\mathrm{Prim}(A) \smallsetminus U_i)$, $i = 1, 2$. Then I_1, I_2 would be nonzero G-invariant ideals such that $I_1 \cdot I_2 = I_1 \cap I_2 = \{0\}$, and then

$$H_\pi = \pi(I_1)H_\pi = \pi(I_1)\big(\pi(I_2)H_\pi\big) = \pi(I_1 \cdot I_2)H_\pi = \{0\},$$

which is a contradiction.

Assume that $G\backslash\mathrm{Prim}(A)$ is almost Hausdorff. If there is no dense orbit $G(P)$ in $\mathrm{Prim}(A)$, then $G\backslash\mathrm{Prim}(A)$ contains an open dense Hausdorff subset which contains at least two different points. But then there exist G-invariant open subsets U_1, U_2 of $\mathrm{Prim}(A)$ with $U_1 \cap U_2 = \emptyset$, which is impossible.

If A is separable, then $G\backslash\mathrm{Prim}(A)$ is second countable (see [23, Chapter 3]) and we find a countable base $\{U_n : n \in \mathbb{N}\}$ for its topology. Since $G\backslash\mathrm{Prim}(A)$ is a Baire space by [23, Chapter 3], it follows that $D := \cap_{n\in\mathbb{N}} U_n$ is also dense in $\mathrm{Prim}(A)$. Note that every open subset of $G\backslash\mathrm{Prim}(A)$ contains D. Hence, if we pick any orbit $G(P) \in D$ then $G(P)$ is dense in $\mathrm{Prim}(A)$, since otherwise D would be a subset of the non-empty open set $G\backslash\big(\mathrm{Prim}(A) \smallsetminus \overline{G(P)}\big)$, which is impossible.

If the dense orbit $G(P)$ is locally closed then $G(P)$ is open in its closure $\mathrm{Prim}(A)$, which implies that $G(P)$ is the unique dense orbit in $\mathrm{Prim}(A)$. This gives the uniqueness assertion of the proposition. $\square$

Suppose that $G(P)$ is a locally closed orbit in $\widehat{A}$. By exactness of the full crossed product functor we can then identify $A_{G(P)} \rtimes G$ with the subquotient $(I_{G(P)} \rtimes$

$G)/(J_{G(P)} \rtimes G)$ of $A \rtimes G$. If G is exact, a similar identification is possible for the reduced crossed product $A_{G(P)} \rtimes_r G$. Using this identification we get

Corollary 10.8. *Suppose that (A, G, α) is smooth. Then we obtain a decomposition of $(A \rtimes G)\widehat{\ }$ (resp. $\mathrm{Prim}(A \rtimes G)$) as the disjoint union of the locally closed subsets $(A_{G(P)} \rtimes G)\widehat{\ }$ (resp. $\mathrm{Prim}(A_{G(P)} \rtimes G)$), where $G(P)$ runs through all G-orbits in $\mathrm{Prim}(A)$. If G is exact, similar statements hold for the reduced crossed products.*

Proof. It follows from Proposition 10.7 that for each $\pi \times U \in (A \rtimes G)\widehat{\ }$, there exists a unique orbit $G(P)$ such that $\ker \pi = P^G = J_{G(P)}$ and then $\pi \times U$ restricts to an irreducible representation of $A_{G(P)} \rtimes G$. Hence

$$(A \rtimes G)\widehat{\ } = \cup\{(A_{G(P)} \rtimes G)\widehat{\ } : G(P) \in G\backslash \mathrm{Prim}(A)\}.$$

To see that this union is disjoint, assume that there exists an element $\rho \times V \in (A_{G(P)} \rtimes G)\widehat{\ }$ (viewed as a representation of $A \rtimes G$) such that $\ker \rho \neq P^G$. Since ρ is a representation of $A_{G(P)} = I_{G(P)}/P^G$ we have $\ker \rho \supseteq P^G$. By Proposition 10.7 there exists a $Q \in \mathrm{Prim}(A)$ such that $\ker \rho = Q^G$. Then $Q^G \supseteq P^G$, which implies that $G(Q) \subseteq \big(\overline{G(P)} \smallsetminus G(P)\big)$. But then

$$\ker \rho = Q^G = \ker G(Q) \supseteq \ker \big(\overline{G(P)} \smallsetminus G(P)\big) = I_{G(P)},$$

which contradicts the assumption that $\rho \times V \in (A_{G(P)} \rtimes G)\widehat{\ }$. □

It is now easy to give a proof of the Mackey-Green-Rieffel theorem, which is the main result of this section. If (A, G, α) is smooth, one can easily check that points in $\mathrm{Prim}(A)$ are automatically locally closed (since they are closed in their orbits). Hence, for each $P \in \mathrm{Prim}(A)$ the restriction $A_P := I_P/P$ of A to $\{P\}$ is a simple subquotient of A. Since I_P and P are invariant under the action of the stabilizer G_P, the action of G_P on A factors through an action of G_P on A_P. It is then straightforward to check (using the same arguments as given in the proof of Corollary 10.8) that there is a canonical one-to-one correspondence between the irreducible representations of $A_P \rtimes G_P$ and the set of all irreducible representations $\rho \times V$ of $A \rtimes G_P$ satisfying $\ker \rho = P$.

Remark 10.9. If A is type I, then $A_P \cong \mathcal{K}(H_\pi)$, the compact operators on the Hilbert space H_π, where $\pi : A \to \mathcal{B}(H_\pi)$ is the unique (up to equivalence) irreducible representation of A with $\ker \pi = P$. To see this we first pass to $A/P \cong \pi(A)$. Since A is type I we know that $\mathcal{K}(H_\pi) \subseteq \pi(A)$. Hence, if we identify $\mathcal{K}(H_\pi)$ with an ideal of A/P, we see (since π does not vanish on this ideal) that this ideal must correspond to the open set $\{\pi\}$ (resp. $\{P\}$) in its closure $\widehat{A/P}$ (resp. $\mathrm{Prim}(A/P)$).

Theorem 10.10 (Mackey-Rieffel-Green)**.** *Suppose that (A, G, α) is smooth. Let $\mathcal{S} \subseteq \mathrm{Prim}(A)$ be a cross-section for the orbit space $G\backslash \mathrm{Prim}(A)$. Then induction of representations and ideals induces bijections*

$$\mathrm{Ind} : \cup_{P\in\mathcal{S}}(A_P \rtimes G_P)\widehat{\ } \to (A \rtimes G)\widehat{\ }; \rho \times V \mapsto \mathrm{ind}_{G_P}^G(\rho \times V) \quad \textit{and}$$

$$\mathrm{Ind} : \cup_{P\in\mathcal{S}} \mathrm{Prim}(A_P \rtimes G_P) \to \mathrm{Prim}(A \rtimes G);\ Q \mapsto \mathrm{ind}_{G_P}^G Q.$$

If G *is exact, these maps restrict to similar bijections*

$$\cup_{P\in\mathcal{S}}(A_P \rtimes_r G_P)\widehat{\ } \stackrel{\mathrm{Ind}}{\to} (A \rtimes_r G)\widehat{\ } \quad \textit{and} \quad \cup_{P\in\mathcal{S}} \mathrm{Prim}(A_P \rtimes_r G_P) \stackrel{\mathrm{Ind}}{\to} \mathrm{Prim}(A \rtimes_r G)$$

for the reduced crossed products.

Proof. We show that $\mathrm{Ind} : \cup_{P\in\mathcal{S}}(A_P \rtimes G_P)\widehat{\ } \to (A \rtimes G)\widehat{\ }$ is a bijection. Bijectivity of the other maps follows similarly.

By Corollary 10.8 it suffices to show that $\mathrm{Ind} : (A_P \rtimes G_P)\widehat{\ } \to (A_{G(P)} \rtimes G)\widehat{\ }$ is an isomorphism for all $P \in \mathcal{S}$. By definition of $A_{G(P)}$ we have $\mathrm{Prim}(A_{G(P)}) \cong G(P)$ and by the smoothness of the action we have $G(P) \cong G/G_P$ as G-spaces. Hence, it follows from Theorem 6.2 that $A_{G(P)} \cong \mathrm{Ind}_{G_P}^G A_P$. Hence induction via Green's $A_{G(P)} \rtimes G - A_P \rtimes G_P$ imprimitivity bimodule $X_P := X_{G_P}^G(A_P)$ gives the desired bijection $\mathrm{ind}^{X_P} : (A_P \rtimes G_P)\widehat{\ } \to (A_{G(P)} \rtimes G)\widehat{\ }$. By Corollary 9.9, induction via X_P coincides with the usual induction for the system $(A_{G(P)}, G, \alpha)$, which by Remark 9.16 is compatible with inducing the corresponding representations for (A, G, α). $\square$

The above result shows that for smooth systems, all representations are induced from the stabilizers for the corresponding action of G on $\mathrm{Prim}(A)$. In fact the above result is much stronger, since it shows that $A \rtimes G$ has a "fibration" over $G\backslash \mathrm{Prim}(A)$ such that the fiber $A_{G(P)} \rtimes G$ over an orbit $G(P)$ is Morita equivalent to $A_P \rtimes G_P$, hence, up to the global structure of the fibration, the study of $A \rtimes G$ reduces to the study of the fibers $A_P \rtimes G_P$. Note that under the assumptions of Theorem 10.10 the algebra A_P is always simple. We shall give a more detailed study of the crossed products $A_P \rtimes G_P$ in the important special case where A is type I in §14 below. The easier situation where $A = C_0(X)$ is treated in §11 below.

Note that the study of the global structure of $A \rtimes G$, i.e., of the global structure of the fibration over $G\backslash \mathrm{Prim}(A)$ is in general quite complicated, even in the situation where $G\backslash \mathrm{Prim}(A)$ is Hausdorff. In general, it is also very difficult (if not impossible) to describe the global topology of $\mathrm{Prim}(A \rtimes G)$ in terms of the bijection of Theorem 10.10. Some progress has been made in the case where A is a continuous-trace C^*-algebra and/or where the stabilizers are assumed to vary continuously, and we refer to [31, 21, 105, 38, 34] and the references given in those papers and books for more information on this problem.

Even worse, the assumption of having a smooth action is a very strong one and for arbitrary systems one cannot expect that one can compute all irreducible representations via induction from stabilizers. Indeed, in general it is not possible to classify all irreducible representations of a non-type I C^*-algebra, and a similar problem occurs for crossed products $A \rtimes G$ if the action of G on $\mathrm{Prim}(A)$ fails to be smooth. However, at least if (A, G, α) is separable and G is amenable, there is a positive result towards the description of $\mathrm{Prim}(A \rtimes G)$ which was obtained by work of Sauvageot and Gootman-Rosenberg, thus giving a positive answer to an earlier formulated conjecture by Effros and Hahn (see [41]). To give precise statements, we need

Definition 10.11. A non-degenerate representation ρ of a C^*-algebra A is called *homogeneous* if all non-trivial subrepresentations of ρ have the same kernel as ρ.

It is clear that every irreducible representation is homogeneous and it is easy to see that the kernel of any homogeneous representation is a prime ideal, and hence it is primitive if A is second countable.

Theorem 10.12 (Sauvageot ([113]))**.** *Suppose that (A,G,α) is a separable system (i.e., A is separable and G is second countable). Let $P \in \mathrm{Prim}(A)$ and let G_P denote the stabilizer of P in G. Suppose that $\rho\times V$ is a homogeneous representation of $A \rtimes G_P$ such that ρ is a homogeneous representation of A with $\ker\rho = P$. Then $\mathrm{ind}_{G_P}^G(\rho\times V)$ is a homogeneous representation of $A\rtimes G$ and $\ker\big(\mathrm{ind}_{G_P}^G(\rho\times V)\big)$ is a primitive ideal of $A\rtimes G$.*

We say that a primitive ideal of $A\rtimes G$ is *induced* if it is obtained as in the above theorem. Note that Sauvageot already showed in [113] that in case where G is amenable, every primitive ideal of $A\rtimes G$ contains an induced primitive ideal and in case where G is discrete every primitive ideal is contained in an induced primitive ideal. Together, this shows that for actions of discrete amenable groups all primitive ideals of $A\rtimes G$ are induced from the stabilizers. Sauvageot's result was generalized by Gootman and Rosenberg in [55, Theorem 3.1]:

Theorem 10.13 (Gootman-Rosenberg)**.** *Suppose that (A,G,α) is a separable system. Then every primitive ideal of $A\rtimes G$ is contained in an induced ideal. As a consequence, if G is amenable, then every primitive ideal of $A\rtimes G$ is induced.*

The condition in Theorem 10.12 that the representations $\rho\times V$ and ρ are homogeneous is a little bit unfortunate. In fact, a somehow more natural formulation of Sauvageot's theorem (using the notion of induced ideals) would be to state that whenever $Q\in\mathrm{Prim}(A\rtimes G_P)$ such that $\mathrm{res}_{\{e\}}^{G_P}(Q)=P$, then $\mathrm{ind}_{G_P}^G(Q)$ is a primitive ideal of $A\rtimes G$. Note that if $\rho\times V$ is as in Theorem 10.12, then $Q=\ker(\rho\times V)$ is an element of $\mathrm{Prim}(A\rtimes G_P)$ which satisfies the above conditions. At present time, we do not know whether this more general statement is true, and we want to take this opportunity to point out that the statement of [31, Theorem 1.4.14] is not correct (or at least not known) as it stands. We are very grateful to Dana Williams for pointing out this error and we refer to the paper [40] for a more elaborate discussion of this problem. But let us indicate here that the problem vanishes if all points in $\mathrm{Prim}(A)$ are locally closed (which is in particular true if A is type I).

Proposition 10.14. *Suppose that (A,G,α) is a separable system such that one of the following conditions is satisfied:*

(i) *All points in $\mathrm{Prim}(A)$ are locally closed (which is automatic if A is type I).*
(ii) *All stabilizers G_P for $P\in\mathrm{Prim}(A)$ are normal subgroups of G (which is automatic if G is abelian).*

Then $\mathrm{ind}_{G_P}^G Q\in\mathrm{Prim}(A\rtimes G)$ for all $P\in\mathrm{Prim}(A)$ and $Q\in\mathrm{Prim}(A\rtimes G_P)$ such that $\mathrm{res}_{\{e\}}^{G_P} Q = P$. If, in addition, G is amenable, then all primitive ideals of $A\rtimes G$ are induced in this way.

Proof. Let us first assume condition (i). Choose $\rho\times V\in(A\rtimes G_P)\widehat{\ }$ such that $\ker(\rho\times V)=Q$ and $\ker\rho=P$. Then we may regard ρ as a representation of A_P, the simple subquotient of A corresponding to the locally closed subset $\{P\}$ of $\mathrm{Prim}(A)$. Since A_P is simple, all nontrivial subrepresentations of ρ have kernel $\{0\}$ in A_P (and hence they have kernel P in A). Hence ρ is homogeneous and the result follows from Theorems 10.12 and 10.13.

Let us now assume (ii). If $N := G_P$ is normal, we may use the theory of twisted actions, which we shall present in §12 below, to pass to the system $((A \rtimes N) \otimes \mathcal{K}, G/N, \beta)$. If $\rho \times V \in (A \rtimes N)\widehat{}$ with $\ker(\rho \times V) = P$, then the corresponding representation of $(A \rtimes N) \otimes \mathcal{K}$ has trivial stabilizer in G/N, and therefore the induced representation has primitive kernel in $A \rtimes G \sim_M \big((A \rtimes N) \otimes \mathcal{K}\big) \rtimes G/N$ by Theorem 10.12. $\square$

Recall that if M is a topological G-space, then two elements $m_1, m_2 \in M$ are said to be in the same *quasi-orbit* if $m_1 \in \overline{G(m_2)}$ and $m_2 \in \overline{G(m_1)}$. Being in the same quasi-orbit is clearly an equivalence relation on M and we denote by $G_q(m)$ the quasi-orbit (i.e., the equivalence class) of m and by $\mathcal{Q}_G(M)$ the set of all quasi-orbits in M equipped with the quotient topology. Note that $\mathcal{Q}_G(M)$ is always a T_0-space. If $G\backslash M$ is a T_0-space, then $\mathcal{Q}_G(M)$ coincides with $G\backslash M$.

If (A, G, α) is a system, it follows from the definition of the Jacobson topology that two elements $P, Q \in \mathrm{Prim}(A)$ are in the same quasi-orbit if and only if $P^G = Q^G$. If the action of G on A is smooth, then all point in $G\backslash \mathrm{Prim}(A)$ are locally closed, which implies in particular that $G\backslash \mathrm{Prim}(A)$ is a T_0-space. Hence in this case we have $\mathcal{Q}_G(\mathrm{Prim}(A)) = G\backslash \mathrm{Prim}(A)$. In what follows, we let

$$\mathrm{Prim}^G(A) := \{P^G : P \in \mathrm{Prim}(A)\} \subseteq \mathcal{I}(A)$$

equipped with the relative Fell topology. Then [52, Lemma on p. 221] gives

Lemma 10.15. *Let (A, G, α) be a system. Then the map*

$$q : \mathrm{Prim}(A) \to \mathrm{Prim}^G(A) : P \mapsto P^G$$

is a continuous and open surjection and therefore factors through a homeomorphism between $\mathcal{Q}_G(\mathrm{Prim}(A))$ and $\mathrm{Prim}^G(A)$.

As a consequence of the previous results we get

Corollary 10.16. *Suppose that (A, G, α) is smooth or that (A, G, α) is separable and G is amenable. Suppose further the action of G on $\mathrm{Prim}(A)$ is free (i.e., all stabilizers are trivial). Then the map*

$$\mathrm{Ind} : \mathrm{Prim}^G(A) \cong \mathcal{Q}_G(A) \to \mathrm{Prim}(A \rtimes G); P \mapsto \mathrm{ind}_{\{e\}}^G P^G$$

is a homeomorphism. In particular, $A \rtimes G$ is simple if and only if every G-orbit is dense in $\mathrm{Prim}(A)$, and $A \rtimes G$ is primitive (i.e., $\{0\}$ is a primitive ideal of A) if and only if there exists a dense G-orbit in $\mathrm{Prim}(A)$.

Proof. It follows from Theorem 10.10 and Theorem 10.13 that the map $\mathrm{ind}_{\{e\}}^G : \mathrm{Prim}(A) \to \mathrm{Prim}(A \rtimes G); P \mapsto \mathrm{ind}_{\{e\}}^G P$ is well defined and surjective. By Corollary 9.13 we know that $\mathrm{ind}_{\{e\}}^G P = \mathrm{ind}_{\{e\}}^G P^G$, so the induction map $\mathrm{Ind} : \mathrm{Prim}^G(A) \to \mathrm{Prim}(A \rtimes G)$ is also well defined and surjective. Equation (9.4) applied to $H = \{e\}$

gives $\operatorname{res}_{\{e\}}^G(\operatorname{ind}_{\{e\}}^G P) = P^G$, which shows that $\operatorname{res}_{\{e\}}^G : \operatorname{Prim}(A \rtimes G) \to \operatorname{Prim}^G(A)$ is the inverse of Ind. Since induction and restriction are continuous by Proposition 9.4 the result follows. $\square$

Remark 10.17. Note that if (A, G, α) is a system with constant stabilizer N for the action of G on $\operatorname{Prim}(A)$, then one can pass to the iterated twisted system $(A \rtimes N, G, N, \alpha^N, \tau^N)$ (see §12 below), and then to an equivariantly Morita equivalent system $(B, G/N, \beta)$ (see Proposition 13.3) to see that induction of primitive ideals gives a homeomorphism between $\mathcal{Q}_{G/N}(\operatorname{Prim}(A \rtimes N))$ and $\operatorname{Prim}(A \rtimes G)$ if one of the following conditions are satisfied:

(i) (A, G, α) is smooth.
(ii) $(A \rtimes N, G, N, \alpha^N, \tau^N)$ is smooth (i.e., the action of G/N on $\operatorname{Prim}(A \rtimes N)$ via α^N satisfies the conditions of Definition 10.6.
(iii) (A, G, α) is separable and G/N is amenable.

A similar result can be obtained for systems with continuously varying stabilizers (see [28]). In the case of constant stabilizers, the problem of describing the topology of $\operatorname{Prim}(A \rtimes G)$ now reduces to the description the topology of $\operatorname{Prim}(A \rtimes N)$ and the action of G/N on $\operatorname{Prim}(A \rtimes N)$. In general, both parts can be quite difficult to perform, but in some interesting special cases, e.g. if A has continuous trace, some good progress has been made for the description of $\operatorname{Prim}(A \rtimes N)$ (e.g. see [37, 38, 34] and the references given there). Of course, if $A = C_0(X)$ is abelian, and N is the constant stabilizer of the elements of $\operatorname{Prim}(A) = X$, then N acts trivially on X and $\operatorname{Prim}(C_0(X) \rtimes N) = \operatorname{Prim}(C_0(X) \otimes C^*(N)) = X \times \operatorname{Prim}(C^*(N))$.

Example 10.18. As an easy application of Corollary 10.16 we get the simplicity of the irrational rotation algebra A_θ, for θ an irrational number in $(0, 1)$. Recall that $A_\theta = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ where $n \in \mathbb{Z}$ acts on $z \in \mathbb{T}$ via $n \cdot z := e^{2\pi i\theta n} z$. Since θ is irrational, the action of $\mathbb{Z}$ on $\operatorname{Prim}(C(\mathbb{T})) = \mathbb{T}$ is free and all $\mathbb{Z}$-orbits are dense in $\mathbb{T}$. Hence, there exists only one quasi-orbit in $\mathbb{T}$ and the crossed product is simple. Of course, there are other more elementary proofs for the simplicity of A_θ which do not use such heavy machinery, but this example illustrates quite well how one can use the above results.

11. The Mackey-machine for transformation groups

Suppose that X is a locally compact G-space and consider the corresponding action of G on $A = C_0(X)$ given by $(s \cdot \varphi)(x) = \varphi(s^{-1}x)$ for $s \in G$, $\varphi \in C_0(X)$. Then $\operatorname{Prim}(A) = X$ and $A_x = \mathbb{C}$ for all $x \in X$, so that $A_x \rtimes G_x = C^*(G_x)$ for all $x \in X$, where G_x denotes the stabilizer of x. Hence, if the action of G on X is smooth in the sense of Definition 10.6, then it follows from Theorem 10.10 that $C_0(X) \rtimes G$ is "fibered" over $G\backslash X$ with fibres $C_0(G(x)) \rtimes G \sim_M C^*(G_x)$ (compare the discussion following Theorem 10.10).

If $V \in \widehat{G_x}$ and if $\varepsilon_x : C_0(X) \to \mathbb{C}$ denotes evaluation at x, then $\varepsilon_x \times V$ is the representation of $C_0(X) \rtimes G_x$ which corresponds to V by regarding $\widehat{G_x} = (A_x \rtimes G_x)\widehat{\ }$

as a subset of $(A \rtimes G_x)\widehat{\ }$ as described in the discussion preceeding Theorem 10.10. In this situation, the result of Theorem 10.12 can be improved by showing:

Proposition 11.1 (cf. [121, Proposition 4.2]). *Let $\varepsilon_x \times V \in \big(C_0(X) \rtimes G_x\big)\widehat{\ }$ be as above. Then $\mathrm{ind}_{G_x}^G(\varepsilon_x \rtimes V)$ is irreducible. Moreover, if $V, W \in \widehat{G_x}$, then $\mathrm{ind}_{G_x}^G(\varepsilon_x \times V) \cong \mathrm{ind}_{G_x}^G(\varepsilon_x \times W)$ if and only if $V \cong W$.*

Combining this with Theorem 10.10 and Theorem 10.13 gives:

Theorem 11.2. *Suppose that X is a locally compact G-space.*

(i) *If G acts smoothly on X, and if $\mathcal{S}$ is a section for $G\backslash X$, then we get a bijection*
$$\mathrm{Ind} : \cup_{x\in\mathcal{S}}\widehat{G_x} \to \big(C_0(X) \rtimes G\big)\widehat{\ }; V \mapsto \mathrm{ind}_{G_x}^G(\varepsilon_x \times V).$$

(ii) *If X and G are second countable and if G is amenable, then every primitive ideal of $C_0(X) \rtimes G$ is the kernel of some induced irreducible representation $\mathrm{ind}_{G_x}^G(\varepsilon_x \times V)$.*

We now want to present some applications to group representation theory:

Example 11.3. Suppose that $G = N \rtimes H$ is the semi-direct product of the abelian group N by the group H. Then, as seen in Example 3.6, we have
$$C^*(N \rtimes H) \cong C^*(N) \rtimes H \cong C_0(\widehat{N}) \rtimes H,$$
where the last isomorphism is given via the Gelfand-transform $C^*(N) \cong C_0(\widehat{N})$. The corresponding action of H on $C_0(\widehat{N})$ is induced by the action of H on $\widehat{N}$ given by $\big(h \cdot \chi\big)(n) := \chi(h^{-1} \cdot n)$ if $h \in H$, $\chi \in \widehat{N}$ and $n \in N$. Thus, if the action of H on $\widehat{N}$ is smooth, we obtain every irreducible representation of $C^*(N \rtimes H) \cong C_0(\widehat{N}) \rtimes H$ as an induced representation $\mathrm{ind}_{H_\chi}^H(\varepsilon_\chi \times V)$ for some $\chi \in \widehat{N}$ and $V \in \widehat{H}_\chi$. The isomorphism $C_0(\widehat{N}) \rtimes H_\chi \cong C^*(N \rtimes H_\chi)$, transforms the representation $\varepsilon_\chi \times V$ to the representation $\chi \times V$ of $N \rtimes H_\chi$ defined by $\chi \times V(n, h) = \chi(n)V(h)$ and one can show that $\mathrm{ind}_{N\rtimes H_\chi}^{N\rtimes H}(\chi \times V)$ corresponds to the representation $\mathrm{ind}_{H_\chi}^H(\varepsilon_\chi \times V)$ under the isomorphism $C^*(N \rtimes N) \cong C_0(\widehat{N}) \rtimes H$. Thus, choosing a cross-section $\mathcal{S} \subseteq \widehat{N}$ for $H\backslash\widehat{N}$, it follows from Theorem 10.10 that
$$\mathrm{Ind} : \cup\{\widehat{H}_\chi : \chi \in \mathcal{S}\} \to \widehat{N \rtimes H}; V \mapsto \mathrm{ind}_{N\rtimes H_\chi}^{N\rtimes H}(\chi \times V)$$
is a bijection.

If the action of H on $\widehat{N}$ is not smooth, but $N \rtimes H$ is second countable and amenable, then we get at least all primitive ideals of $C^*(N \rtimes H)$ as kernels of the induced representations $\mathrm{ind}_{N\rtimes H_\chi}^{N\rtimes H}(\chi \times V)$.

Let us now discuss some explicit examples:

(1) Let $G = \mathbb{R} \rtimes \mathbb{R}^*$ denote the $ax + b$-group, i.e., G is the semi-direct product for the action of the multiplicative group $\mathbb{R}^*$ on $\mathbb{R}$ via dilation. Identifying $\mathbb{R}$ with $\widehat{\mathbb{R}}$ via $t \mapsto \chi_t$ with $\chi_t(s) = e^{2\pi its}$, we see easily that the action of $\mathbb{R}^*$ on $\widehat{\mathbb{R}}$ is also given by dilation. Hence there are precisely two orbits in $\widehat{\mathbb{R}}$: $\{\chi_0\}$ and $\widehat{\mathbb{R}} \smallsetminus \{\chi_0\}$.

Let $\mathcal{S} = \{\chi_0, \chi_1\} \subseteq \widehat{\mathbb{R}}$. Then $\mathcal{S}$ is a cross-section for $\mathbb{R}^*\backslash\widehat{\mathbb{R}}$, the stabilizer of χ_1 in $\mathbb{R}^*$ is $\{1\}$ and the stabilizer of χ_0 is all of $\mathbb{R}^*$. Thus we see that

$$\widehat{G} = \{\chi_0 \times \mu : \mu \in \widehat{\mathbb{R}^*}\} \cup \{\mathrm{ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^*} \chi_1\}.$$

Notice that we could also consider the C^*-algebra $C^*(G)$ as "fibered" over $\mathbb{R}^*\backslash\widehat{\mathbb{R}}$: The open orbit $\widehat{\mathbb{R}} \smallsetminus \{\chi_0\} \cong \mathbb{R}^*$ corresponds to the ideal $C_0(\mathbb{R}^*) \rtimes \mathbb{R}^* \cong \mathcal{K}(L^2(\mathbb{R}^*))$ and the closed orbit $\{\chi_0\}$ corresponds to the quotient $C_0(\widehat{\mathbb{R}^*})$ of $C^*(G)$, so that this picture yields the short exact sequence

$$0 \to \mathcal{K}(L^2(\mathbb{R}^*)) \to C^*(G) \to C_0(\widehat{\mathbb{R}^*}) \to 0$$

(compare also with Example 6.6).

(2) A more complicated example is given by the Mautner group. This group is the semi-direct product $G = \mathbb{C}^2 \rtimes \mathbb{R}$ with action given by

$$t \cdot (z, w) = (e^{-2\pi it}z, e^{-2\pi i\theta t}w),$$

where $\theta \in (0, 1)$ is a fixed irrational number. Identifying $\mathbb{C}^2$ with the dual group $\widehat{\mathbb{C}^2}$ via $(u, v) \mapsto \chi_{(u,v)}$ such that

$$\chi_{(u,v)}(z, w) = \exp(2\pi i\mathrm{Re}(z\bar{u} + w\bar{v})),$$

we get $t \cdot \chi_{(u,v)} = \chi_{(e^{2\pi it}u, e^{2\pi i\theta t}z)}$. The quasi-orbit space for the action of $\mathbb{R}$ on $\widehat{\mathbb{C}^2}$ can then be parametrized by the set $[0, \infty) \times [0, \infty)$: If $(r, s) \in [0, \infty)^2$, then the corresponding quasi-orbit $\mathcal{O}_{(s,t)}$ consists of all $(u, v) \in \mathbb{C}^2$ such that $|u| = r$ and $|v| = s$. Hence, if $s, r > 0$, then $\mathcal{O}_{(s,t)}$ is homeomorphic to $\mathbb{T}^2$ and this homeomorphism carries the action of $\mathbb{R}$ on $\mathcal{O}_{(s,t)}$ to the irrational flow of $\mathbb{R}$ on $\mathbb{T}^2$ corresponding to θ as considered in part (4) of Example 6.6. In particular, $\mathbb{R}$ acts freely but not smoothly on those quasi-orbits. If $r \neq 0$ and $s = 0$, the quasi-orbit $\mathcal{O}_{(s,t)}$ is homeomorphic to $\mathbb{T}$ with action $t \cdot u := e^{2\pi it}u$ and constant stabilizer $\mathbb{Z}$. In particular, all those quasi-orbits are orbits. Similarly, if $r = 0$ and $s \neq 0$, the quasi-orbit $\mathcal{O}_{(r,s)}$ is homeomorphic to $\mathbb{T}$ with action $t \cdot v = e^{2\pi i\theta t}v$ and stabilizer $\frac{1}{\theta}\mathbb{Z}$. Finally, the quasi-orbit corresponding to $(0, 0)$ is the point-set $\{(0, 0)\}$ with stabilizer $\mathbb{R}$.

Since G is second countable and amenable, we can therefore parametrize $\mathrm{Prim}(C^*(G))$ by the set

$$\{(r, s) : r, s > 0\} \cup \big((0, \infty) \times \widehat{\mathbb{Z}}\big) \cup \big((0, \infty) \times \widehat{\tfrac{1}{\theta}\mathbb{Z}}\big) \cup \widehat{\mathbb{R}}.$$

In fact, we can also view $C^*(G)$ as "fibered" over $[0, \infty)^2$ with fibers

$$C^*(G)_{(r,s)} \cong C(\mathbb{T}^2) \rtimes_\theta \mathbb{R} \sim_M A_\theta \qquad \text{for } r, s > 0,$$

where A_θ denotes the irrational rotation algebra,

$$C^*(G)_{(r,0)} \cong C(\mathbb{T}) \rtimes \mathbb{R} \sim_M C(\widehat{\mathbb{Z}}) \cong C(\mathbb{T}) \qquad \text{for } r > 0,$$

$$C^*(G)_{(0,s)} \cong C(\mathbb{T}) \rtimes_\theta \mathbb{R} \sim_M C(\widehat{\frac{1}{\theta}\mathbb{Z}}) \cong C(\mathbb{T}) \qquad \text{for } s > 0,$$

and $C^*(G)_{(0,0)} \cong C_0(\mathbb{R})$. Using continuity of induction and restriction, it is also possible to describe the topology of $\mathrm{Prim}(G)$ in terms of convergent sequences, but we do not go into the details here. We should mention that the Mautner group is the lowest dimensional example of a connected Lie-group G with a non-type I group algebra $C^*(G)$.

Remark 11.4. It follows from Theorems 10.10 and 10.13 that for understanding the ideal structure of $A \rtimes G$, it is necessary to understand the structure of $A_P \rtimes G_P$ for $P \in \mathrm{Prim}(A)$. We saw in this section that this is the same as understanding the group algebras $C^*(G_x)$ for the stabilizers G_x if $A = C_0(X)$ is abelian. In general, the problem becomes much more difficult. However, at least in the important special case where A is type I one can still give a quite satisfactory description of $A_P \rtimes G_P$ in terms of the stabilizers. Since an elegant treatment of that case uses the theory of twisted actions and crossed products, we postpone the discussion of this case to §14 below.

12. Twisted actions and twisted crossed products

One draw-back of the theory of crossed products by ordinary actions is the fact that crossed products $A \rtimes G$ (and their reduced analogues) cannot be written as iterated crossed products $(A \rtimes N) \rtimes G/N$ if N is a normal subgroup such that the extension

$$1 \to N \to G \to G/N \to 0$$

is not topologically split (compare with Example 3.6). In order to close this gap, we now introduce twisted actions and twisted crossed products following Phil Green's approach of [52]. Note that there is an alternative approach due to Leptin and Busby–Smith (see [79, 10, 96] for the construction of twisted crossed products within this theory), but Green's theory seems to be better suited for our purposes.

As a motivation, consider a closed normal subgroup N of the locally compact group G, and assume that $\alpha : G \to \mathrm{Aut}(A)$ is an action. Let $A \rtimes N$ be the crossed product of A by N. Let $\delta : G \to \mathbb{R}^+$ be the module for the conjugation action of G on N, i.e., $\delta(s)\int_N f(s^{-1}ns)\,dn = \int_N f(n)\,dn$ for all $f \in C_c(N)$. A short computation using the formula

$$\int_G g(s)\,ds = \int_{G/N}\left(\int_N g(sn)\,dn\right)\,dsN \tag{12.1}$$

(with respect to suitable choices of Haar measures) shows that $\delta(s) = \Delta_G(s)\Delta_{G/N}(s^{-1})$ for all $s \in G$. Similar to Example 3.6 we define an action $\alpha^N : G \to \mathrm{Aut}(A \rtimes N)$ by

$$\big(\alpha^N_s(f)\big)(n) = \delta(s)\alpha_s\big(f(s^{-1}ns)\big) \tag{12.2}$$

for f in the dense subalgebra $C_c(N, A) \subseteq A \rtimes N$. If we denote by $\tau^N := i_N : N \to U(A \rtimes N)$ the canonical embedding as defined in part (1) of Remark 3.4, then the pair (α^N, τ^N) satisfies the equations

$$\tau^N_n x \tau^N_{n^{-1}} = \alpha^N_n(x) \quad \text{and} \quad \alpha^N_s(\tau^N_n) = \tau^N_{sns^{-1}} \tag{12.3}$$

for all $x \in A \rtimes N$, $n \in N$ and $s \in G$, where in the second formula we extended the automorphism α^N_s of $A \rtimes N$ to $M(A \rtimes N)$. Suppose now that (π, U) is a covariant homomorphism of (A, G, α) into some $M(D)$. Let $(\pi, U|_N)$ denote its restriction to

(A, N, α) and let $\pi \times U|_N : A \rtimes N \to M(D)$ be its integrated form. Then $(\pi \times U|_N, U)$ is a non-degenerate covariant homomorphism of $(A \rtimes N, G, \alpha^N)$ which satisfies

$$\pi \times U|_N(\tau_n^N) = U_n$$

for all $n \in N$ (see Remark 3.4). The pair (α^N, τ^N) is the prototype for a twisted action (which we denote the *decomposition twisted action*) and $(\pi \times U|_N, U)$ is the prototype of a twisted covariant homomorphism as in

Definition 12.1 (Green)**.** Let N be a closed normal subgroup of G. A *twisted action* of (G, N) on a C^*-algebra A is a pair (α, τ) such that $\alpha : G \to \mathrm{Aut}(A)$ is an action and $\tau : N \to UM(A)$ is a strictly continuous homomorphism such that

$$\tau_n a \tau_{n^{-1}} = \alpha_n(a) \quad \text{and} \quad \alpha_s(\tau_n) = \tau_{sns^{-1}}$$

for all $a \in A$, $n \in N$ and $s \in G$. We then say that (A, G, N, α, τ) is a twisted system. A *(twisted) covariant homomorphism* of (A, G, N, α, τ) into some $M(D)$ is a covariant homomorphism (ρ, V) of (A, G, α) into $M(D)$ which *preserves* τ in the sense that $\rho(\tau_n a) = V_n \rho(a)$ for all $n \in N, a \in A$. [14]

Remark 12.2. Note that the kernel of the regular representation $\Lambda_A^N : A \rtimes N \to A \rtimes_r N$ is easily seen to be invariant under the decomposition twisted action (α^N, τ^N) (which just means that it is invariant under α^N), so that (α^N, τ^N) induces a twisted action on the quotient $A \rtimes_r N$. In what follows, we shall make no notational difference between the decomposition twisted actions on the full or the reduced crossed products.

Let $C_c(G, A, \tau)$ denote the set of all continuous A-valued functions on G with compact support mod N and which satisfy

$$f(ns) = f(s)\tau_{n^{-1}} \qquad \text{for all } n \in N,\ s \in G.$$

Then $C_c(G, A, \tau)$ becomes a $*$-algebra with convolution and involution defined by

$$f * g(s) = \int_{G/N} f(t)\alpha_t(g(t^{-1}s))\, dtN \quad \text{and} \quad f^*(s) = \Delta_{G/N}(s^{-1})\alpha_s\big(f(s^{-1})^*\big).$$

If (ρ, V) is a covariant representation of (A, G, N, α, τ), then the equation

$$\rho \times V(f) = \int_{G/N} \rho(f(s))V_s\, dsN$$

defines a $*$-homomorphism $\rho \times V : C_c(G, A, \tau) \to M(D)$, and the *full twisted crossed product* $A \rtimes_{\alpha,\tau} (G, N)$ (or just $A \rtimes (G, N)$ if (α, τ) is understood) is defined as the completion of $C_c(G, A, \tau)$ with respect to

$$\|f\|_{\max} := \sup\{\|\rho \times V(f)\| : (\rho, V) \text{ is a covariant homomorphism of } (A, G, N, \alpha, \tau)\}.$$

Note that the same formulas as given in Remark 3.4 define a twisted covariant homomorphism (i_A, i_G) of (A, G, N, α, τ) into $M(A \rtimes (G, N))$ such that any non-degenerate homomorphism $\Phi : A \rtimes (G, N) \to M(D)$ is the integrated form $\rho \times V$ with $\rho = \Phi \circ i_A$ and $V = \Phi \circ i_G$.

[14] The latter condition becomes $\rho(\tau_n) = V_n$ if (ρ, V) is non-degenerate.

Remark 12.3. It is important to notice that for any twisted action (α,τ) of (G,N) the map

$$\Phi : C_c(G,A) \to C_c(G,A,\tau); \Phi(f)(s) = \int_N f(sn)\tau_{sns^{-1}}\,dn$$

extends to a quotient map $\Phi : A \rtimes G \to A \rtimes (G,N)$ of the full crossed products, such that $\ker\Phi = \cap\{\ker(\pi\times U) : (\pi,U) \text{ preserves } \tau\}$. The ideal $I_\tau := \ker\Phi$ is called the *twisting ideal* of $A \rtimes G$. Note that if $G = N$, then $A \rtimes (N,N) \cong A$ via $f \mapsto f(e); C_c(N,A,\tau) \to A$.

For the definition of the *reduced twisted crossed products* $A \rtimes_{\alpha,\tau,r} (G,N)$ (or just $A \rtimes_r (G,N)$) we define a Hilbert A-module $L^2(G,A,\tau)$ by taking the completion of $C_c(G,A,\tau)$ with respect to the A-valued inner product

$$\langle \xi,\eta\rangle_A := \xi^* * \eta(e) = \int_{G/N} \alpha_{s^{-1}}\big(\xi(s)^*\eta(s)\big)\,dsN.$$

The *regular representation*

$$\Lambda_A^{G,N} : C_c(G,A,\tau) \to \mathcal{L}_A(L^2(G,A,\tau));\;\; \Lambda_A^{G,N}(f)\xi = f * \xi$$

embeds $C_c(G,A,\tau)$ into the algebra of adjointable operators on $L^2(G,A,\tau)$ and then $A \rtimes_r (G,N) := \overline{\Lambda_A^{G,N}\big(C_c(G,A,\tau)\big)} \subseteq \mathcal{L}_A(L^2(G,A,\tau))$. If $N = \{e\}$ is trivial, then $\mathcal{L}_A(L^2(G,A))$ identifies naturally with $M(A \otimes \mathcal{K}(L^2(G)))$, and we recover the original definition of the regular representation Λ_A^G of (A,G,α) and of the reduced crossed product $A \rtimes_r G$ of A by G.

Remark 12.4. The analogue of Remark 12.3 **does not hold** in general for the reduced crossed products, i.e. $A \rtimes_r (G,N)$ is in general not a quotient of $A \rtimes_r G$. For example, if N is not amenable, the algebra $C_r^*(G/N) = \mathbb{C} \rtimes_{\mathrm{id},1,r} (G,N)$ is not a quotient of $C_r^*(G) = \mathbb{C} \rtimes_{\mathrm{id},r} G$ – at least not in a canonical way.

We are now coming back to the decomposition problem

Proposition 12.5 (Green)**.** *Let $\alpha : G \to \mathrm{Aut}(A)$ be an action, let N be a closed normal subgroup of G, and let (α^N,τ^N) be the decomposition twisted action of (G,N) on $A \rtimes N$. Then the map*

$$\Psi : C_c(G,A) \to C_c(G, C_c(N,A), \tau^N);\quad \Psi(f)(n) = \delta(s)f(ns) \tag{12.4}$$

extends to isomorphisms $A\rtimes G \cong (A\rtimes N)\rtimes(G,N)$ and $A\rtimes_r G \cong (A\rtimes_r N)\rtimes_r(G,N)$. In particular, if $A = \mathbb{C}$ we obtain isomorphisms $C^*(G) \cong C^*(N) \rtimes (G,N)$ and $C_r^*(G) \cong C_r^*(N) \rtimes_r (G,N)$. Under the isomorphism of the full crossed products, a representation $\pi \times U$ of $A \rtimes G$ corresponds to the representation $(\pi \times U|_N) \times U$ of $(A \rtimes N) \rtimes (G,N)$.

A similar result holds if we start with a twisted action of (G,M) on A with $M \subseteq N$ (see [52, Proposition 1] and [12]). Note that all results stated in §3 for ordinary crossed products have their complete analogues in the twisted case, where G/N plays the rôle of G. In particular, the full and reduced crossed products coincide if G/N is amenable. Indeed, we shall see in §12 below that there is a convenient way to extend results known for ordinary actions to the twisted case via Morita equivalence (see Theorem 13.4 below).

13. The twisted Morita category and the stabilization trick

As done for ordinary actions in §5 we may consider the *equivariant twisted Morita category* $\mathfrak{M}(G, N)$ (resp. the *compact twisted Morita category* $\mathfrak{M}_c(G, N)$) in which the objects are twisted systems (A, G, N, α, τ) and in which the morphism from (A, G, N, α, τ) and (B, G, N, β, σ) are given by morphisms $[E, \Phi, u]$ from (A, G, α) to (B, G, β) in $\mathfrak{M}(G)$ (resp. $\mathfrak{M}_c(G)$) which preserve the twists in the sense that

$$\Phi(\tau_n)\xi = u_n(\xi)\sigma_n \quad \text{for all } n \in N. \tag{13.1}$$

As for ordinary actions, the crossed product construction $A \rtimes_{(r)} (G, N)$ extend to a descent functor

$$\rtimes_{(r)} : \mathfrak{M}(G, N) \to \mathfrak{M}.$$

If $[E, \Phi, u] \in \mathrm{Mor}(G, N)$ is a morphism from (A, G, N, α, τ) to (B, G, N, β, σ), then the descent $[E \rtimes_{(r)} (G, N), \Phi \rtimes_{(r)} (G, N)]$ can be defined by setting $E \rtimes_{(r)} (G, N) := (E \rtimes G)/\big((E \rtimes G) \cdot I_{(r)}\big)$ with $I_{(r)} := \ker\big(B \rtimes G \to B \rtimes_{(r)} (G, N)\big)$. Alternatively, one can construct $E \rtimes_{(r)} G$ as the closure of $C_c(G, E, \sigma)$, the continuous E-valued functions ξ on G with compact support modulo N satisfying $\xi(ns) = \xi(s)\sigma_{n^{-1}}$ for $s \in G$, $n \in N$, with respect to the $B \rtimes_{(r)} (G, N)$-valued inner product given by

$$\langle \xi, \eta \rangle_{B \rtimes_{(r)} (G,N)}(t) = \int_{G/N} \langle \xi(s), u_s(\eta(s^{-1}t)) \rangle_B \, dsN$$

(compare with the formulas given in §5.4.

There is a natural inclusion functor $\inf : \mathfrak{M}(G/N) \to \mathfrak{M}(G, N)$ given as follows: If $(A, G/N, \alpha)$ is an action of G/N, we let $\inf \alpha : G \to \mathrm{Aut}(A)$ denote the inflation of α from G/N to G and we let $1_N : N \to U(A)$ denote the trivial homomorphism $1_N(s) = 1$. Then $(\inf \alpha, 1_N)$ is a twisted action of (G, N) on A and we set

$$\inf\big((A, G/N, \alpha)\big) := (A, G, N, \inf \alpha, 1_N).$$

Similarly, on morphisms we set $\inf\big([E, \Phi, u]\big) := [E, \Phi, \inf u]$, where $\inf u$ denotes the inflation of u from G/N to G. The dense subalgebra $C_c(G, A, 1_N)$ of the crossed product $A \rtimes_{(r)} (G, N)$ for $(\inf \alpha, 1_N)$ consists of functions which are constant on N-cosets and which have compact supports in G/N, hence it coincides with $C_c(G/N, A)$ (even as a $*$-algebra). The identification $C_c(G, A, 1_N) \cong C_c(G/N, A)$ extends to the crossed products, and we obtain canonical isomorphisms $A \rtimes_{(r)} G/N \cong A \rtimes_{(r)} (G, N)$. A similar observation can be made for the crossed products of morphism and we see that the inclusion $\inf : \mathfrak{M}(G/N) \to \mathfrak{M}(G, N)$ is compatible with the crossed product functor in the sense that the diagram

$$\begin{array}{ccc} \mathfrak{M}(G/N) & \xrightarrow{\inf} & \mathfrak{M}(G, N) \\ {\scriptstyle\rtimes}\big\downarrow & & \big\downarrow{\scriptstyle\rtimes} \\ \mathfrak{M} & = & \mathfrak{M} \end{array}$$

commutes.

In what follows next we want to see that every twisted action is Morita equivalent (and hence isomorphic in $\mathfrak{M}(G,N)$) to some inflated twisted action as above. This will allow us to pass to an untwisted system whenever a theory (like the theory of induced representations, or K-theory of crossed products, etc.) only depends on the Morita equivalence class of a given twisted action.

To do this, we first note that Green's imprimitivity theorem (see Theorem 6.4) extends easily to crossed products by twisted actions: If N is a closed normal subgroup of G such that $N \subseteq H$ for some closed subgroup H of G, and if (α, τ) is a twisted action of (H,N) on A, then we obtain a twisted action $(\operatorname{Ind}\alpha, \operatorname{Ind}\tau)$ of (G,N) on $\operatorname{Ind}_H^G(A,\alpha)$ by defining

$$(\operatorname{Ind}\tau_n f)(s) = \tau_{s^{-1}ns} f(s) \quad \text{for } f \in \operatorname{Ind} A, s \in G \text{ and } n \in N.$$

One can check that the twisting ideals $I_\tau \subseteq A \rtimes H$ and $I_{\operatorname{Ind}\tau} \subseteq \operatorname{Ind} A \rtimes G$ (see Remark 12.3) are linked via the Rieffel correspondence of the $\operatorname{Ind} A \rtimes G - A \rtimes H$ imprimitivity bimodule $X_H^G(A)$. Similarly, the kernels $I_{\tau,r} := \ker\big(A \rtimes H \to A \rtimes_r (H,N)\big)$ and $I_{\operatorname{Ind}\tau,r} := \ker\big(\operatorname{Ind} A \rtimes G \to \operatorname{Ind} A \rtimes_r (G,N)\big)$ are linked via the Rieffel correspondence (we refer to [52] and [70] for the details). Thus, from Proposition 5.4 it follows:

Theorem 13.1. *The quotient $Y_H^G(A) := X_H^G(A)/(X_H^G(A) \cdot I_\tau)$ (resp. $Y_H^G(A)_r := X_H^G(A)/(X_H^G(A) \cdot I_{\tau,r})$) becomes an $\operatorname{Ind}_H^G(A,\alpha) \rtimes (G,N) - A \rtimes (H,N)$ (resp. $\operatorname{Ind}_H^G(A,\alpha) \rtimes r(G,N) - A \rtimes_r (H,N)$) imprimitivity bimodule.*

Remark 13.2. (1) Alternatively, one can construct the modules $Y_H^G(A)$ and $Y_H^G(A)_r$ by taking completions of $Y_0(A) := C_c(G,A,\tau)$ with respect to suitable $C_c(G, \operatorname{Ind} A, \operatorname{Ind}\tau)$- and $C_c(N,A,\tau)$-valued inner products. The formulas are precisely those of (6.1) if we integrate over G/N and H/N, respectively (compare with the formula for convolution in $C_c(G,A,\tau)$ as given in §12).

(2) If we start with a twisted action (α,τ) of (G,N) on A and restrict this to (H,N), then the induced algebra $\operatorname{Ind}_H^G(A,\alpha)$ is isomorphic to $C_0(G/H,A) \cong C_0(G/H) \otimes A$ as in Remark 6.1. The isomorphism transforms the action $\operatorname{Ind}\alpha$ to the action $l \otimes \alpha : G \to \operatorname{Aut}(C_0(G/H,A))$, with $l : G \to \operatorname{Aut}(C_0(G/H))$ being left-translation action, and the twist $\operatorname{Ind}\tau$ is transformed to the twist $1 \otimes \tau : N \to U(C_0(G/H) \otimes A)$. Hence, in this setting, the above theorem provides Morita equivalences

$$A \rtimes_{(r)} (H,N) \sim_M C_0(G/H,A) \rtimes_{(r)} (G,N)$$

for the above described twisted action $(l \otimes \alpha, 1 \otimes \tau)$ of (G,N).

We want to use Theorem 13.1 to construct a functor

$$\mathcal{F} : \mathfrak{M}(G,N) \to \mathfrak{M}(G/N)$$

which, up to a natural equivalence, inverts the inflation functor $\inf : \mathfrak{M}(G/N) \to \mathfrak{M}(G,N)$. We start with the special case of the decomposition twisted actions (α^N, τ^N) of (G,N) on $A \rtimes N$ with respect to a given system (A,G,α) and a normal subgroup N of G (see §12 for the construction). Since A is a G-algebra, it follows from Remark 6.1 that $\operatorname{Ind}_N^G(A,\alpha)$ is isomorphic to $C_0(G/N,A)$ as a G-algebra. Let

$X_N^G(A)$ be Green's $C_0(G/N, A) \rtimes G - A \rtimes N$ imprimitivity bimodule. Since right translation of G/N on $C_0(G/N, A)$ commutes with $\operatorname{Ind}\alpha$, it induces an action

$$\beta^N : G/N \to \operatorname{Aut}\big(C_0(G/N, A) \rtimes G\big)$$

on the crossed product. For $s \in G$ and $\xi \in C_c(G, A) \subseteq X_N^G(A)$ let

$$u_s^N(\xi)(t) := \sqrt{\delta(s)}\alpha_s(\xi(ts)), \quad \xi \in C_c(G, A)$$

where $\delta(s) = \Delta_G(s)\Delta_{G/N}(s^{-1})$. This formula determines an action $u^N : G \to \operatorname{Aut}(X_H^G(A))$ such that $(X_N^G(A), u^N)$ becomes a (G, N)-equivariant $C_0(G/N, A) \rtimes G - A \rtimes N$ Morita equivalence with respect to the twisted actions $(\inf \beta^N, 1_N)$ and (α^N, τ^N), respectively. All these twisted actions pass to the quotients to give also a (G, N)-equivariant equivalence $(X_N^G(A)_r, u^N)$ for the reduced crossed products. Thus we get

Proposition 13.3 (cf. [30, Theorem 1]). *The decomposition action (α^N, τ^N) of (G, N) on $A \rtimes_{(r)} N$ is canonically Morita equivalent to the (untwisted) action β^N of G/N on $C_0(G/N, A) \rtimes_{(r)} G$ as described above.*

If one starts with an arbitrary twisted action (α, τ) of (G, N) on A, one checks that the twisting ideals $I_\tau \subseteq A \rtimes N$ and $I_{\operatorname{Ind}\tau} \subseteq C_0(G/N, A) \rtimes G$ are (G, N)-invariant and that the twisted action on $A \cong (A \rtimes N)/I_\tau$ (cf. Remark 12.3) induced from (α^N, τ^N) is equal to (α, τ). Hence, if β denotes the action of G/N on $C_0(G/N, A) \rtimes (G, N) \cong \big(C_0(G/N, A) \rtimes G\big)/I_{\operatorname{Ind}\tau}$ induced from β^N, then u^N factors through an action u of G on $Y_N^G(A) = X_N^G(A)/(X_N^G(A) \cdot I_\tau)$ such that $(Y_N^G(A), u)$ begomes a (G, N)-equivariant $C_0(G/N) \rtimes (G, N) - A$ Morita equivalence with respect to the twisted actions $(\inf \beta, 1_N)$ and (α, τ), respectively. Following the arguments given in [32] one can show that there is a functor $\mathcal{F} : \mathfrak{M}(G, N) \to \mathfrak{M}(G/N)$ given on objects by the assignment

$$(A, G, N, \alpha, \tau) \overset{\mathcal{F}}{\mapsto} (C_0(G/N, A) \rtimes (G, N),\ G/N,\ \beta)$$

(and a similar crossed-product construction on the morphisms) such that

Theorem 13.4 (cf. [30, Theorem 1] and [32, Theorem 4.1]). *The assignment*

$$(A, G, N, \alpha, \tau) \mapsto (Y_N^G(A), u)$$

is a natural equivalence between the identity functor on $\mathfrak{M}(G, N)$ and the functor $\inf \circ \mathcal{F} : \mathfrak{M}(G, N) \to \mathfrak{M}(G, N)$*, where* $\inf : \mathfrak{M}(G/N) \to \mathfrak{M}(G, N)$ *denotes the inflation functor. In particular, every twisted action of (G, N) is Morita equivalent to an ordinary action of G/N (viewed as a twisted action via inflation).*

Note that a first version of the above Theorem was obtained by Packer and Raeburn in the setting of Busby-Smith twisted actions ([96]). We therefore call it the *Packer-Raeburn stabilization trick*. It allows to extend results known for ordinary actions to the twisted case as soon as they are invariant under Morita equivalence. If A is separable and G is second countable, the algebra $B = C_0(G/N, A) \rtimes (G, N)$ is separable, too. Thus, it follows from Brown-Green-Rieffel theorem of [9] that A and

B are stably isomorphic (a direct isomorphism $B \cong A \otimes \mathcal{K}(L^2(G/N))$ is obtained in [53]). Hence, as a consequence of Theorem 13.4 we get

Corollary 13.5. *If G is second countable and and A is separable, then every twisted action of (G, N) on A is Morita equivalent to some action β of G/N on $A \otimes \mathcal{K}$.*

We want to discuss some further consequences of Theorem 13.4:

13.1. **Twisted Takesaki-Takai duality.** If (A, G, N, α, τ) is a twisted system with G/N abelian, then we define the dual action

$$\widehat{(\alpha, \tau)} : \widehat{G/N} \to \mathrm{Aut}\big(A \rtimes (G, N)\big)$$

as in the previous section by pointwise multiplying characters of G/N with functions in the dense subalgebra $C_c(G, A, \tau)$. Similarly, we can define actions of $\widehat{G/N}$ on (twisted) crossed products of Hilbert bimodules, so that taking dual actions gives a descent functor $\rtimes : \mathfrak{M}(G, N) \to \mathfrak{M}(\widehat{G/N})$. The Takesaki-Takai duality theorem shows that on $\mathfrak{M}(G/N) \subseteq \mathfrak{M}(G, N)$ this functor is inverted, up to a natural equivalence, by the functor $\rtimes : \mathfrak{M}(\widehat{G/N}) \to \mathfrak{M}(G/N)$. Using Theorem 13.4, this directly extends to the twisted case.

13.2. **Stability of exactness under group extensions.** Recall from §8 that a group is called exact if for every short exact sequence $0 \to I \to A \to A/I \to 0$ of G-algebras the resulting sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

of reduced crossed products is exact. We want to use Theorem 13.4 to give a proof of the following result of Kirchberg and S. Wassermann:

Theorem 13.6 (Kirchberg and S. Wassermann [70])**.** *Suppose that N is a closed normal subgroup of the locally compact group G such that N and G/N are exact. Then G is exact.*

The result will follow from

Lemma 13.7. *Suppose that N is a closed normal subgroup of G and that (X, u) is a (G, N)-equivariant Morita equivalence for the twisted actions (β, σ) and (α, τ) of G on B and A, respectively. Let $I \subseteq A$ be a (G, N)-invariant ideal of A, and let $J := \mathrm{Ind}^X I \subseteq B$ denote the ideal of B induced from I via X (which is a (G, N)-equivariant ideal of B).*

Then $J \rtimes_{(r)} (G, N)$ (resp. $(B/J) \rtimes_{(r)} (G, N)$) corresponds to $I \rtimes_{(r)} (G, N)$ (resp. $(A/I) \rtimes_{(r)} (G, N)$) under the Rieffel correspondence for $X \rtimes_{(r)} (G, N)$.

Proof. Let $Y := X \cdot I \subseteq X$. Then the closure $C_c(G, Y, \tau) \subseteq C_c(G, X, \tau)$ is a $B \rtimes_{(r)} (G, N) - A \rtimes_{(r)} (G, N)$ submodule of $X \rtimes_{(r)} (G, N)$ which corresponds to the ideals $J \rtimes_{(r)} (G, N)$ and $I \rtimes_{(r)} (G, N)$ under the Rieffel correspondence. For the quotients observe that the obvious quotient map $C_c(G, X, \tau) \to C_c(G, X/X \cdot I, \tau)$ extends to an imprimitivity bimodule quotient map $X \rtimes_{(r)} (G, N) \to (X/X \cdot I) \rtimes_{(r)} (G, N)$, whose kernel corresponds to the ideals

$K_B := \ker\big(B \rtimes_{(r)} (G,N) \to (B/J) \rtimes_{(r)} (G,N)\big)$ and $K_A := \ker\big(A \rtimes_{(r)} (G,N) \to (A/I) \rtimes_{(r)} (G,N)\big)$ und the Rieffel correspondence (see Remark 5.5). □

As a consequence we get

Lemma 13.8. *Suppose that N is a closed normal subgroup of G such that G/N is exact. Suppose further that $0 \to I \to A \to A/I \to 0$ is a short exact sequence of (G,N)-algebras. Then the sequence*

$$0 \to I \rtimes_r (G,N) \to A \rtimes_r (G,N) \to (A/I) \rtimes_r (G,N) \to 0$$

is exact.

Proof. By Theorem 13.4 there exists a system $(B, G/N, \beta)$ such that $(B, G, N, \inf\beta, 1_N)$ is Morita equivalent to the given twisted system (A, G, N, α, τ) via some equivalence (X, u). If I is a (G,N)-invariant ideal of A, let $J := \operatorname{Ind}^X I \subseteq B$. It follows then from Lemma 13.7 and the Rieffel correspondence, that

$$0 \to I \rtimes_r (G,N) \to A \rtimes_r (G,N) \to (A/I) \rtimes_r (G,N) \to 0$$

is exact if and only if

$$0 \to J \rtimes_r (G,N) \to B \rtimes_r (G,N) \to (B/J) \rtimes_r (G,N) \to 0$$

is exact. But the latter sequence is equal to the sequence

$$0 \to J \rtimes_r G/N \to B \rtimes_r G/N \to (B/J) \rtimes_r G/N \to 0,$$

which is exact since G/N is exact. □

Proof of Theorem 13.6. Suppose that $0 \to I \to A \to A/I \to 0$ is an exact sequence of G-algebras and consider the decomposition twisted action (α^N, τ^N) of (G,N) on $A \rtimes_r N$. Since N is exact, we the sequence

$$0 \to I \rtimes_r N \to A \rtimes_r N \to (A/I) \rtimes_r N \to 0$$

is a short exact sequence of (G,N)-algebras. Since G/N is exact, it follows therefore from Lemma 13.8 that

$$0 \to (I \rtimes_r N) \rtimes_r (G,N) \to (A \rtimes_r N) \rtimes_r (G,N) \to \big((A/I) \rtimes_r N\big) \rtimes_r (G,N) \to 0$$

is exact. But it follows from Proposition 12.5 that this sequence equals

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0.$$

□

13.3. **Induced representations of twisted crossed products.** Using Green's imprimitivity theorem for twisted systems, we can define induced representations and ideals for twisted crossed products $A \rtimes (G,N)$ as in the untwisted case, using the spaces $C_c(G, A, \tau)$ and $C_c(G, \operatorname{Ind} A, \operatorname{Ind}\tau)$ etc. (e.g. see [31, Chapter 1] for this approach). An alternative but equivalent way, as followed in Green's original paper [52] is to define induced representations via the untwisted crossed products: Suppose that (α, τ) is a twisted action of (G,N) on A and let $H \subseteq G$ be a closed subgroup of G such that $N \subseteq H$. Since $A \rtimes (H,N)$ is a quotient of $A \rtimes H$ we can regard every representation of $A \rtimes (H, N)$ as a representation of $A \rtimes H$. We can use the untwisted theory to induce the representation to $A \rtimes G$. But then we have to check that this representation factors through the quotient $A \rtimes (G, N)$ to have a satisfying theory. This has been done in [52, Corollary 5], but one can also obtain it as an easy consequence of Proposition 9.15: Let $I_\tau^N \subset A \rtimes N$ denote the twisting ideal for $(\alpha|_N, \tau)$. It is then clear from the definition of representations $\pi \times U$ of $A \rtimes H$ (resp. $A \rtimes G$) which preserve τ, that $\pi \times U$ preserves τ iff $\pi \times U|_N$ preserves τ as a representation of $A \rtimes N$. Hence, $\pi \times U$ is a representation of $A \rtimes (H, N)$ (resp. $A \rtimes (G, N)$) iff $I_\tau^N \subseteq \ker(\pi \times U|_N)$. Since I_τ^N is easily seen to be a G-invariant ideal of $A \rtimes N$, this property is preserved under induction by Proposition 9.15.

Induction of representations are invariant under passing to Morita equivalent systems. To be more precise: Suppose that (X, u) is a Morita equivalence for the systems (A, G, α) and (B, G, β). If H is a closed subgroup of G and $\pi \times U$ is a representation of $B \rtimes H$, then we get an equivalence

$$\mathrm{Ind}_H^G \left(\mathrm{Ind}^{X \rtimes H}(\pi \times U)\right) \cong \mathrm{Ind}^{X \rtimes G}\left(\mathrm{Ind}_H^G(\pi \times U)\right).$$

This result follows from an isomorphism of $A \rtimes G - B \rtimes H$ bimodules

$$X_H^G(A) \otimes_{A \times H} (XrtimesH) \cong (X \rtimes G) \otimes_{B \rtimes G} (X_H^G(B)),$$

which just means that the respective compositions in the Morita categories coincide. A similar result can be shown for the reduction of representations to subgroups. Both results will follow from linking algebra trick as introduced in [35, §4]:

14. Twisted group algebras, actions on $\mathcal{K}$ and Mackey's little group method

In this section we want to study crossed products of the form $\mathcal{K} \rtimes_{(r)} G$, where $\mathcal{K} = \mathcal{K}(H)$ is the algebra of compact operators on some Hilbert space H.

While there are only trivial actions of groups on the algebra $\mathbb{C}$ of complex numbers, there are usually many nontrivial twisted actions of pairs (G, N) on $\mathbb{C}$. However, in a certain sense they are all equivalent to twisted actions of the following type:

Example 14.1. Assume that $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$ is a central extension of the locally compact group G by the circle group $\mathbb{T}$. Let $\iota : \mathbb{T} \to \mathbb{T}; \iota(z) = z$ denote the identity character on G. Then (id, ι) is a twisted action of $(\tilde{G}, \mathbb{T})$ on $\mathbb{C}$. A (covariant) representations of the twisted system $(\mathbb{C}, \tilde{G}, \mathbb{T}, \mathrm{id}, \iota)$ consists of unitary representation U of $\tilde{G}$ satisfying $U_z = \iota_z \cdot 1$ for all $z \in \mathbb{T} \subseteq \tilde{G}$, i.e., of those representations of $\tilde{G}$ which restrict to a multiple of ι on the central subgroup $\mathbb{T}$ of $\tilde{G}$. Hence, the twisted crossed product $\mathbb{C} \rtimes (\tilde{G}, \iota)$ is the quotient of $C^*(\tilde{G})$ by the ideal $I_\iota = \cap\{\ker U : U \in \mathrm{Rep}(\tilde{G}) \text{ and } U|_{\mathbb{T}} = \iota \cdot 1\}$. Note that the isomorphism class of $\mathbb{C} \rtimes (\tilde{G}, \iota)$ only depends on the isomorphism class of the extension $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$.

If G is second countable[15], we can choose a Borel section $c: G \to \tilde{G}$ in the above extension, and we then obtain a Borel map $\omega: G \times G \to \mathbb{T}$ by

$$\omega(s,t) := c(s)c(t)c(st)^{-1} \in \mathbb{T}.$$

A short computation then shows that ω satisfies the cocycle conditions $\omega(s,e) = \omega(e,s) = 1$ and $\omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r)$ for all $s,t,r \in G$. Hence it is a 2-cocycle in $Z^2(G,\mathbb{T})$ of Moore's group cohomology with Borel cochains (see [87, 88, 89, 90]). The cohomology class $[\omega] \in H^2(G,\mathbb{T})$ then only depends on the isomorphism class of the given extension $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$.[16] Conversely, if $\omega: G \times G \to \mathbb{T}$ is any Borel 2-cocycle on G, let G_ω denote the cartesian product $G \times \mathbb{T}$ with multiplication given by

$$(s,z) \cdot (t,w) = (st, \omega(s,t)zw).$$

By [84,] there exists a unique locally compact topology on G_ω whose Borel structure coincides with the product Borel structure. Then G_ω is a central extension of G by $\mathbb{T}$ corresponding to ω (just consider the section $c: G \to G_\omega; c(s) = (s,1)$) and we obtain a complete classification of the (isomorphism ckasses of) central extensions of G by $\mathbb{T}$ in terms of $H^2(G,\mathbb{T})$. We then write $C^*_{(r)}(G,\omega) := \mathbb{C} \rtimes_{(r)} (G_\omega, \mathbb{T})$ for the corresponding full (resp. reduced) twisted crossed products, which we now call the *twisted group algebra of G corrsponding to ω.*

There is a canonical one-to-one correspondence between the (non-degenerate) covariant representations of the twisted system $(\mathbb{C}, G_\omega, \mathbb{T}, \mathrm{id}, \iota)$ on a Hilbert space H and the *projective ω-representations* of G on H, which are defined as Borel maps

$$V: G \to \mathcal{U}(H) \quad \text{satisfying} \quad V_s V_t = \omega(s,t) V_{st} \quad s,t \in G.$$

Indeed, if $\tilde{V}: G_\omega \to U(H)$ is a unitary representation of G_ω which restricts to a multiple of ι on $\mathbb{T}$, then $V_s := \tilde{V}(s,1)$ is the corresponding ω-representation of G.

A convenient alternative realization of the twisted group algebra $C^*(G,\omega)$ is obtained by taking a completion of the convolution algebra $L^1(G,\omega)$, where $L^1(G,\omega)$ denotes the algebra of all L^1-functions on G with convolution and involution given by

$$f * g(s) = \int_G f(t)g(t^{-1}s)\omega(t,t^{-1}s)\,dt \quad \text{and} \quad f^*(s) = \Delta_G(s^{-1})\overline{\omega(s,s^{-1})f(s^{-1})}.$$

One checks that the $*$-representations of $L^1(G,\omega)$ are given by integrating projective ω-representations and hence the corresponding C^*-norm for completing $L^1(G,\omega)$ to $C^*(G,\omega)$ is given by

$$\|f\|_{\max} = \sup\{\|V(f)\| : V \text{ is an } \omega\text{-representation of } G\}.$$

The map

$$\Phi: C_c(G_\omega, \mathbb{C}, \iota) \to L^1(G,\omega); \Phi(f)(s) := f(s,1)$$

[15] This assumptions is made to avoid measurability problems. With some extra care, much of the following discussion also works in the non-separable case (e.g. see [73])

[16] Two cocycles ω and ω' are in the same class in $H^2(G,\mathbb{T})$ iff they differ by a boundary $\partial f(s,t) := f(s)f(t)\overline{f(st)}$ of some Borel function $f: G \to \mathbb{T}$.

then extends to an isomorphism between the two pictures of $C^*(G,\omega)$.[17] Similarly, we can define the ω-regular representation L_ω of G on $L^2(G)$ by setting $\big(L_\omega(s)\xi\big)(t) = \omega(s,s^{-1}t)\xi(s^{-1}t)$, $\xi \in L^2(G)$, and then realize $C_r^*(G,\omega)$ as $\overline{\mathcal{L}_\omega\big(L^1(G,\omega)\big)} \subseteq \mathcal{B}(L^2(G))$.

We shall see in §14 below, that the theory of twisted group algebras for G is (Morita-) equivalent to the theory of crossed products $\mathcal{K}(H) \rtimes G$, where $\mathcal{K}(H)$ denotes the algebra of compact operators on a Hilbert space H.

Example 14.2. Twisted group algebras appear quite often in C^*-algebra theory. For instance the rational and irrational rotation algebras A_θ for $\theta \in [0,1)$ are isomorphic to the twisted group algebras $C^*(\mathbb{Z}^2,\omega_\theta)$ with $\omega_\theta\big((n,m),(k,l)\big) = e^{i2\pi\theta mk}$. Note that every cocycle on $\mathbb{Z}^2$ is equivalent to ω_θ for some $\theta \in [0,1)$. If $\theta = 0$ we simply get $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, the classical commutative 2-torus. For this reason the A_θ are often denoted as noncommutative 2-tori.

More generally, a *noncommutative n-torus* is a twisted group algebra $C^*(\mathbb{Z}^n,\omega)$ for some cohomology class $[\omega] \in H^2(\mathbb{Z}^n,\mathbb{T})$.

An extensive study of 2-cocycles on abelian groups is given by Kleppner in [73]. In particular, for $G = \mathbb{R}^n$, every cocycle is similar to a cocycle of the form $\omega(x,y) = e^{2\pi i\langle Ax,y\rangle}$, where A is a skew-symmetric real $n \times n$-matrix, and every cocycle of $\mathbb{Z}^n$ is similar to a restiction to $\mathbb{Z}^n$ of some cocycle on $\mathbb{R}^n$. The general structure of the twisted group algebras $C^*(G,\omega)$ for abelian G is studied extensively in [36] in the type I case and in [95] in the general case. If G is abelian, then the *symmetry group* S_ω of ω is defined by

$$S_\omega := \{s \in G : \omega(s,t) = \omega(t,s) \text{ for all } t \in G\}.$$

Then Poguntke shows that $C^*(G,\omega)$ is stably isomorphic to an algebra of the form $C_0(\widehat{S}_\omega) \otimes C^*(\mathbb{Z}^n,\sigma)$, where $C^*(\mathbb{Z}^n,\sigma)$ is some simple noncommutative n-torus.[18]

It follows from Theorems 10.10 and 10.13 that for understanding the ideal structure of $A \rtimes G$, it is necessary to understand the structure of $A_P \rtimes G_P$ for $P \in \mathrm{Prim}(A)$. In the special case $A = C_0(X)$, we saw in the previous section that this is the same as understanding the group algebras $C^*(G_x)$ for the stabilizers G_x, $x \in X$. In general, the problem becomes much more difficult. However, at least in the important special case where A is type I one can still give a quite satisfactory description of $A_P \rtimes G_P$ in terms of the stabilizers. If A is type I, we have $\widehat{A} \cong \mathrm{Prim}(A)$ via $\sigma \mapsto \ker\sigma$ and if $P = \ker\sigma$ for some $\sigma \in \widehat{A}$, then the simple subquotient A_P of A corresponding to P is isomorphic to $\mathcal{K}(H_\sigma)$. Thus, we have to understand the structure of the crossed products $\mathcal{K}(H_\sigma) \rtimes G_\sigma$, where G_σ denotes the stabilizer of $\sigma \in \widehat{A}$.

Hence, in what follows we shall always assume that G is a locally compact group acting on the algeba $\mathcal{K}(H)$ of compact operators on some Hilbert space H. In order to avoid measerability problems, we shall always assume that G is second countable

[17] Use the identity $\overline{\omega(t,t^{-1})}\omega(t^{-1},s)\omega(t,t^{-1}s) = 1$ in order to check that Φ preserves multiplication.

[18] Two C^*-algebras A and B are called *stably isomorphic* if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(l^2(\mathbb{N}))$. We refer to §5 below for a more detailed discussion

and that H is separable (for a discussion of the non-separable case we refer to [52, Theorem 18]). Moreover, after stabilization if necessary (which does not change the ideal and representation spaces), we may assume that H is infinite dimensional and then we write $\mathcal{K} := \mathcal{K}(H)$. Since every automorphism of $\mathcal{K}$ is given by conjugation with some unitary $U \in \mathcal{B}(H)$, it follows that the automorphism group of $\mathcal{K}$ is isomorphic (as topological groups) to the group $\mathcal{P}U := \mathcal{U}/\mathbb{T}\cdot 1$, where $\mathcal{U} = \mathcal{U}(H)$ denotes the group of unitary operators on H equipped with the strong operator topology. We choose a Borel map $V_\alpha : G \to \mathcal{P}U$ such that $\alpha_s = \operatorname{Ad} V_\alpha(s)$ for all $s \in G$. Since $V_\alpha(s)V_\alpha(t)$ and $V_\alpha(st)$ give the same automorphism of $\mathcal{K}$, they must differ by some element in $\mathbb{T}$, and hence there exists a Borel map $\omega_\alpha : G \times G \to \mathbb{T}$ such that

$$V_\alpha(s)V_\alpha(t) = \omega(s,t)V_\alpha(st) \quad \text{for all } s,t \in G.$$

Comparing $V_\alpha(st)V_\alpha(r)$ with $V_\alpha(s)V_\alpha(tr)$ then shows that ω_α satisfies the cocycle identity

$$\omega_\alpha(s,t)\omega_\alpha(st,r) = \omega_\alpha(s,tr)\omega_\alpha(t,r) \quad \text{for all } s,t,r \in G,$$

and $\omega_\alpha(e,t) = \omega_\alpha(t,e) = 1$ for all $t \in G$. Hence ω_α determines a class $[\omega_\alpha]$ in Moore's Borel cohomology group $H^2(G,\mathbb{T})$. We refer to [87, 89] for the definition and the general properties of this cohomology theory, but we mention here that two cocycles $\omega, \omega' \in Z^2(G,\mathbb{T})$ are cohomologuous iff they differ by some trivial cocycle $\partial f(s,t) := f(s)f(t)\overline{s(st)}$ for some Borel map $f : G \to \mathbb{T}$. It is then esay to check that the class $[\omega_\alpha]$ does not depend on the choice of V_α. The following result goes back to early work of Mackey (e.g. see [84, 85]).

If we define

$$\tilde{G} := \{(s,U) \in G \times \mathcal{U}(H) : \alpha_s = \operatorname{Ad}(U)\}$$

then we obtain a central extension

$$1 \longrightarrow \mathbb{T} \xrightarrow{z \mapsto (e, \bar{z}\cdot 1)} \tilde{G} \xrightarrow{(s,U)\mapsto s} G \longrightarrow 1,$$

and then a corresponding twisted system $(\mathbb{C}, \tilde{G}, \mathbb{T}, \mathrm{id}, \iota)$ as in Example 14.1, where $\iota(z) = z$ is the identity character on $\mathbb{T}$. Recall from Example 14.1 that

$$(\mathbb{C}, \tilde{G}, \mathbb{T}, \mathrm{id}, \iota)\widehat{\ } = \{V \in \widehat{\tilde{G}} : V|_{\mathbb{T}} = \iota \cdot 1_{H_V}\}.$$

Lemma 14.3. *Let $U : \tilde{G} \to \mathcal{U}(H)$ denote the second projection. Then the pair (H,U) is a $(\tilde{G},\mathbb{T})$-equivariant Morita equivalence between $(\mathcal{K}(H), G, \alpha)$ (viewed as a twisted $(\tilde{G},\mathbb{T})$-system via inflation) and $(\mathbb{C}, \tilde{G}, \mathbb{T}, \mathrm{id}, \iota)$. Therefore, induction via H gives a homeomorphism*

$$V \mapsto (\mathrm{id} \otimes 1_{H_V}, U \otimes V)$$

between $\{V \in \widehat{\tilde{G}} : V|_{\mathbb{T}} = \iota \cdot 1_{H_V}\}$ and $(\mathcal{K}(H) \rtimes G)\widehat{\ }$, where we remark that $U \otimes V$ is trivial on $\mathbb{T} \subseteq \tilde{G}$, and hence may be regarded as a unitary representation of G.

Proof. The proof follows directly from the definitions. ∎

Choose a Borel section $c : \mathcal{P}U \to \mathcal{U}$. If $\alpha : G \to \mathcal{P}U$ is a continuous homomorphism, let $V_\alpha := c \circ \alpha : G \to \mathcal{U}$. Since $V_\alpha(s)V_\alpha(t)$ and $V_\alpha(st)$ both implement the automorphism α_{st}, there exists a number $\omega_\alpha(s,t) \in \mathbb{T}$ with

$$\omega_\alpha(s,t) \cdot 1 = V_\alpha(st)V_\alpha(t)^* V_\alpha(s)^*.$$

A short computation shows that ω_α is a Borel 2-cocycle on G as in Example 14.1 and that V_α is an ω_α^{-1}-projective representation of G on H.

The class $[\omega_\alpha] \in H^2(G,\mathbb{T})$ only depends on α and it vanishes if and only if α is *unitary* in the sense that α is implemented by a strongly continuous homomorphism $V : G \to \mathcal{U}$. [19] Therefore, the class $[\omega_\alpha] \in H^2(G,\mathbb{T})$ is called the *Mackey obstruction for α being unitary.* An easy computation gives:

Lemma 14.4. *Let $\alpha : G \to \mathrm{Aut}(\mathcal{K})$, $V_\alpha : G \to \mathcal{U}$ and ω_α be as above. Let G_{ω_α} denote the central extension of G by $\mathbb{T}$ corresponding to ω_α as described in Example 14.1 and let $\iota : \mathbb{T} \to \mathbb{C}$ denote the inclusion. Let*

$$\tilde{V}_\alpha : G_{\omega_\alpha} \to \mathcal{U}(H); \quad \tilde{V}_\alpha(s,z) = \bar{z}V_\alpha(s).$$

Then $(H, \tilde{V}_\alpha)$ is a $(G_{\omega_\alpha}, \mathbb{T})$-equivariant Morita equivalence between the action α of $G \cong G_{\omega_\alpha}/\mathbb{T}$ on $\mathcal{K}$ and the twisted action (id, ι) of $(G_{\omega_\alpha}, \mathbb{T})$ on $\mathbb{C}$.

We refer to §5.4 for the definition of twisted equivariant Morita equivalences. Since Morita equivalent twisted systems have Morita equivalent full and reduced crossed products, it follows that $\mathcal{K} \rtimes_\alpha G$ is Morita equivalent to the twisted group algebra $C^*(G, \omega_\alpha)$ (and similarly for $\mathcal{K} \rtimes_r G$ and $C_r^*(G,\omega)$). Recall from Example 14.1 that there is a one-to-one correspondence between the representations of $C^*(G, \omega_\alpha)$ (or the covariant representations of $(\mathbb{C}, G_{\omega_\alpha}, \mathbb{T}, \mathrm{id}, \iota)$) and the projective ω_α-representations of G. Using the above lemma and induction of covariant representations via the bimodule $(H, \tilde{V}_\alpha)$ then gives:

Theorem 14.5. *Let $\alpha : G \to \mathrm{Aut}(\mathcal{K})$ be an action and let ω_α and $V : G \to \mathcal{U}(H)$ be as above. Then the assignment*

$$L \mapsto (\mathrm{id} \otimes 1, V_\alpha \otimes L)$$

gives a homeomorphic bijection between the (irreducible) ω_α-projective representations of G and the (irreducible) non-degenerate covariant representations of $(\mathcal{K}, G, \alpha)$.

Remark 14.6. **(1)** It is actually quite easy to give a direct isomorphism between $C^*(G,\omega_\alpha) \otimes \mathcal{K}$ and the crossed product $\mathcal{K} \rtimes_\alpha G$. If $V_\alpha : G \to U(H)$ is as above, then one easily checks that

$$\Phi : L^1(G, \omega_\alpha) \odot \mathcal{K} \to L^1(G, \mathcal{K}); \Phi(f \otimes k)(s) = f(s)kV_s^*.$$

is a $*$-homomorphism with dense range such that

$$(\mathrm{id} \otimes 1) \times (V_\alpha \otimes L)\big(\Phi(f \otimes k)\big) = L(f) \otimes k$$

[19] To see this one should use the fact that any measurable homomorphism between polish groups is automatically continuous by [89, Proposition 5].

for all $f \in L^1(G,\omega)$ and $k \in \mathcal{K}$, and hence the above theorem implies that Φ is isometric with respect to the C^*-norms. A similar argument also shows that $\mathcal{K} \rtimes_r G \cong C_r^*(G,\omega_\alpha) \otimes \mathcal{K}$.

(2) The separability assumptions made above are not really necessary: Indeed, if $\alpha : G \to \mathrm{Aut}(\mathcal{K}(H)) \cong \mathcal{P}U(H)$ is an action of any locally compact group on the algebra of compact operators on any Hilbert space H, then

$$\tilde{G} := \{(s,U) \in G \times \mathcal{U}(H) : \alpha_s = \mathrm{Ad}(U)\}$$

fits into the central extension

$$1 \longrightarrow \mathbb{T} \xrightarrow{z \mapsto (e, z\cdot 1)} \tilde{G} \xrightarrow{(s,U)\mapsto s} G \longrightarrow 1.$$

Combining the previous results (and using the identification $\widehat{A} \cong \mathrm{Prim}(A)$ if A is type I) with Theorem 10.10 now gives:

Theorem 14.7 (Mackey's little group method)**.** *Suppose that (A,G,α) is a smooth separable system such that A is type I. Let $S \subseteq \widehat{A}$ be a section for the quotient space $G\backslash\widehat{A}$ and for each $\pi \in S$ let $V_\pi : G \to \mathcal{U}(H_\pi)$ be a measurable map such that $\pi(\alpha_s(a)) = V_\pi(s)\pi(a)V_\pi(s)^*$ for all $a \in A$ and $s \in G$ (such map always exists). Let $\omega_\pi \in Z^2(G_\pi,\mathbb{T})$ be the 2-cocycle satisfying*

$$\omega_\pi(s,t)\cdot 1_{H_\pi} := V_\pi(st)V_\pi(t)^*V_\pi(s)^*.$$

Then

$$\mathrm{IND} : \cup_{\pi\in S} C^*(G_\pi,\omega_\pi)\widehat{\ } \to (A \rtimes G)\widehat{\ }; \mathrm{IND}(L) = \mathrm{ind}_{G_\pi}^G(\pi\otimes 1)\times(V_\pi \otimes L).$$

is a bijection, which restricts to homeomorphisms between $C^*(G_\pi,\omega_\pi)\widehat{\ }$ and it's image $(A_{G_\pi} \rtimes G)\widehat{\ }$ for each $\pi \in S$.

Remark 14.8. **(1)** If G is exact, then a similar result holds for the reduced crossed product $A \rtimes_r G$, if we also use the reduced twisted group algebras $C_r^*(G_\pi,\omega_\pi)$ of the stabilizers.

(b) If (A,G,α) is a type I smooth system which is not separable, then the action of G_π on $\mathcal{K}(H_\pi)$ induced from

Notice that the above result in particular applies to all systems (A,G,α) with A type I and G compact, since actions of compact groups on type I algebras are always smooth in the sense of Definition 10.6. Since the central extensions G_ω of a compact group G by $\mathbb{T}$ are compact, and since $C^*(G,\omega)$ is a quotient of $C^*(G_\omega)$ (see Example 14.1), it follows that the twisted group algebras $C^*(G,\omega)$ are direct sums of matrix algebras if G is compact. Using this, we easily get from Theorem 14.7:

Corollary 14.9. *Suppose that (A,G,α) is a separable system with A type I and G compact. Then $A \rtimes G$ is type I. If, moreover, A is CCR, then $A \rtimes G$ is CCR, too.*

Proof. Since the locally closed subset $(A_{G_\pi} \rtimes G)\widehat{\ }$ corresponding to some orbit $G(\pi) \subseteq \widehat{A}$ is homeomorphic (via Morita equivalence) to $(\mathcal{K}(H_\pi) \rtimes G)\widehat{\ } \cong C^*(G_\pi,\omega_\pi)\widehat{\ }$, it follows that $(A_{G(\pi)} \rtimes G)\widehat{\ }$ is a discrete set in the induced topology. This implies that all points in $(A \rtimes G)\widehat{\ }$ are locally closed. Moreover, if A is CCR, then the points in $\widehat{A}$ are closed. Since G is compact, it follows then that the G-orbits in $\widehat{A}$ are closed, too. But then the discrete set $(A_{G_\pi} \rtimes G)\widehat{}$ is closed in $(A \rtimes G)\widehat{}$, which implies that the points in $(A \rtimes G)\widehat{}$ are closed. $\square$

Notice that it is not very difficult to remove the separability assumption from the above result.

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