KK-theory as a triangulated category

Notes from the lectures by Ralf Meyer

Focused Semester on KK-Theory and its Applications
Münster 2009

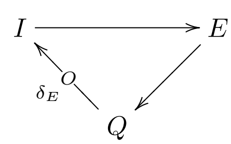
1 Lecture 1

Triangulated categories formalize the properties needed to do homotopy theory in a category, mainly the properties needed to manipulate long exact sequences. In addition, localization of functors allows the construction of interesting new functors. This is closely related to the Baum-Connes assembly map.

1.1 What additional structure does the category KK have?

Suspension automorphism Define $A[n] = C_0(\mathbf{R}^{-n}, A)$ for all $n \leq 0$. Note that by Bott periodicity $A[-2] = A$, so it makes sense to extend the definition of $A[n]$ to $\mathbf{Z}$ by defining $A[n] = A[-n]$ for $n > 0$.

Exact triangles Given an extension $I \rightarrowtail E \twoheadrightarrow Q$ with a completely positive contractive section, let $\delta_E \in \mathsf{KK}_1(Q, I)$ be the class of the extension. The diagram



where $\circ\!\!\rightarrow$ denotes a degree one map is called an *extension triangle*. An alternate notation is

$$Q[-1] \xrightarrow{\delta_E} I \longrightarrow E \longrightarrow Q.$$

An *exact triangle* is a diagram in KK isomorphic to an extension triangle. Roughly speaking, exact triangles are the sources of long exact sequences of KK-groups.

Remark 1.1. There are many other sources of exact triangles besides extensions.

Definition 1.2. A *triangulated category* is an additive category with a suspension automorphism and a class of exact triangles satisfying the axioms (TR0), (TR1), (TR2), (TR3), and (TR4). The definition of these axiom will appear in due course.

Example 1.3. The category KK with suspension and exact triangles as above is triangulated. ◊

Warning 1.4. The $\mathbf{Z}/2$-graded KK-category is not a triangulated category. ■

1.2 Cones and cylinders

Definition 1.5. Let $f\colon A \to B$ be a $*$-homomorphism. Then the cone of f is $\operatorname{cone}(f) = \Big\{(a,b) \in A \oplus C_0\big((0,1],B\big) \;\Big|\; b(1) = f(a)\Big\}$, while the cylinder of f is $\operatorname{cyl}(f) = \Big\{(a,b) \in A \oplus C_0\big([0,1],B\big) \;\Big|\; b(1) = f(a)\Big\}$. Note that any $*$-homomorphism $f\colon E \to Q$ gives rise to an extension triangle

$$Q[-1] \longrightarrow \operatorname{cone}(f) \rightarrowtail \operatorname{cyl}(f) \twoheadrightarrow Q$$

and that the cone of $\mathrm{id}_A\colon A \to A$ is the cone of A.

Definition 1.6. Let $f\colon A \to B$ be a $*$-homomorphism. Then there is an extension $B[-1] \rightarrowtail \operatorname{cone}(f) \twoheadrightarrow A$ with a completely positive contractive section. The class of this extension in $\mathsf{KK}_1(A, B[-1])$ turns out to be $f[-1]\colon A[-1] \to B[-1]$, so this gives a extension triangle

$$A[-1] \xrightarrow{f[-1]} B[-1] \longrightarrow \operatorname{cone}(f) \longrightarrow A.$$

Such an extension triangle is called an *mapping cone triangle.*

Warning 1.7. It is possible that in Definition 1.6, the class of the extension in $\mathsf{KK}_1(A, B[-1])$ should be $-f[-1]$. This sign error is due to the fact that $\operatorname{cone}(f)' = \Big\{(a,b) \in A \oplus C_0\big([0,1),B\big) \;\Big|\; b(0) = f(a)\Big\}$, is an alternate definition of the mapping cone. ■

Lemma 1.8. *A triangle is exact if and only if it is isomorphic to a mapping cone triangle.*

Proof. Assume $I \rightarrowtail E \twoheadrightarrow Q$ is an extension with a completely positive contractive section, and consider the diagram

$$\begin{array}{ccccccc}
Q[-1] & & I & \rightarrowtail & E & \overset{f}{\twoheadrightarrow} & Q \\
\| & & \downarrow{\scriptstyle\alpha} & & \| & & \| \\
Q[-1] & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & E & \overset{f}{\longrightarrow} & Q.
\end{array}$$

The proof of excision in KK shows that α exists, and that α is invertible in KK. Moreover, $\delta_E\colon Q[-1] \to I$ makes the resulting diagram commutative, so exact triangles are isomorphic to mapping cone triangles.

For the converse consider the diagram

$$\begin{array}{ccccccc}
Q[-1] & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & E & \overset{f}{\longrightarrow} & Q \\
\| & & \| & & & & \| \\
Q[-1] & \longrightarrow & \operatorname{cone}(f) & \rightarrowtail & \operatorname{cyl}(f) & \twoheadrightarrow & Q
\end{array}$$

and the definition of $\operatorname{cyl}(f)$. □

Remark 1.9. By the previous lemma, it is clear that one can define exact triangles in terms of mapping cone triangles. It is worth mentioning that by using this approach, one can prove excision with Puppe sequences. ■

1.3 Properties of exact triangles - The axioms (TR0), (TR1), (TR2), and (TR3)

Definition 1.10 ((TR0))**.** Triangles isomorphic to exact ones are exact and

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is exact. The last part can be reformulated as

$$0 \longrightarrow A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0$$

is exact.

Definition 1.11 ((TR1))**.** Any morphism in the category is part of an exact triangle, i.e. for any $f\colon A \to B$, there is an exact triangle

$$B[-1] \longrightarrow C \longrightarrow A \xrightarrow{f} B.$$

KK *satisfies (TR1).* By using the Cuntz picture, one can represent $f \in \mathsf{KK}_0(A, B)$ by a $*$-homomorphism $\tilde{f}\colon qA \to B \otimes \mathrm{K}$. Then form the diagram

$$\begin{array}{ccccccc}
(B \otimes \mathrm{K})[-1] & \longrightarrow & \mathrm{cone}(\tilde{f}) & \longrightarrow & qA & \xrightarrow{\tilde{f}} & B \otimes \mathrm{K} \\
\downarrow \sim & & \| & & \downarrow \sim & & \downarrow \sim \\
B[-1] & \longrightarrow & \mathrm{cone}(\tilde{f}) & \longrightarrow & A & \longrightarrow & B
\end{array}$$

where $\sim$ denotes KK-equivalence.

Another point of view is to note that $f \in \mathsf{KK}_0(A, B) \cong \mathsf{KK}_1\big(C_0(\mathbf{R}, A), B\big)$ and then look at $B[-1] \xrightarrow{f[-1]} A[-1] \longrightarrow E \longrightarrow B$. □

Definition 1.12 ((TR2))**.** The triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is exact if and only if the triangle

$$B \xrightarrow{-v} C \xrightarrow{-w} A[1] \xrightarrow{-u[1]} B[1]$$

is exact.

Remark 1.13. In the last part of (TR2), it is only required that an odd number of the maps change sing. Thus it is enough that one of maps has a change of sign. ■

KK *satisfies (TR2).* Given $f\colon A \to B$, we have extension triangle

$$\begin{array}{ccccccccc}
A[-1] & \xrightarrow{f[-1]} & B[-1] & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & A & \xrightarrow{f} & B \\
 & & \| & & \| & & \downarrow \sim & \ominus & \| \\
 & & B[-1] & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & \mathrm{cyl}(f) & \longrightarrow & B
\end{array}$$

where the last triangle commutes up to sign. □

Definition 1.14 ((TR3))**.** Given an commutative diagram of with exact rows

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle \alpha[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

there exist a morphism $\gamma\colon C \to C'$ such that the resulting diagram is commutative.

KK *satisfies (TR3).* In a special case, this is the functorality of the mapping cone. For the general case one can use the Cuntz picture. ☐

1.4 General facts about triangulated categories

For any category D and A, B objects in D, one denote by $\mathsf{D}(A, B)$ the group of morphisms from A to B.

Proposition 1.15. *Let* T *be a triangulated category,* D *an object of* T *and*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

an exact triangle. Then there is an induced long exact sequence

$$\longrightarrow \mathsf{T}(D, A[n]) \longrightarrow \mathsf{T}(D, B[n]) \longrightarrow \mathsf{T}(D, C[n]) \longrightarrow \mathsf{T}(D, A[n+1]) \longrightarrow$$

and an induced dual long exact sequence

$$\longrightarrow \mathsf{T}(A[n+1], D) \longrightarrow \mathsf{T}(C[n], D) \longrightarrow \mathsf{T}(B[n], D) \longrightarrow \mathsf{T}(A[n], D) \longrightarrow \cdot$$

Proof. Since the arguments for the two long exact sequences are similar, we only show the first part. The proof of this follows a similar argument as the one for KK. Note that it suffices to show exactness at one point in the long exact sequence. By using (TR0) and (TR3) one can show that the composition of any two consecutive morphisms in an exact triangle is zero, and this shows that "im $\subset$ ker" in the long exact sequence. To show "ker $\subset$ im" consider

$$\begin{array}{ccccccc} D & \longrightarrow & 0 & \longrightarrow & D[1] & \xrightarrow{-\mathrm{id}_{D[1]}} & D[1] \\ \downarrow & & \downarrow & & & & \downarrow \\ B & \longrightarrow & C & \longrightarrow & A[1] & \longrightarrow & B[1] \end{array}$$

and use (TR3). ☐

Definition 1.16. A functor from a triangulated category is called *(co)-homological* if it creates long exact sequences from exact triangles as in Proposition 1.15.

Theorem 1.17. *If*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

is a morphism of exact triangles in T *with* α *and* β *invertible, then* γ *is invertible.*

Proof. Observe that for all objects D in T, the morphisms $\mathsf{T}(D,\alpha)\colon \mathsf{T}(D,A) \to \mathsf{T}(D,A')$ and $\mathsf{T}(D,\beta)\colon \mathsf{T}(D,B) \to \mathsf{T}(D,B')$ are invertible. By the Yoneda lemma it suffices to prove that $\mathsf{T}(D,\gamma)$ is invertible for all D in T. This follows from the 5-lemma applied to the diagram

$$\begin{array}{ccccccc} \mathsf{T}(D,A) & \longrightarrow & \mathsf{T}(D,B) & \longrightarrow & \mathsf{T}(D,C) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \mathsf{T}(D,A') & \longrightarrow & \mathsf{T}(D,B') & \longrightarrow & \mathsf{T}(D,C') & \longrightarrow & \cdots \end{array}$$

where "$\cdots$" denotes the continuing long exact sequence. □

Corollary 1.18 (Uniquness of mapping cones)**.** *The exact triangle*

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

containing $f\colon A \to B$ is unique up to isomorphism.

Warning 1.19*.* The isomorphism in Corollary 1.18 is not a natural isomorphism. A "workaround" for this is to use model categories. ■

Exercise 1 For all objects A, B in a triangulated category the diagram

$$A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B \xrightarrow{0} A[1]$$

is an exact triangle.

Exercises 2 If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is an exact triangle with $w = 0$, then it is isomorphic to the exact triangle in Exercise 1.

Exercises 3 The diagram $A \xrightarrow{f} B \longrightarrow 0 \longrightarrow A[1]$ is an exact triangle if and only if f is invertible.

Exercises 4 The morphism f is invertible if and only if $\operatorname{cone}(f) = 0$.

1.5 Other kinds of exact sequences

1.5.1 Mayer-Vietoris

Theorem 1.20. *If f_1, f_2 have completely positive contractive section, then there is an exact triangle*

$$A_1 \oplus_B A_2 \longrightarrow A_1 \oplus A_2 \longrightarrow B \longrightarrow (A_1 \oplus_B A_2)[1]$$

where $A_1 \oplus_B A_2$ is the pullback of $\begin{array}{ccc} & & A_2 \\ & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & B \end{array}$.

Proof. Let $H = \Big\{(a_1, a_2, b) \in A_1 \oplus A_2 \oplus C\big([0,1], B\big) \;\Big|\; f_1(a_1) = b(0), f_2(a_2) = b(1)\Big\}$, i.e. H is the homotopy pullback, and use the fact that $A_1 \oplus_B A_2 \to H$ is an KK-equivalence. □

1.5.2 Crossed products

Let $\alpha\colon A \to A$ be an automorphism, and define the mapping torus of α by $T_\alpha = \Big\{ f \in C\big([0,1],A\big) \;\Big|\; f(1) = a\big(f(0)\big) \Big\}$. By considering the two exact triangles $A \xrightarrow{\mathrm{id}_A - \alpha} A \xrightarrow{\iota_\alpha} A \rtimes_\alpha \mathbf{Z} \longrightarrow A[1]$ and $T_\alpha \longrightarrow A \xrightarrow{\mathrm{id}_A - \alpha} A \longrightarrow T_\alpha[1]$ one get that $A \rtimes_\alpha \mathbf{Z}$ and $T_\alpha[1]$ are KK-equivalent.

1.5.3 Homotopy colimit

Consider the inductive system $(A_n, \alpha_n\colon A_n \to A_{n+1})$ of C*-algebras. The homotopy colimit of this system, $\mathrm{hocolim}(A_n, \alpha_n)$, is the mapping cone of $\mathrm{id} - S\colon \bigoplus_{n=0}^\infty A_n \to \bigoplus_{n=0}^\infty A_n$ where S is the shift map, $(a_i)_{i=0}^\infty \mapsto (a'_i)_{i=0}^\infty$ with $a'_0 = 0$ and $a'_i = \alpha_{i-1}(a_{i-1})$ for $i > 0$.

Note that $\operatorname{coker} S = \varinjlim(A_n, \alpha_n)$. In some nice cases we have that $\varinjlim(A_n, \alpha_n)$ is KK-equivalent to the homotopy colimit.

1.5.4 Observation

We have replaced questions about exactness of sequences with questions about isomorphisms.

2 Lecture 2

We have seen that triangulated categories are efficient for manipulating a single exact sequence. But this is limited to one dimensional diagrams. For $\mathbf{Z}^2$-actions things get more complicated. One way to proceed is step-by-step, i.e. using the Pimsner-Voiculescu sequence twice. One can then ask: "Is there some structure behind such iterative constructions?" and "Can we do homological algebra in triangulated categories?". The answer is to both questions is "Yes".

2.1 Motivation and definitions

We are motivated by the following picture:

$$\begin{array}{ccc}
\text{Triangulated category } \mathsf{T} & \xrightarrow{\text{Homological functor F}} & \text{Abelian category } \mathsf{C} \\
\cup\!| & & \cup\!| \\
\text{Projective objects in } \mathsf{T} & & \text{Projective objects in } \mathsf{C}
\end{array}$$

where in nice cases F gives an equivalence on the projective objects (e.g. there is an adjoint $\mathrm{F}^\dagger$ from the projectives in C to the projectives in T. Moreover, by taking projective resolutions, one can get more information about T. For instance, if all projective resolutions have length one, we obtain the Universal Coefficient Theorem (UCT).

Henceforth T will be a triangulated category, C an Abelian category with suspension (i.e. an automorphism $\mathsf{C} \to \mathsf{C}$), and $\mathrm{F}\colon \mathsf{T} \to \mathsf{C}$ a homological functor that commutes with suspensions.

Warning 2.1. The functor $\mathsf{K}_*\colon \mathsf{KK} \to \mathsf{Ab}^{\mathbf{Z}/2}$ satisfies the requirement for functors F, while $\mathsf{K}_0\colon \mathsf{KK} \to \mathsf{Ab}$ does not. Here the suspension in $\mathsf{Ab}^{\mathbf{Z}/2}$ maps (G_0, G_1) to (G_1, G_0). ■

Definition 2.2. A morphism $f \in \mathsf{T}(A, B)$ is F-*monic* if $\mathrm{F}(f)$ is monic, F-*epic* if $\mathrm{F}(f)$ is epic, F-*phantom* if $\mathrm{F}(f) = 0$, and an F-*equivalence* if $\mathrm{F}(f)$ is invertible. A chain complex $C^\bullet\colon \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$ is F-*exact* if $\mathrm{F}(C^\bullet)$ is exact. An object A of T is F-*contractible* if $\mathrm{F}(A) = 0$. A resolution $0 \leftarrow A \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots$ in T is a F-*resolution* if it is F-exact.

Lemma 2.3. *Let* $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ *be an exact triangle in* T. *The following are equivalent:*

1. u *is* F-*monic.*
2. w *is* F-*phantom.*
3. v *is* F-*epic.*
4. $\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0 \longrightarrow \cdots$ *is* F-*exact.*

Proof. Use the long exact sequence

$$\cdots \longrightarrow \mathrm{F}_{n+1}(C) \xrightarrow{\mathrm{F}_{n+1}(w)} \mathrm{F}_n(A) \xrightarrow{\mathrm{F}_n(u)} \mathrm{F}_n(B) \xrightarrow{\mathrm{F}_n(v)} \mathrm{F}_n(C) \longrightarrow \cdots$$

to show (1) ⇒ (2) ⇒ (3) ⇒ (1). Then use the sequence to show that (1), (2, and (3) are equivalent to (4). □

Exercise 5 A morphism f is an F-equivalence if and only if the object $\mathrm{cone}(f)$ is F-contractible.

Exercise 6 The F-phantom morphisms determine the F-monic morphisms, the F-epic morphism, the F-equivalences, and the F-contractible objects.

Exercise 7 (hard) The F-phantom maps determine F-exact chain complexes.

Remark 2.4. Observe that the F-phantom maps form an ideal and are closed under suspensions. ■

Definition 2.5. Let D be an Abelian category. A homological functor $\mathrm{G}\colon \mathsf{T} \to \mathsf{D}$ is called F-*exact* if it maps F-exact chain complexes to exact chain complexes.

Exercise 8 A homological functor G is F-exact if and only if $\mathrm{F}(f) = 0$ for each F-phantom map f.

Definition 2.6. An object A of T is F-*projective* if the functor $\mathsf{T}(A, \_)$ is F-exact. Similarly, the object A is F-*injective* if the functor $\mathsf{T}(\_, A)$ is F-exact.

Example 2.7. In KK, $\mathbf{C}$ is K_*-projective since $\mathsf{KK}_*(\mathbf{C}, B) \cong \mathsf{K}_*(B)$. ◊

Remark 2.8. There is a general fact that direct sums, suspensions and retracts of F-projectives remain F-projective. ■

2.2 Two digressions

1. Let $\mathrm{Ho}(\mathsf{T})$ be the homotopy category of chain complexes of T. Then $\mathrm{Ho}(\mathsf{T})$ is triangulated since T is additive. Moreover the F-exact chain complexes form a thick subcategory. Localizing at this thick subcategory yields a derived category, $\mathrm{Der}(\mathsf{T}, \mathrm{F})$. The derived category $\mathrm{Der}(\mathsf{T}, \mathrm{F})$ has a canonical t-structure because we may define F-exactness of chain complexes in a fixed degree. The heart of this t-structure is an Abelian category C, and the functor $\mathsf{T} \to \mathrm{Ho}(\mathsf{T})$ gives a homological functor $\tilde{\mathrm{F}}\colon \mathsf{T} \to \mathsf{C}$ with the same kernel as F.

2. Homotopy fixed points can be defined for any $\mathbf{Z}$-action on a object of a triangulated category T. This fail for $\mathbf{Z}/2$ or $\mathbf{Z}^2$ actions ("the classifying space of G should be one dimensional"). To work with such thing one need $\mathsf{KK}^{\mathbf{Z}/2}$ or $\mathsf{KK}^{\mathbf{Z}^2}$ respectively.

2.3 The Universal Coefficient Theorem

From Example 2.7 and Remark 2.8 it is clear that for countable sets I_0, I_1, the object $\big(\bigoplus_{I_0} \mathbf{C}\big) \oplus \big(\bigoplus_{I_1} C_0(\mathbf{R})\big)$ in KK is K_*-exact. The next theorem shows that there are no more K_*-projective objects in KK than these. But first a tiny bit of notation: $\mathsf{Ab}_c^{\mathbf{Z}/2}$ is the category of countable $\mathbf{Z}/2$-graded Abelian groups.

Theorem 2.9. *Define* $\mathsf{K}_*^\dagger\colon \big\{$*Countable free* $\mathbf{Z}/2$*-graded Abelian group*$\big\} \to \mathsf{KK}$ *by* $\big(\bigoplus_{I_0} \mathbf{Z}[0]\big) \oplus \big(\bigoplus_{I_1} \mathbf{Z}[1]\big) \mapsto \big(\bigoplus_{I_0} \mathbf{C}\big) \oplus \big(\bigoplus_{I_1} C_0(\mathbf{R})\big)$. *Then* $\mathsf{K}_*^\dagger$ *is a functor,* $\mathsf{K}_*\big(\mathsf{K}_*^\dagger(H)\big) = H$, $\mathsf{KK}\big(\mathsf{K}_*^\dagger(H), B\big) = \mathrm{Hom}_{\mathbf{Z}/2}\big(H, \mathsf{K}_*(B)\big)$, *and* $\mathsf{K}_*^\dagger$ *sets up an equivalence between the projective objects in* $\mathsf{Ab}_c^{\mathbf{Z}/2}$ *and the* K_**-projective objects in* KK.

Now let A be a separable C*-algebra. Then $\mathsf{K}_*(A)$ is a countable $\mathbf{Z}/(2)$-graded Abelian group, so we have the following free resolution

$$0 \longleftarrow \mathsf{K}_*(A) \xleftarrow{\epsilon} H_0 \xleftarrow{d} H_1 \longleftarrow 0 \tag{A}$$

with both H_0 and H_1 countable. By applying $\mathsf{K}_*^\dagger$ to this resolution and defining $\epsilon^\dagger$ in a suitable way (this can be done by Theorem 3.1) we obtain

$$0 \longleftarrow A \xleftarrow{\epsilon^\dagger} \mathsf{K}_*^\dagger(H_0) \xleftarrow{\mathsf{K}_*^\dagger(d)} \mathsf{K}_*^\dagger(H_1) \longleftarrow 0 \tag{B}$$

with $\epsilon^\dagger \circ \mathsf{K}_*^\dagger(d) = 0$. Note that one obtain the resolution (A) by applying K_* to (B). Furthermore, if $\epsilon^\dagger$ splits then A is isomorphic to a retreat of $\mathsf{K}_*^\dagger(H_0)$. In this case there is an idempotent $p\colon H_0 \to H_0$ such that $A \cong \mathrm{range}\big(\mathsf{K}_*^\dagger(p)\big) \cong \mathsf{K}_*^\dagger\big(\mathrm{range}(p)\big)$.

Recall that $\mathsf{KK}(\_, A)$ is a cohomological functor. From this, the fact that $\epsilon^\dagger \circ \mathsf{K}_*^\dagger(d) = 0$ in the sequence (B), and the mapping cone triangle of $\mathsf{K}_*^\dagger(d)$,

$$\mathsf{K}_*^\dagger(H_1) \xrightarrow{\mathsf{K}_*^\dagger(d)} \mathsf{K}_*^\dagger(H_0) \xrightarrow{d'} \mathrm{cone}(\mathsf{K}_*^\dagger(d)) \longrightarrow \mathsf{K}_*^\dagger(H_1)[1],$$

it follows that there is a map $\phi\colon \mathrm{cone}(\mathsf{K}_*^\dagger(d)) \to A$ such that $\phi \circ d' = \epsilon^\dagger$. It can be shown that $\mathsf{K}_*(\phi)$ is an isomorphism. Thus there is a 6-term exact sequence

$$\begin{array}{ccccc} \mathsf{KK}_0(\tilde{A}, B) & \longrightarrow & \mathsf{KK}_0(\mathsf{K}_*^\dagger(H_0), B) & \longrightarrow & \mathsf{KK}_0(\mathsf{K}_*^\dagger(H_1), B) \\ \uparrow & & & & \downarrow \\ \mathsf{KK}_1(\mathsf{K}_*^\dagger(H_1), B) & \longleftarrow & \mathsf{KK}_1(\mathsf{K}_*^\dagger(H_0), B) & \longleftarrow & \mathsf{KK}_1(\tilde{A}, B) \end{array}$$

where $\mathsf{K}_*(A) = \mathsf{K}_*(\tilde{A})$. Using Theorem 2.9 one can replace $\mathsf{KK}_0(\mathsf{K}_*^\dagger(H_0), B) \to \mathsf{KK}_0(\mathsf{K}_*^\dagger(H_0), B)$ with $\mathrm{Hom}\big(H_0, \mathsf{K}_*(B)\big) \to \mathrm{Hom}\big(H_1, \mathsf{K}_*(B)\big)$. From the above we get the Universal Coefficient Theorem for KK:

Theorem 2.10 (The Universal Coefficient Theorem (UCT))**.** *For* A, B *objects in* KK *we have*

$$\mathrm{Ext}\big(\mathsf{K}_*(A), \mathsf{K}_*(B)\big) \rightarrowtail \mathsf{KK}_0(\tilde{A}, B) \twoheadrightarrow \mathrm{Hom}\big(\mathsf{K}_*(A), \mathsf{K}_*(B)\big)$$

for some $\tilde{A}$ *with* $\mathsf{K}_*(A) = \mathsf{K}_*(\tilde{A})$.

Definition 2.11. The *UCT class* is the class of objects A in KK such that for objects $\tilde{A}$ as in Theorem 2.10, one has $\tilde{A} \cong A$.

Exercise 9 Let A_1, A_2 belong to the UCT class. If $\mathsf{K}_*(A_1) \cong \mathsf{K}_*(A_2)$, then $A_1 \cong A_2$. Thus any isomorphisms $\mathsf{K}_*(A_1) \to \mathsf{K}_*(A_2)$ lifts to an KK-equivalence $A_1 \to A_2$.

Remark 2.12. The Universal Coefficient Theorem followed from the fact the K_*-projective resolutions in KK had length one and that we had an equivalence $\mathsf{K}_*^\dagger$ from the projectives in $\mathsf{Ab}_c^{\mathbf{Z}/2}$ to the K_*-projectives in KK.

2.4 Another example

For objects (A, α) in $\mathsf{KK}^{\mathbf{Z}}$ define F' by

$$\mathrm{F}'(A, \alpha) = \big\{\mathsf{K}_*(A) \text{ with } \mathbf{Z}\text{-action induced by } \alpha\big\}.$$

Then F is a functor from $\mathsf{KK}^{\mathbf{Z}}$ to the category of countable $\mathbf{Z}/2$-graded $\mathbf{Z}[\mathbf{Z}]$-modules. The only thing different from the previous section is that there are F'-projective resolutions of length two. Consequentially, there is no Universal Coefficient Theorem in this case.

In order to get an Universal Coefficient Theorem for $\mathsf{KK}^{\mathbf{Z}}$ we have to look at other functors. So consider $\mathrm{F}\colon \mathsf{KK}^{\mathbf{Z}} \to \mathsf{KK}[Z]$ defined by

$$\mathrm{F}(A, \alpha) = \big\{A \in \mathsf{KK} \text{ with } \alpha\colon \mathbf{Z} \to \mathsf{KK}(A, A)\big\}.$$

In this case there is a correspondence between the projectives obtained by letting $\mathrm{F}^\dagger(A \otimes \mathbf{Z}[\mathbf{Z}]) = C_0(\mathbf{Z}) \otimes A$. From the resolution

$$0 \longleftarrow \mathsf{K}_*(A) \longleftarrow A \otimes \mathbf{Z}[\mathbf{Z}] \longleftarrow A \otimes \mathbf{Z}[\mathbf{Z}] \longleftarrow 0$$

we obtain length one F-projective resolutions in $\mathsf{KK}^{\mathbf{Z}}$. In this case the resulting Universal Coefficient Theorem is the Pimsner-Voiculescu sequence comparing $\mathsf{KK}^{\mathbf{Z}}(A, B)$ to $\mathsf{KK}(A, B)$.

3 Lecture 3

Recall that for KK we had

$$\begin{array}{ccc}
\mathsf{KK} & \xrightarrow{\mathsf{K}_*} & \mathsf{Ab}_c^{\mathbf{Z}/2} \\
\cup\!\!\mid & & \cup\!\!\mid \\
\mathscr{P} & \xleftarrow[\sim]{\mathsf{K}_*^\dagger} & \mathrm{Free}\mathsf{Ab}_c^{\mathbf{Z}/2}
\end{array}$$

where $\mathsf{Ab}_c^{\mathbf{Z}/2}$ are the category of countable $\mathbf{Z}/2$-graded Abelian groups and $\mathscr{P}$ are the K_*-projective objects in KK. The functor $\mathsf{K}_*^\dagger$ was an adjoint to K_* on the projective objects, and from the projective resolution $0 \leftarrow \mathsf{K}_*(A) \leftarrow H_0 \leftarrow H_1 \leftarrow 0$ we obtained an exact triangle $\mathsf{K}_*^\dagger(H_1) \to \mathsf{K}_*^\dagger(H_0) \to \tilde{A} \to \mathsf{K}_*^\dagger(H_1)[1]$. This gave us the Universal Coefficient Theorem by applying $\mathrm{Hom}(\_, B)$. Another homological functor we could have used is $\mathsf{K}_*(\_ \otimes B)$. By applying this functor to the exact triangle, and rewriting the resulting long exact sequence, the result would have been some kind of Künneth formula. Now, this all hinged on the ability to lift a projective resolution in $\mathsf{Ab}_c^{\mathbf{Z}/2}$ to an K_*-projective resolution in KK. We can do this because of Theorem 3.1:

Theorem 3.1. *Let* T *be a triangulated category,* C *a stable Abelian category (i.e. it has a suspension automorphism) with enough projective objects, and* $\mathrm{F}\colon \mathsf{T} \to \mathsf{C}$ *be a stable homological functor (i.e. it commutes with the suspension automorphisms). Denote by* $\mathscr{P}_{\mathsf{C}}$ *the projective objects in* C*, and let* $\mathrm{F}^{\dagger}$ *be a with a left adjoint of* F *defined on* $\mathscr{P}_{\mathsf{C}}$ *such that* $\mathrm{F} \circ \mathrm{F}^{\dagger} \cong \mathrm{id}_{\mathscr{P}_{\mathsf{C}}}$*. Then the categories of* F*-projective objects in* T *and the projective objects in* C *are equivalent. Moreover, for* A *an object in* T*, the projective resolutions of* $\mathrm{F}(A)$ *in* C *corresponds up to isomorphism bijectively with the* F*-projective* F*-resolutions of* A *in* T.

Example 3.2. For G a compact (quantum) group, let $\mathsf{T} = \mathsf{KK}^G$, $\mathsf{C} = \mathrm{Rep}(G) - \mathrm{Mod}_c^{\mathbf{Z}/2}$ where $\mathrm{Rep}(G)$ is the representation ring of G, and $\mathrm{F} = \mathsf{K}_*^G\colon \mathsf{T} \to \mathrm{Rep}(G) - \mathrm{Mod}_c^{\mathbf{Z}/2}$. In this case $\mathrm{F}^{\dagger}(\mathrm{Rep}(G)) = \mathbf{C}$, $\mathsf{K}_*^G = \mathsf{KK}_*^G(\mathbf{C}, \_)$ and $\mathsf{KK}_*^G(\mathbf{C}, B) = K_*^G(B) = \mathrm{Hom}_{\mathrm{Rep}(G)}(\mathrm{Rep}(G), K_*^G(B)$. Here projective resolutions have length more than one, so one must use some more argumentation to get an UCT. This was done by Rosenberg and Schochet in [RS86]. $\lozenge$

For the remainder of this lecture we will use the definitions of Theorem 3.1. so T will be a triangulated category, C an Abelian category with suspension (i.e. an automorphism $\mathsf{C} \to \mathsf{C}$), and $\mathrm{F}\colon \mathsf{T} \to \mathsf{C}$ a homological functor that commutes with suspensions. Moreover, $\mathscr{P}_{\mathrm{F}}$ will denote F-projective objects, while $\mathscr{N}_{\mathrm{F}}$ will denoted the F-contractible objects. Our goal is to relate homological algebra in the Abelian category C to things happening in the original triangulated category T.

3.1 The F-phantom tower over A

Let

$$0 \longleftarrow A \xleftarrow{\pi_0} P_0 \xleftarrow{\delta_1} P_0 \xleftarrow{\delta_2} P_0 \xleftarrow{\delta_3} P_0 \longleftarrow \cdots$$

be an F-projective resolution of an object A in T. The F-phantom tower of A is

$$\begin{array}{ccccccccccccc}
A = N_0 & \xrightarrow{\iota_0^1} & N_1 & \xrightarrow{\iota_1^2} & N_2 & \xrightarrow{\iota_2^3} & N_3 & \longrightarrow & \cdots \\
\pi_0 \nwarrow & \swarrow\circ & \pi_1 \nwarrow & \swarrow\circ & \pi_2 \nwarrow & \swarrow\circ & \pi_3 \nwarrow & \swarrow\circ & \\
P_0 & \xleftarrow{\delta_1} & P_1 & \xleftarrow{\delta_2} & P_2 & \xleftarrow{\delta_3} & P_3 & \longleftarrow & \cdots
\end{array}$$

where N_{i+1} is the mapping cone of π_i, each π_i is F-epic, and each ι_i^{i+1} is F-phantom (recall that $\xrightarrow{\circ}$ denotes a map of degree one).

To construct this tower inductively, we start by looking at the leftmost triangle. So let $N_0 = A$, N_1 be the mapping cone of π_0 and consider

$$P_0 \xrightarrow{\pi_0} N_0 \xrightarrow{\iota_0^1} N_1 \longrightarrow P_0[1].$$

Since π_0 is F-epi, the map ι_0^1 is F-phantom by Lemma 2.3. So to do our inductive construction it is enough to find a F-epic map $\pi_1\colon P_1 \to N_1$. Since $\mathsf{T}(P_1, \_)$ is a F-exact functor and $\pi_0 \circ \delta_1 = 0$ we have

$$\begin{array}{ccccc}
\mathsf{T}(P_1, N_1[0]) & \rightarrowtail & \mathsf{T}(P_1, P_0[1]) & \twoheadrightarrow & \mathsf{T}(P_1, N_0[1]) \\
 & & \Psi & & \Psi \\
 & & \delta_1 & \longmapsto & 0
\end{array}$$

Thus there is a π_1 in $\mathsf{T}(P_1, N_1)$ mapping to δ_1. It is even possible to show that π_1

is unique. To show that π_1 is F-epic, consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{F}(N_1[0]) & \rightarrowtail & \mathrm{F}(P_0[1]) & \twoheadrightarrow & \mathrm{F}(N_0[1]) \\ {\scriptstyle \mathrm{F}(\pi_1)}\uparrow & & \| & & \| \\ \mathrm{F}(P_1[0]) & \longrightarrow & \mathrm{F}(P_0[1]) & \twoheadrightarrow & \mathrm{F}(N_0[1]) \end{array}$$

with exact rows and do a diagram chase.

Definition 3.3. Let $\mathscr{I}$ be an ideal in an additive category D. A morphism $f\colon A \to B$ in D is *$\mathscr{I}$-versal* if for any morphism $j\colon A \to X$ in $\mathscr{I}$ there exist a morphism $B \to X$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \swarrow{\scriptstyle \exists} & \\ X & & \end{array}$$

commutative.

Remark 3.4. If $\mathscr{I}$ is an ideal in an additive category D, then for any $n > 0$ the collection all composition of n morphisms from $\mathscr{I}$, $\mathscr{I}^n$, is also an ideal. ∎

Now consider the ideal of F-phantom maps, $\mathscr{I} = \ker \mathrm{F}$ and consider the diagram

$$\begin{array}{ccccc} P_0 & \xrightarrow{\pi_0} & A & \xrightarrow{\iota_0^1} & N_1 \\ & {\scriptstyle 0}\searrow & {\scriptstyle j}\downarrow & \swarrow{\scriptstyle \exists} & \\ & & X & & \end{array}$$

with $j \in \mathscr{I}$. Since $\mathscr{I}(P, \_) = 0$ if P is F-projective and $j \circ \pi_0 \in \mathscr{I}$, it follows that $j \circ \pi_0 = 0$. Since $\mathsf{T}(\_, X)$ is cohomological there is a morphism $N_1 \to X$ making the resulting diagram commutative, so ι_0^1 is $\mathscr{I}$-versal. Similarly, if $\iota_0^n = \iota_0^1 \circ \cdots \iota_{n-1}^n\colon A \to N_n$ then ι_0^n is $\mathscr{I}^n$-versal.

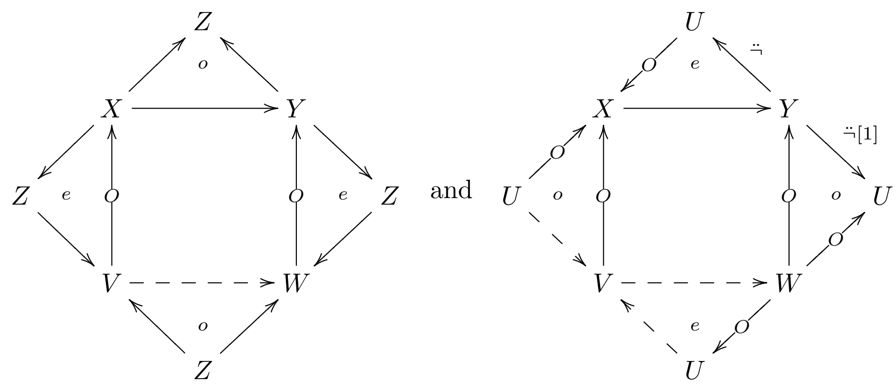
Now assume that the F-projective resolutions have finite length, that is ι_n^{n+1} is invertible for large n (i.e. n larger then the length of the resolution). So $\mathrm{F}(\iota_n^{n+1}) = 0$ and $\mathrm{F}(N_n) = 0$ for large n. Thus the N_i's are eventually F-contractible objects. The assignment $A \mapsto N_n$ for large n gives a functor from T to $\mathscr{N}_\mathrm{F}$ up to F-equivalence*. Now since the map $A \to N_n$ is $\mathscr{I}$-versal and eventually $N_n \cong N_{n+1}$ we get $\mathscr{I}^n(A, \_) \cong \mathscr{I}^{n+1}(A, \_)$. By defining $\mathscr{I}^\infty(A, \_) = \bigcap \mathscr{I}^n(A, \_)$ of can show that $\mathscr{I}^\infty(A, B) \cong \mathsf{T}(N_n, B)$.

3.2 The octahedron axiom

Before we state the axiom (TR4), we need some notation: Let $\begin{array}{ccc} A & \longrightarrow & B \\ & {\scriptstyle \nwarrow\!\circ}\ \ e \ \swarrow & \\ & C & \end{array}$ and $\begin{array}{ccc} A & \longrightarrow & C \\ & \searrow\ o \ \nearrow & \\ & B & \end{array}$ denote an exact and a commutative triangle respectively (recall that $\xrightarrow{\ \circ\ }$ denotes a degree one morphism).

Definition 3.5 ((TR4) - The octahedron axiom)**.** Given the solid lines in the diagrams

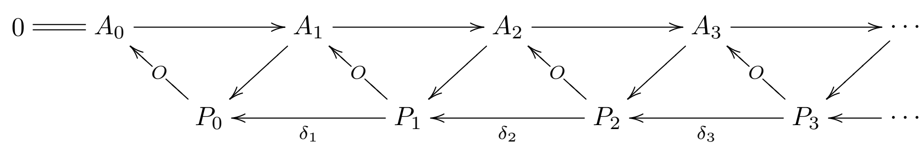
* If the F-projective resolutions does not have finite length, one can use $\mathrm{hocolim}(N_i, \iota_i^{i+1})$ under some assumptions on direct sums.



(where each morphism is repeated twice except $\ddot{\neg}\colon Y \to U$ and $\ddot{\neg}[1]\colon Y[1] \to U[1]$), there are morphisms $U \to V$ and $V \to W$ (each given twice by dotted lines) such that the resulting triangles are exact or commutative as indicated.

3.3 The approximation tower

The approximation tower is in some sense analogues to creating a cellular complex. The skeletons will be called A_n, while the cells of the complex are the P_n's from the F-projective resolution. We construct of the approximation tower

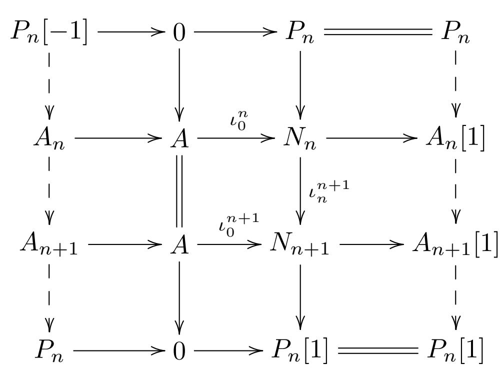


by induction using the F-phantom tower and the octahedron axiom.

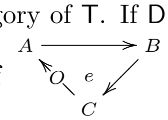
By looking at the exact triangles containing the map ι_0^n,

$$N_n[-1] \longrightarrow A_n \longrightarrow A \xrightarrow{\iota_0^n} N_n,$$

we can form the commutative diagram



where the downward arrows $\dashrightarrow$ exist and form exact triangles by (TR4).

Definition 3.6. Let D be a subcategory of T. If D is closed under isomorphisms, suspensions and exact triangles (i.e. if  is an exact triangle where two of the objects are in D, so is the third), then D is a *triangulated subcategory*.

Remark 3.7. The collection of F-contractible objects, $\mathscr{N}_{\mathrm{F}}$, form a triangulated subcategory. ■

Definition 3.8. Let $\mathscr{P}_{\mathrm{F}}$ be the collection of F-projective objects in T. The *triangulated subcategory of* T *generated by* $\mathscr{P}_{\mathrm{F}}$ is the smallest triangulated subcategory $\langle \mathscr{P}_{\mathrm{F}} \rangle$ of T that contains $\mathscr{P}$.

Remark 3.9. All the A_n's from the approximation tower belong to $\langle \mathscr{P}_{\mathrm{F}} \rangle$. ■

In the case of finite F-projective resolutions, we have that $\iota_n^{n+1}\colon N_n \to N_{n+1}$ is a F-equivalence. It follows that the assignment $A \mapsto A_n$ is a functor up to F-equivalence. Our next goal is to show that $A \mapsto A_n$ is left adjoint to the inclusion $\langle \mathscr{P}_{\mathrm{F}} \rangle \to \mathsf{T}$.

3.4 Localization

Observe that if $P \in \mathscr{P}_{\mathrm{F}}$ and $N \in \mathscr{N}_{\mathrm{F}}$ then $\mathsf{T}(P, N) = 0$. Thus for $P' \in \langle \mathscr{P}_{\mathrm{F}} \rangle$ one has $\mathsf{T}(P', N) = 0$. Now assume as before that the F-projective resolutions have finite length. This implies that there is an exact triangle

$$\begin{array}{ccccc} A_n & \longrightarrow & A & \longrightarrow & N_n \quad (N_n \overset{\circ}{\longrightarrow} A_n) \\ \cap & & & & \cap \\ \langle \mathscr{P}_{\mathrm{F}} \rangle & & & & \mathscr{N}_{\mathrm{F}} \end{array}$$

for large n. Since we are only interested in $\langle \mathscr{P}_{\mathrm{F}} \rangle$ and $\mathscr{N}_{\mathrm{F}}$ up to F-equivalence, it follows that the functors $A \mapsto A_n$ and $A \mapsto N_n$ are F-exact.

Now let B be a F-contractible object. By applying $\mathsf{T}(\_, B)$ to the exact triangle above we obtain the exact sequence

$$\begin{array}{ccccccc} \mathsf{T}(A_n, B) & \longrightarrow & \mathsf{T}(A, B) & \longrightarrow & \mathsf{T}(N_n, B) & \longrightarrow & \mathsf{T}(A_n[1], B) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

Consequentially the functor $A \mapsto A_n$ is left adjoint to the embedding $\langle \mathscr{P}_{\mathrm{F}} \rangle \to \mathsf{T}$, so T decomposes into $\langle \mathscr{P}_{\mathrm{F}} \rangle$ and $\mathscr{N}_{\mathrm{F}}$ in a controlled way. This is an example of localization.

Definition 3.10. The *localization* of a functor $\mathrm{G}\colon \mathsf{T} \to \mathsf{D}$ at $\mathscr{N}_{\mathrm{F}}$ is the functor $\mathbb{L}\mathrm{G}\colon \mathsf{T} \to \mathsf{D}$ given by the composition $\mathsf{T} \xrightarrow{A \mapsto A_n} \langle \mathscr{P}_{\mathrm{F}} \rangle \xrightarrow{\mathrm{G}} \mathsf{D}$.

Given a functor $\mathrm{G}\colon \mathsf{T} \to \mathsf{D}$, then the maps $A_n \to A$ induce a natural transformation from $\mathbb{L}\mathrm{G}$ to G. The Baum-Connes assembly map is a special case of this.

4 Lecture 4

For a triangulated category T, a stable Abelian category C, and a stable homological functor $\mathrm{F}\colon \mathsf{T} \to \mathsf{C}$ we have the collections $\mathscr{P}_{\mathrm{F}}$ of F-projective objects and $\mathscr{N}_{\mathrm{F}}$ of F-contractible objects. Moreover, we have the localizing* category $\langle \mathscr{P}_{\mathrm{F}} \rangle$. The pair $\big(\langle \mathscr{P}_{\mathrm{F}} \rangle, \mathscr{N}_{\mathrm{F}}\big)$ is complementary in the sense that $\mathsf{T}\big(\langle \mathscr{P}_{\mathrm{F}} \rangle, \mathscr{N}_{\mathrm{F}}\big) = 0$ and for each $A \in \mathsf{T}$ there is an exact triangle $P \longrightarrow A \longrightarrow N \longrightarrow P[1]$ with $P \in \langle \mathscr{P}_{\mathrm{F}} \rangle$ and $N \in \mathscr{N}_{\mathrm{F}}$.

* A category is localizing if is is triangulated and closed under countable direct sums.

Warning 4.1. Usually the case is that $\mathsf{T}\big(\mathscr{N}_{\mathrm{F}}, \langle \mathscr{P}_{\mathrm{F}} \rangle\big) \neq 0$. ■

Under mild technical assumptions on direct sums (or if all F-resolutions in T has finite length), the categories $\langle \mathscr{P}_{\mathrm{F}} \rangle$ and $\mathscr{N}_{\mathrm{F}}$ generates T, that is T is the smallest category containing $\langle \mathscr{P}_{\mathrm{F}} \rangle$ and $\mathscr{N}_{\mathrm{F}}$ that is closed in exact triangles and countable direct sums. We can in principle understand how objects in $\langle \mathscr{P}_{\mathrm{F}} \rangle$ are built out of generators (by using the approximation tower). This in turn leads to spectral sequences, so studying $\langle \mathscr{P}_{\mathrm{F}} \rangle$ is a "topological" problem.

4.1 Triangulated categories for the Baum-Connes conjecture

In order to apply our machinery to the Baum-Connes conjecture, we must first fix the category T and the functor F. However, one often start with a functor F′ and then modifies his functor to F such that $\ker(\mathrm{F}') = \ker(\mathrm{F})$. So let G be a locally compact group, $\mathsf{T} = \mathsf{KK}^G$ and $\mathscr{F}' = \{H \subseteq G \mid H \text{ is a compact subgroup of } G\}$ (Note that the Baum-Connes conjecture holds for compact groups). Let $\mathrm{F}' \colon \mathsf{KK}^G \to \prod_{H \in \mathscr{F}'} \mathsf{KK}^H$ be the exact functor which in each component "forgets" some of the G-action (i.e. the G-action becomes an H-action). Then F′ is a triangulated functor*, and $\ker(\mathrm{F}') = \{f \text{ a morphism in } \mathsf{KK}^G \mid f \text{ becomes } 0 \text{ in } KK^H \text{ for all } H \in \mathscr{F}'\}$.

The next step is to find and adjoint. Since F′ does not have an adjoint, we modify F′ to F without changing the kernel. To do this we need some more definitions.

Definition 4.2. Given a compact subgroup H of G, the *restriction functor* is $\mathrm{Res}_G^H \colon \mathsf{KK}^G \to \mathsf{KK}^H$, and the *induction functor* is $\mathrm{Ind}_H^G \colon \mathsf{KK}^H \to \mathsf{KK}^G$.

Proposition 4.3. *Let H be an open compact subgroup of G. Then the functors Ind_H^G and Res_G^H are adjoint so* $\mathsf{KK}^G\big(\mathrm{Ind}_H^G(A), B\big) \cong \mathsf{KK}^H\big(A, \mathrm{Res}_G^H(B)\big)$.

Now if G is totally disconnected, then any compact subgroup of G is contained in an compact open subgroup. Thus we do not change $\ker(\mathrm{F}')$ if we replace $\mathscr{F}'$ with the family of compact open subgroups. Since the set of such subgroups are countable, Proposition 4.3 shows that the functor F so obtained has an adjoint.

In the general case, any locally compact group G contains an open almost connected subgroup. Moreover, any compact subgroup of G is contained in one of these almost connected subgroups. If L is an almost connected subgroup of G, then L contains a maximal compact subgroup K, any compact subgroup of L is subconjugate to K (i.e. conjugate to a subgroup of K), and L/K is a smooth manifold. Thus we might replace $\mathscr{F}'$ with $\mathscr{F} = \mathscr{F}' \cap \mathscr{M}$ where $\mathscr{M} = \{H \subseteq G \mid G/H \text{ is a smooth manifold with a smooth } G\text{-action}\}$.

Remark 4.4. We have the following relationship between categories:

$$\begin{array}{ccc} \mathsf{KK}^G & \xrightarrow{\mathrm{Res}_G^H} & \mathsf{KK}^H \\ {\scriptstyle \mathrm{p}^*_{G/H}}\downarrow & \underset{G \ltimes G/H \sim H}{\overset{\cong}{\nearrow}} & \\ \mathsf{RKK}^G(G/H) & & \end{array}$$

where $\mathrm{p}^*_{G/H}$ is a kind of pullback. Thus if A and B are G-C*-algebras one has $KK^H(A, B) \cong \mathsf{RKK}^G(G/H; A, B)$. ■

Now let X be a smooth proper G-manifold. In this case there is a left adjoint for the functor $\mathrm{p}_X^* \colon \mathsf{KK}_*^G \to \mathsf{RKK}_*^G(X)$, namely $\mathfrak{p}_X \otimes \_$ for $\mathfrak{p}_X = C_0(\mathcal{T}X)$ or $\mathfrak{p}_X = C_0\big(X; \mathrm{Cliff}(\mathcal{T}X)\big)$. Consequentially, for all G-C*-algebras A and B one have $\mathsf{RKK}_*^G(X; A, B) \cong \mathsf{KK}_*^G(\mathfrak{p}_X \otimes A, B)$. This generalizes the duality isomorphism since we have $\mathsf{RK}^*(X) \cong \mathsf{KK}_*\big(C_0(\mathcal{T}X\big) = \mathsf{KK}_X^{\mathrm{top}}(\mathcal{T}X)$. For more see the lectures by Kasparov (the lecture notes are available online).

* Although F′ is not a homological functor, is is exact so everything works out fine.

Proposition 4.5. *Let G be a locally compact group and $\mathscr{F}$ be a countable family of compact smooth subgroups in G (i.e. G/H is a smooth manifold with smooth G-action for each $H \in \mathscr{F}$). Then the functor $\mathsf{KK}^G \xrightarrow{\mathrm{F}} \prod_{H\in\mathscr{F}} \mathsf{KK}^H$ defined by "forgetting part of the G-action in each factor" has a left adjoint defined on the range of F. Therefore KK^G has enough F-projective objects (i.e. any object in KK^G has a F-projective resolution). Any F-projective object is a direct summand of a direct sum of G-C^*-algebras of the form $\mathrm{Ind}_H^G(A)$ with $H \subset G$ compact and A a H-C^*-algebra, and all objects of this form are F-projective.*

Remark 4.6. Recall that for H a subgroup of G and A a H-C*-algebra we have $G \ltimes \mathrm{Ind}_H^G(A) \sim_{\mathrm{Morita}} H \ltimes A$. ■

Question (open) What is the universal homological functor describing $\ker(\mathrm{F})$?
Equivalent: Describe the F-derived functors.

The above question is easy when G is torsion free. Then the universal homological functor is $\mathrm{F}\colon \mathsf{KK}^G \to \mathsf{KK}[G/G_0]$ (with $G/G_0 \to \mathsf{KK}_0(A,A)$ for $A \in \mathsf{KK}[G/G_0]$).

4.2 Localization and the Baum-Connes conjecture

By the choices in the previous section we obtain the collections

$$\mathscr{N}_{\mathrm{F}} = \{A \in \mathsf{KK}^G \mid A \cong 0 \text{ in } \mathsf{KK}^H \text{ for all compact } H \subseteq G\}$$

and

$$\langle \mathscr{P}_{\mathrm{F}} \rangle = \big\langle \{\mathrm{Ind}_H^G(A) \mid H \subseteq G \text{ compact and } A \text{ a } H\text{-C}^*\text{-algebra}\} \big\rangle.$$

Moreover $\big(\langle \mathscr{P}_{\mathrm{F}} \rangle, \mathscr{N}_{\mathrm{F}}\big)$ form a complementary pair, and the assumptions on direct sums are satisfied, so $\mathscr{N}_{\mathrm{F}}$ and $\langle \mathscr{P}_{\mathrm{F}} \rangle$ generates KK^G.

Theorem 4.7. *With the definitions above, the localization of KK^G to $\mathscr{N}_{\mathrm{F}}$ (denoted by $\mathsf{KK}^G/\mathscr{N}_{\mathrm{F}}$) is equivalent to the category $\mathsf{RKK}^G(\mathcal{E}G)$ where $\mathcal{E}G$ is the universal proper G-space. Moreover the diagram*

$$\begin{array}{ccc} \mathsf{KK}^G(A,B) & \longrightarrow & \mathsf{KK}^G/\mathscr{N}_{\mathrm{F}}(A,B) \\ {\scriptstyle \mathrm{p}^*_{\mathcal{E}G}}\big\downarrow & \overset{\cong}{\nearrow} & \\ \mathsf{RKK}^G(\mathcal{E}G;A,B) & & \end{array}$$

commutes, where $\mathrm{p}\colon \mathcal{E}G \to \{pt\}$.

Let us now assume that $\mathcal{E}G$ has a dual, i.e. $\mathsf{RKK}^G_*(\mathcal{E}G;A,B) \cong \mathsf{KK}^G_*(\mathfrak{p}_{\mathcal{E}G} \otimes A, B)$. It is a fact that if $B \in \mathscr{N}_{\mathrm{F}}$ then $\mathrm{p}^*_{\mathcal{E}G}(B) \cong 0$ in $\mathsf{RKK}^G(\mathcal{E}G)$. This implies that $\mathsf{KK}^G_*(\mathfrak{p}_{\mathcal{E}G} \otimes A, B) = 0$ for all $B \in \mathscr{N}_{\mathrm{F}}$, whence $\mathfrak{p}_{\mathcal{E}G} \in \langle \mathscr{P}_{\mathrm{F}} \rangle$. It is worth mentioning that one can often construct $\mathfrak{p}_{\mathcal{E}G}$ by hand to see this directly.

Now assume even more, namely that $\mathfrak{p}_{\mathcal{E}G}$ is a proper G-C*-algebra, that is a $G \ltimes \mathcal{E}G$-C*-algebra. The duality isomorphism above can then be described using the Dirac element $D \in \mathsf{KK}^G(\mathfrak{p}_{\mathcal{E}G}, \mathbf{C}) \cong \mathsf{RKK}^G(\mathcal{E}G; \mathbf{C}, \mathbf{C}) \ni 1$ and the dual Dirac element $\Theta \in \mathsf{RKK}^G(\mathcal{E}G; \mathbf{C}, \mathfrak{p}_{\mathcal{E}G}) \cong \mathsf{KK}^G(\mathfrak{p}_{\mathcal{E}G}, \mathfrak{p}_{\mathcal{E}G}) \ni 1$. This gives conditions on D and Θ that insure invertibility of the duality map. In particular $\mathrm{p}^*_{\mathcal{E}G}(D)$ and Θ must be inverse of each other. Now $\mathrm{p}^*_{\mathcal{E}G}(\mathrm{id}_A \otimes D)$ is invertible, whence $\mathrm{p}^*_X(\mathrm{id}_A \otimes D)$ is invertible for all proper G-spaces X. By letting $X = G/H$ it follows that $\mathrm{Res}_G^H(\mathrm{id}_A \otimes D)$ is invertible, so $A \otimes \mathfrak{p}_{\mathcal{E}G} \xrightarrow{\mathrm{id}_A \otimes D} A$ is an F-equivalence. Consequentially

$$A \otimes \mathfrak{p}_{\mathcal{E}G} \xrightarrow{\mathrm{id}_A \otimes D} A \longrightarrow \mathrm{cone}(\mathrm{id}_A \otimes D) \longrightarrow A \otimes \mathfrak{p}_{\mathcal{E}G}[1]$$

is a triangle as needed for localization. Thus $\mathsf{KK}^G/\mathscr{N}_{\mathrm{F}}(A,B) \cong \mathsf{KK}^G(A\otimes \mathfrak{p}_{\mathcal{E}G}, B) \cong \mathsf{RKK}^G(\mathcal{E}; A, B)$.

Let us finally localize $\mathsf{K}_*(A \rtimes_r G)$. We get

$$\mathsf{K}_*\big((A\otimes \mathfrak{p}_{\mathcal{E}G}) \rtimes_r G\big) \cong_{\text{duality}} \varinjlim_{\substack{X\subseteq \mathcal{E}G\\ X \text{ compact}}} \mathsf{KK}^G\big(C_0(X), A\big).$$

The localization $\mathbb{L}\mathrm{G}$ at $\mathscr{N}_{\mathrm{F}}$ of a functor G has the following properties

- $\mathbb{L}\mathrm{G}$ is exact,
- $\mathbb{L}\mathrm{G}|_{\mathscr{N}_{\mathrm{F}}} = 0$, and
- $\mathbb{L}\mathrm{G} \to \mathrm{G}$ is invertible on $\mathscr{P}_{\mathrm{F}}$.

If there is a functor G' and a natural transformation $\mathrm{G}' \to \mathrm{G}$ with these properties, then it is equivalent to $\mathbb{L}\mathrm{G} \to \mathrm{G}$. An example of a natural transformation with these properties is the Baum-Connes assembly map, so it must be the localization map.

4.3 Remark on the Farrell-Jones conjecture

To get the Farrell-Jones conjecture, replace KK^G by a category of equivalent spectra. The functor F becomes the restriction functor for the family of virtually cyclic subgroups.

References

[Bel00] Apostolos Beligiannis. Relative homological algebra and purity in triangulated categories. *Journal of Algebra*, 227(1):268–361, 2000.

[Chr98] J. Daniel Christensen. Ideals in triangulated categories: phantoms, ghosts and skeleta. *Advances in Mathematics*, 136(2):284–339, 1998.

[Mey08] Ralf Meyer. Homological algebra in bivariant K-theory and other triangulated categories. II, 2008, arXiv:0801.1344v3 [math.KT].

[MN03] Ralf Meyer and Ryszard Nest. The Baum-Connes Conjecture via Localisation of Categories, 2003, arXiv:0312292v3 [math.KT].

[MN07] Ralf Meyer and Ryszard Nest. Homological algebra in bivariant K-theory and other triangulated categories, 2007, arXiv:0702146v2 [math.KT].

[RS86] Jonathan Rosenberg and Claude Schochet. The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory. *Memoirs of the American Mathematical Society*, 62(348), 1986.