Elaborate version of a talk given on

The Kasparov Product

at Prof. Echterhoff's postgraduate seminar

- revised version with less typos -

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Contents

In this talk, all C^*-algebras denoted by A, B, C, ... are assumed to be σ-unital and graded. Some statements are also true if not all of the C^*-algebras involved are σ-unital, but we would like to keep the formulation of our theorems and propositions simple.

1 Invitation to the Kasparov product

The aim of this talk is to define the Kasparov product

$$\otimes_B\colon \operatorname{KK}(A,B)\times\operatorname{KK}(B,C)\longrightarrow\operatorname{KK}(A,C)$$

for C^*-algebras A, B and C. We will have to assume that the algebra A is separable. In the first part of the talk, we will discuss the Kasparov product for some special cases giving us some desirable conditions for the general product. Before we dive into the technicalities, an outline of the product is given as a goal for the construction.

1.1 The Kasparov product in some special cases

1.1.1 Homomorphisms

Definition 1.1. Let A and B be C^*-algebras and $f\colon A\longrightarrow B$ be a $*$-homomorphism. Then we define

$$(f) := (B,f,0)\in\mathbb{E}(A,B)\quad\text{and}\quad [f] := [(B,f,0)]\in\operatorname{KK}(A,B).$$

(f) is indeed an element of $\mathbb{E}(A,B)$ because B is σ-unital (and hence countably generated as a B-Hilbert-module) and $\operatorname{K}_B(B)=B$ (so f factors through the compact operators).

Obviously, we should define the Kasparov product in a way that ensures the formula

$$[g\circ f] = [f]\otimes_B[g], \tag{1}$$

where we denote the Kasparov product with a tensor product notation which will soon be plausible. If A is a C^*-algebra, then $(\operatorname{Id}_A) = (A,\operatorname{Id}_A,0)$. Now $[\operatorname{Id}_A]$ seems to be a natural candidate for a left unit element for $\operatorname{KK}(A,B)$ and $[\operatorname{Id}_B]$ should act as a right unit:

$$\forall x\in\operatorname{KK}(A,B):\quad [\operatorname{Id}_A]\otimes_A x = x\otimes_B[\operatorname{Id}_B] = x. \tag{2}$$

In the following section we are going to analyze a more general, but still comparatively simple situation:

1.1.2 Kasparov cycles with trivial operator

What do these Kasparov cycles look like?

Proposition 1.2. Let A and B be C^*-algebras. Then $(E,\phi,0)$ is a Kasparov cycle if and only if E is a countably generated graded Hilbert B-module and $\phi\colon A\longrightarrow\operatorname{L}_B(E)$ is a graded $*$-homomorphism such that $\phi(A)\subseteq\operatorname{K}_B(E)$.

For such modules there is an obvious definition of a product on the level of cycles:

Definition 1.3. Let A, B and C be C^*-algebras. Then we define for every $\mathcal{E}_1=(E_1,\phi_1,0)\in\mathbb{E}(A,B)$ and $\mathcal{E}_2=(E_2,\phi_2,0)\in\mathbb{E}(B,C)$:

$$\mathcal{E}_1\otimes_B\mathcal{E}_2 = (E_1,\phi_1,0)\otimes_B(E_2,\phi_2,0) := (E_1\otimes_B E_2,\phi_1\otimes 1,0)\in\mathbb{E}(A,C)$$

The module $E_1\otimes_B E_2$ is countably generated because E_1 and E_2 are. Because $\phi_2(B)\subseteq\operatorname{K}_C(E_2)$ one can show that $\operatorname{K}_B(E_1)\otimes 1\subseteq\operatorname{K}_C(E_1\otimes_B E_2)$ (but be careful: this is not true in general). The grading of $E_1\otimes_B E_2$ was given in the preceding talk.

The Kasparov product should surely satisfy

$$[\mathcal{E}_1]\otimes_B[\mathcal{E}_2] = [\mathcal{E}_1\otimes_B\mathcal{E}_2]. \tag{3}$$

There is a link of the above definition to strong Morita equivalences of C^*-algebras:

Proposition 1.4. Let A and B be C^*-algebras, $\mathcal{E}_1 := (E_1, \phi_1, 0) \in \mathbb{E}(A, B)$ and $\mathcal{E}_2 := (E_2, \phi_2, 0) \in \mathbb{E}(B, A)$ such that $\mathcal{E}_1 \otimes_B \mathcal{E}_2 \cong (\mathrm{Id}_A) = (A, \mathrm{Id}_A, 0)$ and $\mathcal{E}_2 \otimes_A \mathcal{E}_1 \cong (\mathrm{Id}_B)$. Then $\phi_1 \colon A \longrightarrow \mathrm{K}_B(E_1)$ is an isomorphism, E_1 is an A-B-imprimitivity bimodule and $E_2 \cong \overline{E_1}$.

This was proved in a more general setting (but without the grading which should not cause any problems) in [EKQR02], Lemma 2.4.

Of course also the opposite is true:

Proposition 1.5. Let A and B be C^*-algebras and E be an A-B-imprimitivity bimodule. Let ϕ denote the action of A on E. Then $\mathcal{E} := (E, \phi, 0) \in \mathbb{E}(A, B)$. Moreover, $\overline{E}$ is an B-A-imprimitivity bimodule, and if ψ is the action of B on $\overline{E}$, then $\overline{\mathcal{E}} := (\overline{E}, \psi, 0) \in \mathbb{E}(B, A)$ and $\mathcal{E} \otimes_B \overline{\mathcal{E}} \cong (\mathrm{Id}_A)$ as well as $\overline{\mathcal{E}} \otimes_A \mathcal{E} \cong (\mathrm{Id}_B)$.

Note that E and $\overline{E}$ are automatically countably generated because A and B are σ-unital.

1.1.3 No problems so far

It's worth a thought to check that these requirements for the Kasparov product are not contradictory. In particular, we would like to show the following proposition which implies (3) $\Rightarrow$ (1).

Proposition 1.6. Let A, B and C be C^*-algebras, $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$. Then

$$[g \circ f] = [(f) \otimes_B (g)].$$

Proof. Note that $(g \circ f) = (C, g \circ f, 0)$ and $(f) \otimes_B (g) = (B \otimes_g C, f \otimes 1, 0)$. There is a canonical map μ from $B \otimes_g C$ to C, given on simple tensors by

$$b \otimes_g c \mapsto g(b)c.$$

A short calculation shows that this is a well-defined isometric C-linear map respecting the inner products. If g is non-degenerate, then μ is unitary and the cycles in question are isomorphic. For general g, the image of μ is $g(B)C$, the non-degenerate part of the left Banach B-module C. This is also a Hilbert B-module and the result follows from the following observation that we are going to prove later also for non-zero operators. □

Lemma 1.7. *Let A and B be C^*-algebras and $(E, \phi, 0) \in \mathbb{E}(A, B)$. Then $E_0 := \phi(A)E = \overline{\phi(A)E}$ is a graded Hilbert sub-B-module of E, invariant under $\phi(A)$, such that $(E_0, \phi, 0)$ is in $\mathbb{E}(A, B)$ and homotopic to $(E, \phi, 0)$. The homotopy may be chosen with vanishing operator.*

Proof. We have to construct the homotopy and we do this analogously to the construction in 18.3.6 of [Bla98]. Define $\overline{E} := E[0, 1]$ and the sub-$B[0, 1]$-module $\overline{E}_0 := \{f \in E[0, 1] : f(1) \in E_0\}$. Let $\gamma \colon \mathrm{L}_B(E) \longrightarrow \mathrm{L}_{B[0,1]}(E[0, 1]) \cong \mathrm{L}_B(E)[0, 1]$ be the embedding of $\mathrm{L}_B(E)$ as constant functions. Note that γ maps $\mathrm{K}_B(E)$ into $\mathrm{K}_{B[0,1]}(E[0, 1]) \cong \mathrm{K}_B(E)[0, 1]$. Then $\gamma \circ \phi$ is a graded $*$-homomorphism with image in the compact operators on $E[0, 1]$, so $(E[0, 1], \gamma \circ \phi, 0) \in \mathbb{E}(A, B[0, 1])$ (note that $E[0, 1]$ is countably generated). Obviously, $\overline{E}_0$ is $\gamma(\phi(A))$-invariant so that $(\overline{E}_0, \gamma \circ \phi, 0)$ is in $\mathbb{E}(A, B[0, 1])$ as well (note that every element of $\overline{E}_0$ can be written as a sum of an element of $E[0, 1)$ and of an element of $E_0[0, 1]$ with both $B[0, 1]$-modules being countably generated). Now

$$\psi_t \colon \overline{E}_0 \otimes_{\mathrm{ev}_t} B \longrightarrow E, f \otimes b \mapsto f(t)b$$

is an isometric B-linear map respecting the inner product for every $t \in [0, 1]$. It is surjective for every $t < 1$ and has image E_0 for $t = 1$, and $(\overline{E}_0, \gamma \circ \phi, 0)$ is a homotopy from E to E_0.

□

1.1.4 Homotopies

Proposition 1.8. Let A, B be C^*-algebras and E be a countably generated Hilbert B-module. Let $\phi\colon A \longrightarrow \mathrm{K}_B(E)[0,1] \cong \mathrm{K}_{B[0,1]}(E[0,1])$ be a graded $*$-homomorphism. For every $t \in [0,1]$ let

$$\phi_t\colon A \longrightarrow \mathrm{K}_B(E),\ a \mapsto \phi(a)(t).$$

Then

$$\mathrm{ev}_{t,*}(E[0,1], \phi, 0) \cong (E, \phi_t, 0).$$

In particular, $(E[0,1], \phi, 0)$ is a homotopy from $(E, \phi_0, 0)$ to $(E, \phi_1, 0)$.

Corollary 1.9. Let A and B be C^*-algebras and $(f_t)_{t\in[0,1]}$ be a homotopy of graded $*$-homomorphisms from A to B. Then (f_0) is homotopic to (f_1), i.e. if f_0 and $f_1\colon A \longrightarrow B$ are homotopic, then $[f_0] = [f_1]$.

1.2 A picture of the Kasparov product

Theorem 1.10. *Let A, B, C be C^*-algebras, A separable. Then there exists a map*

$$\otimes_B\colon \mathrm{KK}(A,B) \times \mathrm{KK}(B,C) \longrightarrow \mathrm{KK}(A,C),$$

called the Kasparov product. It has the following properties:

(Biadditivity) *The Kasparov product is additive in the first component*

$$\forall x_1, x_2 \in \mathrm{KK}(A,B)\ \forall y \in \mathrm{KK}(B,C) : \quad (x_1 + x_2)\otimes_B y = x_1 \otimes_B x_y + x_2 \otimes_B y,$$

as well as in the second:

$$\forall x \in \mathrm{KK}(A,B)\ \forall y_1, y_2 \in \mathrm{KK}(B,C) : \quad x\otimes_B (y_1 + y_2) = x\otimes_B y_1 + x\otimes_B y_2.$$

(Associativity) *Let D be another graded C^*-algebra and assume that B is separable, too. Then*

$$\forall x \in \mathrm{KK}(A,B)\ \forall y \in \mathrm{KK}(B,C)\ \forall z \in \mathrm{KK}(C,D) : \quad x\otimes_B (y\otimes_C z) = (x\otimes_B y)\otimes_C z.$$

(Unit elements) *If we define $1_A := [\mathrm{Id}_A]$ and $1_B := [\mathrm{Id}_B]$, then*

$$\forall x \in \mathrm{KK}(A,B) : \quad 1_A \otimes_A x = x \otimes_B 1_B = x.$$

(Functoriality) *If $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ are graded $*$-homomorphisms, then*

$$\forall x \in \mathrm{KK}(A,B) :\ x \otimes_B [g] = g_*(x) \quad \textit{and} \quad \forall y \in \mathrm{KK}(B,C) :\ [f] \otimes_B y = f^*(y).$$

("Triviality") *If $(E_1, \phi_1, 0) \in \mathbb{E}(A,B)$ and $(E_2, \phi_2, 0) \in \mathbb{E}(B,C)$, then*

$$[(E_1, \phi_1, 0)] \otimes_B [(E_2, \phi_2, 0)] = [(E_1 \otimes_B E_2, \phi_1 \otimes 1, 0)].$$

Remark 1.11. Note that if we restrict ourselves to separable C^*-algebras, we can take the KK-elements as morphisms and obtain a category with the Kasparov product as composition. (To be more precise: you first have to flip the variables of the product.)

Definition 1.12. Let A and B be separable C^*-algebras. Then $x \in \mathrm{KK}(A,B)$ is called an isomorphism (in KK-theory) if there is a $y \in \mathrm{KK}(B,A)$ such that $x \otimes_B y = 1_A$ and $y \otimes_A x = 1_B$.

Remark 1.13. Obviously, isomorphisms of separable C^*-algebras induce isomorphisms in KK-theory. In the situation of proposition 1.5 we have in particular that $[\mathcal{E} \otimes_B \overline{\mathcal{E}}] = 1_A$ and $[\overline{\mathcal{E}} \otimes_B \mathcal{E}] = 1_B$, so it follows that A-B-imprimitivity bimodules induce "isomorphisms in KK-theory " as well.

2 Three technical tools

Before we introduce the technical devices that we are going to use in the proof of the existence as well as in the proof of the properties of the Kasparov product, we would like to sketch the basic ideas of the construction in order to point out the technical problems that we have to face.

Of course we are going to try to define the product on the level of Kasparov cycles. Let A, B and C be C^*-algebras and $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$, $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. Then there is an obvious choice of the module and the action of A for the product: $E_{12} := E_1 \otimes_B E_2$ and $\phi_{12} := \phi_1 \otimes 1$.

Now we have to find a suitable operator on $E_1 \otimes_B E_2$. A first idea would be $F_1 \otimes_B F_2$. Despite the problem that this isn't a well-defined operator (as F_2 is not B-linear on the left), it would be an operator of degree zero rather than of degree one anyway.

So what about $F_1 \otimes_B 1 + 1 \otimes_B F_2$. The first part makes sense and is of degree one. But the second definitely causes some problems: As already pointed above, F_2 isn't B-linear on the left. On the other hand, there is at least the commutation relation $[F_2, \phi_2(b)] \in \mathrm{K}_C(E_2)$ for every $b \in B$. We will use this relation to construct a substitute for the operator $1 \otimes_B F_2$, called an F_2-connection for E_1. This construction is one of the three technical tools presented in this talk.

Suppose for a moment that the expression $1 \otimes_B F_2$ makes sense. Then we still have to check that the operator $F_1 \otimes_B 1 + 1 \otimes_B F_2$ satisfies the conditions from the definition of a Kasparov cycle. In particular, we have to analyze its square. But

$$(F_1 \otimes_B 1 + 1 \otimes_B F_2)^2 = (F_1^2) \otimes_B 1 + (F_1 \otimes_B 1)(1 \otimes_B F_2) + (1 \otimes_B F_2)(F_1 \otimes_B 1) + 1 \otimes_B (F_2^2).$$

To some extend, we are able to handle the first term because we have

$$((F_1^2) \otimes_B 1)\phi_{12}(a) = (F_1^2 \phi_1(a)) \otimes_B 1 = (1 \otimes_B 1)\phi_{12}(a)$$

for all $a \in A$, at least up to some operator in $\mathrm{K}_B(E) \otimes_B 1$ that we might disregard for the time being. Similarly, we might get the last term under control. But the middle terms are problematic.

The solution is to find suitable operators $M, N \geq 0$ in $\mathrm{L}_C(E_{12})$ such that $M + N = 1$ in order to consider the operator

$$M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}}(1 \otimes_B F_2).$$

The idea is to choose M and N in a way that the middle terms of the square of the operator are small. The result that ensures the existence of such coefficient operators is known as Kasparov's technical lemma and constitutes the second tool that is going to be presented in this section.

The way we are going to construct a Kasparov product on the level of Kasparov cycles will involve many choices; the construction thus cannot be expected to give a well-defined function on the level of cycles. But at least we will be able to show that whatever choices we make, we will end up with operator homotopic cycles, allowing us to define a function on the level of KK-elements. This aim is achieved by means of a criterion for operator homotopy that forms the first technical tool given in this exposition.

2.1 A sufficient condition for operator homotopy

Definition 2.1. Let B be a C^*-algebra and I be a closed ideal in B. Let $q\colon B \longrightarrow B/J,\ b \mapsto b + J$. Let $a, b \in B$. We say that

1. a is orthogonal to $b \mod J$ if $ab \in J$, i.e. if $q(a)q(b) = 0$.
2. $a = b \mod J$ if $a - b \in J$, i.e. if $q(a) = q(b)$.
3. $a \leq b \mod J$ if $q(a) \leq q(b)$.

Remark 2.2. Let B be a C^*-algebra and J be a closed ideal in A. Then

$$a \geq 0 \mod J \quad \Leftrightarrow \quad \exists j \in J : \ a + j \geq 0.$$

Proof. Let $q\colon A \longrightarrow A/J$ be the quotient map. Suppose that $q(a) \geq 0$. Then we can find a $b \in A$ such that $q(b)^*q(b) = q(a)$. Then we have $j := b^*b - a \in J$ and $a + j = b^*b \geq 0$.

On the other hand, let $j \in J$ such that $a + j \geq 0$. Then $q(a) = q(a + j) \geq 0$. □

Definition 2.3. Let E be a graded (A, B)-Hilbert-bimodule where $\phi\colon A \longrightarrow \mathrm{L}_B(E)$ denotes the action of A on E. Define

$$Q := Q_A(E) := \{T \in \mathrm{L}_B(E) : \ \forall a \in A : \ [T, \phi(a)] \in \mathrm{K}_B(E)\}$$

and

$$J := J_A(E) := \{T \in Q_A(E) : \ \forall a \in A : \ T\phi(a) \in \mathrm{K}_B(E)\}.$$

Then it's easy to check that $Q_A(E)$ is a graded sub-C^*-algebra of $\mathrm{L}_B(E)$ and $J_A(E)$ is a closed $*$-invariant, graded ideal in $Q_A(E)$ containing the compact operators. By definition, if $(E, \phi, F) \in \mathbb{E}(A, B)$ then $F \in Q^{(1)}$ and $(F - F^*), (F^2 - 1) \in J$. So $Q_A(E)$ and $J_A(E)$ can be used to rephrase the definition of $\mathbb{E}(A, B)$.

Lemma 2.4. *Let $q \in Q_A(E)^{(0)}$ satisfying $\forall a \in A : \ \phi(a)q\phi(a)^* \geq 0 \mod \mathrm{K}_B(E)$. Then $q \geq 0 \mod J_A(E)$.*

Proof. We first show that $q - q^* \in J$, i.e. q is self-adjoint modulo J. Let $b \in A$ be positive and find $a \in A$ such that $b = aa^*$. Because $[q - q^*, \phi(a)] \in \mathrm{K}_B(E)$ we have

$$(q-q^*)\phi(b) = (q-q^*)\phi(a)\phi(a^*) = \phi(a)(q-q^*)\phi(a)^* = \phi(a)q\phi(a)^* - (\phi(a)q\phi(a)^*)^* = 0 \mod \mathrm{K}_B(E).$$

As every element of A can be written as the sum of four positive elements we have shown that q is self-adjoint modulo $J_A(E)$. So w.l.o.g. let q be self-adjoint. Then there are unique $q_+, q_- \in Q_A(E)$ such that $q_\pm \geq 0$, $q_+ - q_- = q$ and $q_+q_- = q_-q_+ = 0$. But $\phi(a)q_\pm\phi(a)^* \geq 0 \mod \mathrm{K}_B(E)$ and

$$(\phi(a)q_\pm\phi(a)^*)(\phi(a)q_\mp\phi(a)^*) = \phi(a)\phi(a^*a)q_\pm q_\mp\phi(a)^* = 0 \mod \mathrm{K}_B(E)$$

for every $a \in A$. So $\phi(a)q_-\phi(a)^*$ is the negative part of $\phi(a)q\phi(a)^*$ for every $a \in A$. Therefore $\phi(a)q_-\phi(a)^*$ is in $\mathrm{K}_B(E)$. As above we conclude that $q_- \in J$. So $q \geq 0 \mod J$.

Note that it suffices to have $\forall a \in A : \ \phi(a)q\phi(a)^* \geq 0 \mod J$ □

Proposition 2.5. [1] Let A and B be C^*-algebras, $\mathcal{E} = (E, \phi, F)$ and $\mathcal{E}' = (E, \phi, F')$ elements of $\mathbb{E}(A, B)$. Then the following condition is sufficient for $\mathcal{E}$ and $\mathcal{E}'$ to be operator-homotopic:

(4) $$\forall a \in A : \ \phi(a)[F, F']\phi(a)^* \geq 0 \mod \mathrm{K}_B(E).$$

Proof. Let $Q := Q_A(E)$ and $J := J_A(E)$. As a first approximation let's define for every $t \in [0, 1]$:

$$\tilde{F}_t := \sqrt{t}F' + \sqrt{1-t}F \in Q.$$

We choose the square root because we would like to facilitate the calculations involving $\tilde{F}_t^2$. One could also use the functions sin and cos. The family $(\tilde{F}_t)_{t\in[0,1]}$ has surely the property that $\tilde{F}_0 = F$ and $\tilde{F}_1 = F'$. Moreover, $\tilde{F}_t$ is of degree 1 for every $t \in [0, 1]$ and

$$\tilde{F}_t - \tilde{F}_t^* = \sqrt{t}(F' - F'^*) + \sqrt{1-t}(F - F^*) \in J.$$

To prove that $(E, \phi, \tilde{F}_t)$ the only thing that is left to check is the condition on the square of the operator:

$$\tilde{F}_t^2 = tF'^2 + (1-t)F^2 + \sqrt{t(1-t)}\,\underbrace{(FF' + F'F)}_{=[F,F']} = (t + (1-t))\,1 + \sqrt{t(1-t)}[F, F'] \mod J.$$

So $\tilde{F}_t$ will not do, but the idea is to normalize it. This requires that $1 + \sqrt{t(1-t)}[F, F']$ is positive and invertible so that we can take $(\cdot)^{-1/2}$. To this end we approximate $[F, F']$ by a positive operator, commuting with F and F' modulo J.

[1] cf. lemma 2.1.18 in [JT91].

We have $[F, F'] \in Q$ because

$$[[F, F'], \phi(a)] = -(-1)^{\deg a}[[F', \phi(a)], F] - [[\phi(a), F], F']$$

for every homogeneous $a \in A$. So we can apply the preceding lemma to get that $[F, F'] = p + j$ where $p \in Q^{(0)}$, $p \geq 0$ and $j \in J^{(0)}$. But p commutes with F and $F' \mod J$ because:

$$Fp = F[F, F'] = FFF' + FF'F \overset{F^2=1 \mod J}{=} F' + FF'F \mod J$$

and

$$pF = [F, F']F = FF'F + F'FF \overset{F^2=1 \mod J}{=} FF'F + F' \mod J.$$

Similarly for F'. Now define

$$F_t := (1 + \sqrt{t(1-t)}p)^{-1/2}\tilde{F}_t \in Q$$

for every $t \in [0, 1]$. Then $F_t^2 = 1 \mod J$. We also have

$$F_t - F_t^* = (1 + \sqrt{t(1-t)}p)^{-1/2}(\tilde{F}_t - \tilde{F}_t^*) = 0 \mod J.$$

□

2.2 The technical lemma

Definition 2.6. If B is a graded C^*-algebra with grading automorphism β_B, and A is a sub-C^*-algebra of B we say that A is a graded sub-C^*-algebra if $\beta_B(A) \subseteq A$. Note that a graded sub-C^*-algebra is itself a graded C^*-algebra with the induced grading.

In the following, all sub-C^*-algebras are supposed to be graded.

Definition 2.7. Let B be a C^*-algebra, A a sub-C^*-algebra and $\mathcal{F}$ a subset of B. We say that $\mathcal{F}$ derives A if

$$\forall a \in A\, \forall f \in \mathcal{F} : \; [f, a] \in A.$$

Here we use the graded commutator.

The following theorem is known as Kasparov's technical lemma:

Theorem 2.8. [2] *Let B be a graded σ-unital C^*-algebra, let A_1, A_2 be σ-unital sub-C^*-algebras of $\mathrm{M}(B)$ and let $\mathcal{F}$ be a separable closed linear subspace of $\mathrm{M}(B)$ such that $\beta_B(\mathcal{F}) = \mathcal{F}$. Assume that*

1. *$A_1A_2 \subseteq B$, i.e. A_1 and A_2 are orthogonal $\mod B$, and*
2. *$[\mathcal{F}, A_1] \subseteq A_1$, i.e. $\mathcal{F}$ derives A_1.*

Then there exist elements $M, N \in \mathrm{M}(B)$ of degree 0 such that $M + N = 1$, $M, N \geq 0$, $MA_1 \subseteq B$, $NA_2 \subseteq B$ and $[N, \mathcal{F}] \subseteq B$.

Remark 2.9. Note that the larger A_1, A_2 and $\mathcal{F}$, the stronger the lemma. For example, we can always assume that $B \subseteq A_1, A_2$ because we can replace A_i by $A_i + B$. Note that $A_i + B$ is a graded sub-C^*-algebra and that it is σ-unital, because if b is a strictly positive element in B and a_i is a strictly positive element in A_i then $b + a_i$ is a strictly positive element in $A_i + B$. To see this note that $b + a_i$ is positive and $(a_i + b)(A_i + B) \supseteq a_iA_i + bB$ where the latter set is dense in $A_i + B$.

So the interesting part of A_i is the part not contained in B.

Following [Bla98] one can rephrase the technical lemma as follows:

If D_1 and D_2 are orthogonal σ-unital sub-C^*-algebras of $\mathrm{Q}(B) := \mathrm{M}(B)/B$, i.e. of the outer multiplier algebra of B, and if $\mathcal{F}$ is a separable subspace of $\mathrm{Q}(B)$ which derives D_1, then there is a positive element M of $\mathrm{Q}(B)$, of norm 1, commuting with $\mathcal{F}$, which is a unit for D_2 and orthogonal to D_1.

[2] cf. [JT91], Theorem 2.2.1

2.2.1 Special cases

Corollary 2.10. [3] Let B be a graded C^*-Algebra and E be a countably generated graded Hilbert B-module. Let $\mathrm{L}_B(E)$ have the grading β_E induced by the grading of E.

Let A_1 and A_2 be graded σ-unital sub-C^*-algebras of $\mathrm{L}_B(E)$ and $\mathcal{F} \subseteq \mathrm{L}_B(E)$ be a separable closed linear subspace such that $\beta_E(\mathcal{F}) = \mathcal{F}$. Assume

1. $A_1 A_2 \subseteq \mathrm{K}_B(E)$.
2. $[\mathcal{F}, A_1] \subseteq A_1$.

Then there exist $M, N \in \mathrm{L}_B(E)$ of degree 0 such that $M + N = 1$, $M, N \geq 0$, $MA_1 \subseteq \mathrm{K}_B(E)$, $NA_2 \subseteq \mathrm{K}_B(E)$ and $[\mathcal{F}, M] \subseteq \mathrm{K}_B(E)$.

Proof. The C^*-algebra $\mathrm{K}_B(E)$ is σ-unital, because E is countably generated. Now apply the technical lemma to $(\mathrm{K}_B(E), A_1, A_2, \mathcal{F})$ instead of $(B, A_1, A_2, \mathcal{F})$ where we identify $\mathrm{L}_B(E)$ with $\mathrm{M}(\mathrm{K}_B(E))$. □

Proposition 2.11. Let X be a topological space. Then the following are equivalent:

1. Every two disjoint open sets have disjoint closures.
2. The closures of open sets are open.
3. For every two disjoint open sets U and V there is a clopen set W containing U such that W and V are disjoint.
4. For every two disjoint open sets U and V there is a continuous function on X taking values in $[0, 1]$ that vanishes on U and equals 1 on V.

Proof. 1. $\Rightarrow$ 2.: Let $U \subseteq X$ be open. The interior V of $X \setminus U$ is open. The closure of U is $X \setminus V$. Because U and V are disjoint, so are their closures. Hence V is closed and therefore $\overline{U}$ is open.

The rest is completely trivial. □

Definition 2.12. A topological space X is called stonean if one of the equivalent conditions of the preceding proposition are satisfied.

Corollary 2.13. If X is a locally compact, σ-compact topological space then the corona space $\partial X := \beta X \setminus X$ is stonean.

Proof. Let $B := \mathcal{C}_0(X)$. Then $\mathrm{M}(B) = \mathcal{C}(\beta X)$. Let U_1 and U_2 be disjoint open sets in ∂X. Then there are open sets $\tilde{U}_1$ and $\tilde{U}_2$ in βX such that $U_i = \partial X \cap \tilde{U}_i$ and therefore $\tilde{U}_1 \cap \tilde{U}_2 \subseteq X$. Define $A_i := \{f \in \mathcal{C}(\beta X) : \operatorname{supp} f \subseteq \tilde{U}_i\}$ where $\operatorname{supp} f := \{x \in \beta X : f(x) \neq 0\}$. Then $A_1 A_2 \subseteq \{f \in \mathcal{C}(\beta X) : \operatorname{supp} f \subseteq \tilde{U}_1 \cap \tilde{U}_2\} \subseteq \mathcal{C}_0(X)$. So the conditions of the technical lemma are satisfied with $\mathcal{F} = 0$ and trivial grading. Thus we can find $M, N \in \mathcal{C}(\beta X)$ such that $M, N \geq 0$, $M + N = 1$, $MA_1 \subseteq \mathcal{C}_0(X)$ and $NA_2 \subseteq \mathcal{C}_0(X)$. Define $m := M|_{\partial X}$ and $n := N|_{\partial X}$. For every $x \in U_1$ there exists an $f \in A_1$ such that $f(x) \neq 0$ we can deduce that $m|_{U_1} = 0$. Similarly, $n|_{U_2} = 0$ and therefore $m|_{U_2} = 1$. So we have shown that ∂X is stonean. □

2.2.2 The proof of the lemma

Lemma 2.14. *Let D be a C^*-algebra.*

1. *D is separable if and only if it is generated as a C^*-algebra by a countable subset of D.*
2. *If D is separable, then every approximate unit for D contains a countable approximate unit.*

Lemma 2.15. *Let D be a C^*-algebra with grading automorphism β_D.*

1. *If d is a strictly positive element of D, then so is $d + \beta_D(d)$.*

[3] cf. [JT91], Corollary 2.2.3

2. *If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for D, then so is $\frac{1}{2}\left(u_\lambda + \beta_D(u_\lambda)\right)_{\lambda\in\Lambda}$.*

Definition 2.16. Suppose that D is a C^*-algebra, C a closed ideal of D. Then an approximate unit $(u_\lambda)_{\lambda\in\lambda}$ for C is called quasi-central for D if

$$\forall d \in D : \lim_{\lambda\in\Lambda} [u_\lambda, d] = 0$$

Here we use the graded commutator.

Lemma 2.17. [4] *Let C be a C^*-algebra, contained as a closed ideal in a C^*-algebra D. There exists an approximate unit for C consisting of elements of degree 0 which is quasi-central for D. If D is separable, then the approximate unit can be chosen to be a sequence.*

Lemma 2.18. [5] *Let C be a C^*-algebra. Then for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x, y \in C$, $\|x\|, \|y\| \leq 1$, $x \geq 0$, $\deg x = 0$:*

$$\|[x, y]\| < \delta(\varepsilon) \;\Rightarrow\; \left\|[\sqrt{x}, y]\right\| < \varepsilon.$$

Lemma 2.19. [6] *Let C be a C^*-algebra, $(x_n)_{n\in\mathbb{N}}$ a bounded sequence of self-adjoint elements in $\mathrm{M}(C)$ and S subset of C such that the closed right ideal spanned by S in C is C itself (this holds for example if S contains a strictly positive element). Then $(x_n)_{n\in\mathbb{N}}$ converges strictly in $\mathrm{M}(C)$ if and only if $(x_n s)_{n\in\mathbb{N}}$ is a norm-Cauchy sequence in C for all $s \in S$. If all x_n are positive or of degree 0, then so is their limit.*

Proof of the technical lemma. Define $\mathcal{G} := \mathrm{C}^*(\mathcal{F} \cup \{1\})$. Then $\mathcal{G}$ is separable. The norm closure A_1' of $\mathcal{G}A_1$ is a β_E-invariant C^*-algebra containing A_1. Moreover, A_1' satisfies $A_1'A_2 \subseteq B$ as well as $\mathcal{F}A_1' \subseteq \mathcal{G}A_1' \subseteq A_1'$. Furthermore, A_1' is σ-unital since every approximate unit for A_1 is also an approximate unit for A_1'. So if we replace A_1 by A_1', we have improved $[\mathcal{F}, A_1] \subseteq A_1$ to $\mathcal{F}A_1 \subseteq A_1$ and we can even assume that $\mathcal{F}$ is a separable C^*-algebra.

Let b, a_1, a_2 be strictly positive elements of norm ≤ 1 of B, A_1 and A_2, respectively. Let $F := \{x_1, x_2, \ldots\}$ be a countable subset of $\mathcal{F}$ which spans a dense subspace of $\mathcal{F}$ such that $\forall n \in \mathbb{N} : \|x_n\| \leq 1$ and $\lim_{n\longrightarrow\infty} x_n = 0$. Now it suffices to find a degree 0 element $N \in \mathrm{M}(B)$ such that $0 \leq N \leq 1$, $a_1 - Na_1 \in B$, $Na_2 \in B$, and $[N, F] \subseteq B$.

Define $A := \mathrm{C}^*(\mathcal{F} \cup \{a_1\})$, and let I be the closed two-sided ideal in A generated by a_1. Then A is separable since $\mathcal{F}$ is, I is separable since A is, and $I \subseteq A_1$ since $\mathcal{F}A_1 \subseteq A_1$. So I contains a countable approximate unit $(u_n)_{n\in\mathbb{N}}$ for I of degree 0 elements which is quasi-central for A.

By passing to a subsequence we can assume that for all $n \in \mathbb{N}$:

(a) $\|u_n a_1 - a_1\| < 2^{-n}$, and

(b) $\forall f \in F : \|[u_n, f]\| < 2^{-n}$.

Define $C := \mathrm{C}^*(F \cup \{b, a_1, a_2, u_1, u_2, \ldots\})$ and let J be the closed two-sided ideal in C generated by b. Then C and J are separable and J is contained in B. Then we can find a countable approximate unit $(v_n)_{n\in\mathbb{N}}$ for J of degree 0 elements which is quasi-central for C. Note that $(v_n)_{n\in\mathbb{N}}$ is also an approximate unit for B because $b \in J$.

By passing to a subsequence we can assume that for all $n \in \mathbb{N}$:

(c) $\forall x \in \{b, a_2u_n, a_2u_{n+1}\} \subseteq B \cap C : \|v_n x - x\| < 2^{-2n}$, and

(d) $\forall x \in F \cup \{a_1, a_2, b\} : \|[v_n, x]\| < \delta(2^{-(n+1)})/2$,

where $\delta(2^{-n}) > \delta(2^{-(n+1)}) > 0$ is the δ from lemma 2.18.

Define $d_1 := v_1$ and for all $n \in \mathbb{N}_{>1}$:

$$d_n := (v_n - v_{n-1})^{1/2}.$$

Then by lemma 2.18 we have

[4] cf. [Bla98], 12.4.1 and [Ped79], 3.12.14.

[5] cf. [JT91], 2.2.2.

[6] cf. [Bla98], proposition 12.1.2

(d') $\forall x \in F \cup \{a_1, a_2, b\} : \; \|[d_n, x]\| < 2^{-n}$.

We want to define
$$N := \sum_{n\in\mathbb{N}} d_n u_n d_n.$$

For every $k \in \mathbb{N}$ we have
$$0 \leq \sum_{n=1}^{k} d_n u_n d_n \leq \sum_{n=1}^{k} d_n d_n = v_k,$$
so the partial sums are bounded in norm by 1. Note that
$$\|(v_n - v_{n-1})b\| = \|v_n b - b + b - v_{n-1} b\| \leq 2^{-2n} + 2^{-2(n-1)} = 5 \cdot 2^{-2n}.$$

This yields
$$\|d_n u_n b d_n\|^2 = \left\|d_n u_n b d_n^2 b u_n d_n\right\| = \|d_n u_n b(v_n - v_{n-1}) b u_n d_n\| \leq 5 \cdot 2^{-2n}$$
and hence
$$\|d_n u_n b d_n\| \leq \sqrt{5} \cdot 2^{-n}.$$

It follows that
$$\|d_n u_n d_n b\| \leq \|d_n u_n b d_n\| + \|d_n u_n\| \, \|[b, d_n]\| \leq (\sqrt{5} + 1) 2^{-n}.$$

So $\sum_{n=1}^{\infty} d_n u_n d_n b$ converges in norm. So $\sum_{n=1}^{\infty} d_n u_n d_n$ converges strictly to some operator $N \in \mathrm{M}(B)$, where $\deg N = 0$ and $0 \leq N \leq 1$. Since multiplication is separately strictly continuous we have that
$$a_1 - N a_1 = \sum_{n=1}^{\infty} (d_n^2 - d_n u_n d_n) a_1,$$
$$N a_2 = \sum_{n=1}^{\infty} d_n u_n d_n a_2,$$
and
$$\forall x \in F : \; [N, x] = \sum_{n=1}^{\infty} [d_n u_n d_n, x].$$

We show that these series converge in norm, because all the terms which are summed up are contained in B ($d_n \in B$!). To this end we rewrite the terms:

(I) $(d_n^2 - d_n u_n d_n) a_1 = d_n(1 - u_n)[d_n, a_1] + d_n(a_1 - u_n a_1) d_n$,

(II) $d_n u_n d_n e_2 = d_n u_n [d_n, e_2] + d_n u_n e_2 d_n$, and

(III) $[d_n u_n d_n, x] = d_n u_n [d_n, x] + [d_n, x] u_n d_n + d_n [u_n, x] d_n$.

The norm of (I) is $\leq 2 \cdot 2^{-n}$ by (a) and (d'). The norm of (II) is $\leq (1 + \sqrt{5}) 2^{-n}$ by (d') and (c). Finally, the norm of (III) is $\leq 3 \cdot 2^{-n}$ by (d') and (b). □

2.3 Connections

2.3.1 Definition

In this section, let B, C be graded C^*-algebras, E_1 a Hilbert B-module, E_2 an Hilbert B-C-bimodule where $\phi : B \longrightarrow \mathrm{L}_C(E_2)$ denotes the action of B. Let $E_{12} := E_1 \otimes_B E_2$ be the graded tensor product of E_1 and E_2.

Definition 2.20. Let F_2 be an operator on E_2. Then we say that F_2 is B-linear (in the graded sense) if

$$\forall b \in B:\ [F_2, \phi(b)] = 0.$$

We say that F_2 is B-linear up to compact operators if

$$\forall b \in B:\ [F_2, \phi(b)] \in \mathrm{K}_C(E_2).$$

So one of the conditions of (E_2, ϕ, F_2) being a Kasparov triple is exactly that F_2 is B-linear up to compact operators.

Remark 2.21. If F_2 is homogeneous and B-linear, then indeed we have $F_2(bx) - (-1)^{\deg b \deg F_2} bF_2(x) = 0$ for every homogeneous $b \in B$, $x \in E_2$, i.e.

$$F_2(bx) = bF_2(x) \text{ if } F_2 \text{ is even} \quad \text{and} \quad F_2(bx) = \beta_B(b)F_2(x) \text{ if } F_2 \text{ is odd.}$$

Here we abbreviate $\phi(b)x$ by bx.

If F_2 is even, the operator $1 \otimes_B F_2$ is well-defined. If F_2 is odd, we can at least make sense of the expression $S_{E_1} \otimes_B F_2$ because for every $x_1 \in E_1$, $x_2 \in E_2$, $b \in B$:

$$S_{E_1}(x_1 b) \otimes_B F_2(x_2) = S_{E_1}(x_1)\beta_B(b) \otimes_B F_2(x_2) = S_{E_1}(x_1) \otimes_B \beta_B(b)F_2(x_2) = S_{E_1}(x_1) \otimes_B F_2(bx_2).$$

If $F_2 = F_2^{(0)} + F_2^{(1)}$, where $\deg F_2^{(i)} = i$, we define

$$1 \otimes_B F_2 := 1 \otimes_B F_2^{(0)} + S_{E_1} \otimes_B F_2^{(1)}$$

as a short-hand notation.

Remark 2.22. If F_2 is just B-linear up to compact operators, we cannot expect the expressions $1 \otimes_B F_2^{(0)}$ or $S_{E_1} \otimes_B F_2^{(1)}$ to make sense, but we can at least try to get a substitute for these operators. That is: We have to list some of the properties that $1 \otimes_B F_2$ possesses in the B-linear case and to then construct some operator that has these properties in the general case. The topic of this section is to describe this construction and to say to what extend the result is unique.

Definition 2.23. For any $x \in E_1$ set

$$T_x \colon E_2 \longrightarrow E_{12},\ e_2 \mapsto x \otimes e_2.$$

T_x is called an E_2-tensor operator for E_1

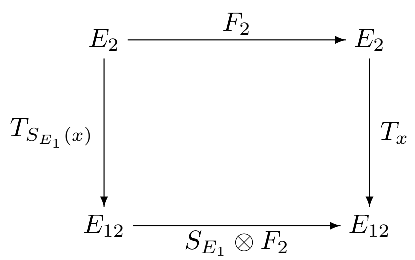
Remark 2.24. For all $x \in E_1$ we have $T_x \in \mathrm{L}_C(E_2, E_{12})$. The operator T_x^* is given by $T_x^*(e_1 \otimes e_2) = \phi(\langle x, e_1\rangle)e_2$. If we regard $\mathrm{L}_C(E_2, E_{12})$ as a right B-module, then the map $x \mapsto T_x$ is B-linear, $\|T_x\| \le \|x\|$ and T_x has the same degree as x whenever x is homogeneous. Note that whenever E_1' is another Hilbert B-module and $S \in \mathrm{L}_B(E_1, E_1')$ then

$$(5) \qquad \forall x \in E_1:\ T_{Sx} = (S \otimes_B 1)T_x \quad \text{and} \quad \forall x' \in E_1':\ T_{x'}^*(S \otimes_B 1) = T_{S^*x'}^*.$$

Now how do the operators T_x and $1 \otimes_B F_2$ interact if F_2 is B-linear? If F_2 is odd and $x \in E_1$, $y \in E_2$ then

$$T_x F_2(y) = x \otimes F_2(y) = (S_{E_1} \otimes F_2)(S_{E_1}(x) \otimes y) = (S_{E_1} \otimes F_2)T_{S_{E_1}(x)}(y),$$

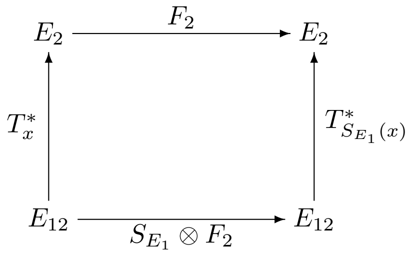
i.e the following diagram is commutative:



Similarly,

$$F_2(T_x^*(y\otimes z)) = F_2(\langle x, y\rangle z) = \beta_B(\langle x, y\rangle)F_2(z) = T^*_{S_{E_1}(x)}\left(S_{E_1}(y)\otimes F_2(z)\right) = T^*_{S_{E_1}(x)}\left(S_{E_1}\otimes F_2\right)(y\otimes z),$$

or equivalently, the diagram



commutes.

For an elegant description of this define

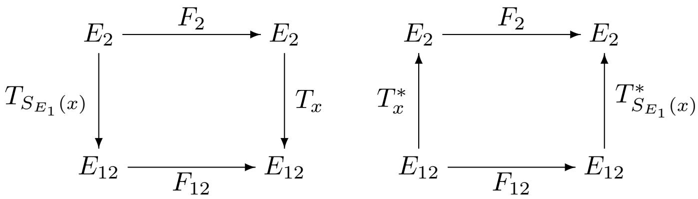
$$\tilde{T}_x := \begin{pmatrix} 0 & T_x^* \\ T_x & 0 \end{pmatrix} \quad \text{and} \quad \tilde{F_{12}} := \begin{pmatrix} F_2 & 0 \\ 0 & F_{12} \end{pmatrix} \in \mathrm{L}_C(E_2 \oplus E_{12}),$$

for every $x \in E_1$ and $F_{12} \in \mathrm{L}_C(E_{12})$. Then the commutativity of the two diagrams is equivalent to the formula $\left[\widetilde{1\otimes_B F_2}, \tilde{T}_x\right] = 0$ for every $x \in E_1$. A short calculation shows that this is also true if F_2 is even. Hence the following definition:

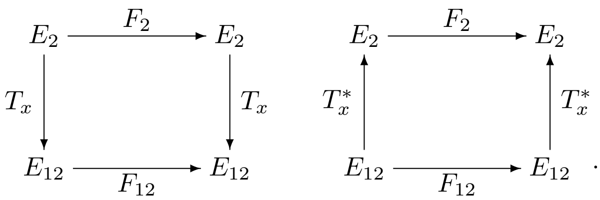
Definition 2.25 (Connection)**.** Let $F_2 \in \mathrm{L}_C(E_2)$. Then an operator $F_{12} \in \mathrm{L}_C(E_{12})$ is called an F_2-connection for E_1 (or an F_2-connection on E_{12}) if, for all $x \in E_1$,

$$\left[\tilde{F}_{12}, \tilde{T}_x\right] \in \mathrm{K}_C\left(E_2 \oplus E_{12}\right).$$

If F_2 is odd, it's equivalent to saying that the following diagrams commute for every $x \in E_1$ modulo compact operators



If F_2 is even, it's equivalent to saying that the following diagrams commute for every $x \in E_1$ modulo compact operators



2.3.2 Uniqueness and properties

We start with a lemma that proves useful in general:

Lemma 2.26. [7] *Let D be a C^*-algebra (not necessary σ-unital), X_1, X_2 be Hilbert D-modules. Then*

$$\mathrm{K}_D(X_1, X_2) = \{m \in \mathrm{L}_D(X_1, X_2) : \; mm^* \in \mathrm{K}_D(X_2)\}.$$

[7] cf. [JT91], lemma 1.1.10.

Proof. Obviously, $\mathrm{K}_D(X_1, X_2)$ is contained in the right-hand side. For the other inclusion note that

$$\forall m \in \mathrm{L}_D(X_1, X_2): \ \|m\|^2 = \|mm^*\| .$$

This is easily poved as in the case that $X_1 = X_2$. Now let $(v_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $\mathrm{K}_D(X_2)$. Then for every $m \in \mathrm{L}_D(X_1, X_2)$ and every $\lambda \in \Lambda$:

$$\|v_\lambda m - m\|^2 = \|v_\lambda mm^* v_\lambda - v_\lambda mm^* - mm^* v_\lambda + mm^*\| .$$

This shows that if $mm^* \in \mathrm{K}_D(X_2)$ then $\lim_{\lambda \in \Lambda} v_\lambda m = m$. Since $v_\lambda m \in \mathrm{K}_D(X_1, X_2)$ for all $\lambda \in \Lambda$ this yields $m \in \mathrm{K}_D(X_1, X_2)$. □

Remark 2.27. We can view $\mathrm{L}_D(X_1, X_2)$ as a left Hilbert $\mathrm{L}_D(X_2)$-module. The lemma says that if one only considers those operators m for which the inner product $\langle m, m \rangle = mm^*$ is in $\mathrm{K}_D(X_2)$, then one gets the left Hilbert $\mathrm{K}_D(X_2)$-module $\mathrm{K}_D(X_1, X_2)$.

Proposition 2.28. An operator $F_{12} \in \mathrm{L}_C(E_{12})$ is a 0-connection on E_{12} if and only if

$$\forall T \in \mathrm{K}_B(E_1): \quad F_{12}(T \otimes 1),\ (T \otimes 1)F_{12} \ \in \mathrm{K}_C(E_{12}). \tag{6}$$

Proof. $F_{12} \in \mathrm{L}_C(E_{12})$ is a zero-connection for E_1 if and only if $F_{12}T_x, T_x^* F_{12}$ are compact for all $x \in E_1$. Note that for every $x, y \in E_1$ we have

$$T_x T_y^* = \Theta_{x,y} \otimes_B 1 .$$

Thus if F_{12} is a zero-connection then for all $x, y \in E_1$:

$$F_{12}(\Theta_{x,y} \otimes_B 1) = (F_{12}T_x)T_y^* \in \mathrm{K}_C(E_2),$$

and similarly $(\Theta_{x,y} \otimes_B 1)F_{12} \in \mathrm{K}_C(E_2)$. By linearity and continuity F_{12} satisfies (6).

Conversely, if F_{12} satisfies condition (6), then $F_{12}T_x T_x^* F_{12}^* = F_{12}(\Theta_{x,x} \otimes_B 1)F_{12}^* \in \mathrm{K}_C(E_{12})$ for all $x \in E_1$. Because of lemma 2.26 this yields $F_{12}T_x \in \mathrm{K}_C(E_2, E_{12})$. Similarly for $T_x^* F_{12}$. Thus F_{12} is a zero-connection for E_1. □

Proposition 2.29. Let $F_2, F_2' \in \mathrm{L}_C(E_2)$ and F_{12} be an F_2-connection, and F_{12}' be an F_2'-connection. Then

1. F_{12}^* is an F_2^*-connection, $F_{12}^{(0)}$ is an $F_2^{(0)}$-connection, and $F_{12}^{(1)}$ is an $F_2^{(1)}$-connection.
2. $F_{12}+F_{12}'$ is an (F_2+F_2')-connection and $F_{12}F_{12}'$ is an (F_2F_2')-connection. If F_2 and F_{12} are normal, then $f(F_{12})$ is an $f(F_2)$-connection for every continuous function f such that the spectra of F_2 and F_{12} are contained in its domain of definition.
3. The set of all F_2-connections is an affine space parallel to the space of all 0-connections.
4. If $T \in \mathrm{K}_B(E_1)$, then $[F_{12}, T \otimes 1] \in \mathrm{K}_C(E_{12})$.
5. If F_2 is a "compact perturbation" of zero, i.e. $F_2\phi(B), \phi(B)F_2 \subseteq \mathrm{K}_C(E_2)$, then F_{12} is also a 0-connection.
6. Suppose that E_3 is a Hilbert D-module, $\psi\colon C \longrightarrow \mathrm{L}_D(E_3)$ is a $*$-homomorphism, and $F_3 \in \mathrm{L}_D(E_3)$ with $[F_3, \psi(C)] \subseteq \mathrm{K}_D(E_3)$. If F_{23} is an F_3-connection on $E_2 \otimes_C E_3$, and F is an F_{23}-connection on $E = E_1 \otimes_B (E_2 \otimes_C E_3)$, then F is an F_3-connection on $E \cong (E_1 \otimes_B E_2) \otimes_C E_3$.

Proposition 2.30. Assume that $E_1 = E_1' \oplus E_1''$. Let $\iota'\colon E_1' \longrightarrow E_1$ be the canonical embedding and $\pi'\colon E_1 \longrightarrow E_1'$ be the canonical projection. Note that ι', π' have degree zero and $\pi'\iota' = \mathrm{Id}_{E_1'}$. Moreover we have $\pi'^* = \iota'$. Similar statements hold for ι'' and π''.

Assume that $F_2 \in \mathrm{L}_C(E_2)$.

1. If F_{12} is an F_2-connection for E_1, then

$$F'_{12} := (\pi' \otimes_B 1)F_{12}(\iota' \otimes_B 1) \in \mathrm{L}_C(E'_1 \otimes_B E_2)$$

is an F_2-connection for E'_1. Similarly one can define an F_2-connection F''_{12} for E''_1.

2. If F'_{12} and F''_{12} are F_2-connections for E'_1 and E''_1, respectively, then

$$F_{12} := (\iota' \otimes_B 1)F'_{12}(\pi' \otimes_B 1) + (\iota'' \otimes_B 1)F''_{12}(\pi'' \otimes_B 1)$$

is an F_2 connection for E_1.

Proof. W.l.o.g. let F_2 be homogeneous.

1. Let F_{12} be an F_2-connection for E_1. Assume w.l.o.g. that F_{12} is homogeneous. Let F'_{12} be defined as above. Then F'_{12} is homogeneous and of the same degree as F_2 and F_{12}. Let $x \in E'_1$ be homogeneous. Then

$$\begin{aligned} F'_{12}T_x &= (\pi' \otimes_B 1)F_{12}(\iota' \otimes_B 1)T_x = (\pi' \otimes_B 1)F_{12}T_{\iota' x} \\ &= (-1)^{\deg x \deg F_2}(\pi' \otimes_B 1)T_{\iota' x}F_2 = (-1)^{\deg x \deg F_2}T_xF_2 \mod \mathrm{K}_C(E_2, E_{12}) \end{aligned}$$

and

$$\begin{aligned} T_x^*F'_{12} &= T_x^*(\pi' \otimes_B 1)F_{12}(\iota' \otimes_B 1) = T^*_{\iota'(x)}F_{12}(\iota' \otimes_B 1) \\ &= (-1)^{\deg x \deg F_2}F_2T^*_{\iota'(x)}(\iota' \otimes_B 1) = (-1)^{\deg x \deg F_2}F_2T_x \mod \mathrm{K}_C(E_{12}, E_2). \end{aligned}$$

2. Let F'_{12}, F''_{12} and F_{12} be as in the second statement of the proposition. Without loss of generality we can assume that they are all homogenous. Let $x \in E_1$ be homogeneous. Then $x = \iota'(\pi'(x)) + \iota''(\pi''(x))$ and

$$\begin{aligned} (\iota' \otimes_B 1)F'_{12}(\pi' \otimes_B 1)T_x &= (\iota' \otimes_B 1)F'_{12}T_{\pi'(x)} = (-1)^{\deg x \deg F_2}(\iota' \otimes_B 1)T_{\pi'(x)}F_2 \\ &= (-1)^{\deg x \deg F_2}T_{\iota'(\pi'(x))}F_2 \mod \mathrm{K}_C(E_2, E_{12}) \end{aligned}$$

as well as

$$\begin{aligned} T_x^*(\iota' \otimes_B 1)F'_{12}(\pi' \otimes_B 1) &= T^*_{\pi'(x)}F'_{12}(\pi' \otimes_B 1) = (-1)^{\deg x \deg F_2}F_2T^*_{\pi'(x)}(\pi' \otimes_B 1) \\ &= (-1)^{\deg x \deg F_2}F_2T^*_{\iota'(\pi'(x))} \mod \mathrm{K}_C(E_{12}, E_2). \end{aligned}$$

and similar for the E''_1 part. It follows that

$$\begin{aligned} F_{12}T_x &= (\iota' \otimes_B 1)F'_{12}(\pi' \otimes_B 1)T_x + (\iota'' \otimes_B 1)F''_{12}(\pi'' \otimes_B 1)T_x. \\ &= (-1)^{\deg x \deg F_2}T_{\iota'(\pi'(x))}F_2 + (-1)^{\deg x \deg F_2}T_{\iota''(\pi''(x))}F_2 \\ &= (-1)^{\deg x \deg F_2}\left(T_{\iota'(\pi'(x))} + T_{\iota''(\pi''(x))}\right)F_2 \\ &= (-1)^{\deg x \deg F_2}\left(T_{\iota'(\pi'(x))+\iota''(\pi''(x))}\right)F_2 \\ &= (-1)^{\deg x \deg F_2}T_xF_2 \mod \mathrm{K}_C(E_2, E_{12}). \end{aligned}$$

Analogously for the other equation.

□

2.3.3 Existence

Example 2.31. If $F_2 \in \mathrm{L}_C(E_2)$ is a B-linear operator, i.e. $[F_2, \phi(B)] = 0$, then $1 \otimes_B F_2 \in \mathrm{L}(E_2 \otimes E_{12})$ defined as above is an F_2-connection for E_1 (and in particular, 0 is a 0-connection for every E_1). If $B = \mathbb{C}$ and ϕ is unital, then this applies in particular.

Example 2.32. Let $\phi\colon B \longrightarrow \mathrm{L}_C(E_2)$ be non-degenerate, $F_2 \in \mathrm{L}_C(E_2)$ be B-linear up to compact operators, and $E_1 = B$. Define

$$\Phi\colon B \otimes_B E_2 \longrightarrow E_2,\ b \otimes_B x \mapsto bx.$$

Because ϕ is non-degenerate Φ is an isomorphism; in particular we have $\Phi^* = \Phi^{-1}$. Note that

$$\phi(b) = \Phi \circ T_b, \quad \text{and hence} \quad \Phi^* \circ \phi(b) = T_b.$$

If we define $F_{12} := \Phi^* F_2 \Phi$, then

$$F_{12}T_b = \Phi^* F_2 \Phi T_b = \Phi^* F_2 \phi(b) = (-1)^{\deg b \deg F_2}\Phi^* \phi(b) F_2 = (-1)^{\deg b \deg F_2} T_b F_2 \mod \mathrm{K}_C(E_2, E_{12}),$$

and

$$T_b^* F_{12} = T_b^* \Phi^* F_2 \Phi = \phi(b^*) F_2 \Phi = (-1)^{\deg b \deg F_2} F_2 \phi(b^*)\Phi = (-1)^{\deg b \deg F_2} F_2 T_b^* \mod \mathrm{K}_C(E_{12}, E_2).$$

Example 2.33. Assume that B is unital, ϕ is non-degenerate, i.e. unital, and $E_1 = \hat{\mathbb{H}}_B$. Let F_2 be B-linear up to compact operators. Then there is a standard F_2-connection. There are two ways of constructing it: directly or by reduction to the previous example. For a direct construction define the isomorphism

$$\Phi\colon \hat{\mathbb{H}}_B \otimes_B E_2 \longrightarrow \hat{\mathbb{H}} \otimes_{\mathbb{C}} B \otimes_B E_2 \longrightarrow \hat{\mathbb{H}} \otimes_{\mathbb{C}} E_2,$$

Note that $\Phi \in \mathrm{L}_C\left(\hat{\mathbb{H}}_B \otimes_B E_2, \hat{\mathbb{H}} \otimes_{\mathbb{C}} E_2\right)$ is unitary. Now there is an F_2-connection G for $\hat{\mathbb{H}}$ using example 2.31. So G is an element of $\mathrm{L}_C\left(\hat{\mathbb{H}} \otimes_{\mathbb{C}} E_2\right)$. W.l.o.g. we can assume that G and F_2 are homogeneous and of the same degree. Define

$$F := \Phi^{-1} \circ G \circ \Phi.$$

Now for every $\xi \in \hat{\mathbb{H}}$ and every $b \in B$ we have

$$\Phi \circ T_{\xi\otimes b} = T_\xi \circ \phi(b) \quad \text{and hence} \quad T^*_{\xi\otimes b} \circ \Phi^{-1} = \phi(b^*) \circ T^*_\xi.$$

Moreover, the degree of $\xi \otimes b$ is $\deg \xi + \deg b$ if b and ξ are homogeneous. Now

$$\begin{aligned}
F \circ T_{\xi\otimes b} &= \Phi^{-1} \circ G \circ \Phi \circ T_{\xi\otimes b} = \Phi^{-1} \circ G \circ T_\xi \circ \phi(b) \\
&= (-1)^{\deg F_2 \deg \xi} \Phi^{-1} \circ T_\xi \circ F_2 \circ \phi(b) \\
&= (-1)^{\deg F_2 \deg \xi} (-1)^{\deg F_2 \deg b} \Phi^{-1} \circ T_\xi \circ \phi(b) \circ F_2 \\
&= (-1)^{\deg F_2 \deg(\xi\otimes b)} \circ T_{\xi\otimes b} \circ F_2 \mod \mathrm{K}_C(E_2, \hat{\mathbb{H}} \otimes_B E_2),
\end{aligned}$$

and

$$\begin{aligned}
T^*_{\xi\otimes b} \circ F &= T^*_{\xi\otimes b} \circ \Phi^{-1} \circ G \circ \Phi = \phi(b^*) \circ T^*_\xi \circ G \circ \Phi \\
&= (-1)^{\deg F_2 \deg \xi} \phi(b^*) \circ F_2 \circ T^*_\xi \circ \Phi \\
&= (-1)^{\deg F_2 \deg(\xi\otimes b)} F_2 \circ \phi(b^*) \circ T^*_\xi \circ \Phi \\
&= (-1)^{\deg F_2 \deg(\xi\otimes b)} F_2 \circ T^*_{\xi\otimes b} \mod \mathrm{K}_C(\hat{\mathbb{H}} \otimes_B E_2, E_2).
\end{aligned}$$

For a prove that uses example 2.32 note that

$$\hat{\mathbb{H}}_B \otimes_B E_2 \cong (\hat{\mathbb{H}} \otimes_{\mathbb{C}} B) \otimes_B E_2 \cong \hat{\mathbb{H}} \otimes_{\mathbb{C}} (B \otimes_B E_2).$$

Then by example 2.32 there is an F_2-connection G on $B \otimes_B E_2$. By 2.31 there is a G-connection F on $\hat{\mathbb{H}} \otimes_{\mathbb{C}} (B \otimes_B E_2)$ and by proposition 2.29 the operator F may be regarded as an F_2-connection.

Proposition 2.34. Let E_1 be a countably generated Hilbert B-module, E_2 a Hilbert B-C-bimodule with B-action ϕ, and $F_2 \in \mathrm{L}_C(E_2)$ such that $[F_2, \phi(b)] \in \mathrm{K}_C(E_2)$ for every $b \in B$. Then there exists an F_2-connection for E_1.

Proof. W.l.o.g. let B and ϕ be unital. A reduction argument for this is given in [JT91], proposition 2.2.5.. Now we can just collect what we have already done: Because E_1 is countably generated we can assume that E_1 is a direct summand of $\hat{\mathbb{H}}_B$. By proposition 2.30 it suffices to consider the case where $E_1 = \hat{\mathbb{H}}_B$. But we have already covered this case in example 2.33. □

Remark 2.35. A careful revision of the above construction shows that we can extend the last proposition in the following way: If $t \mapsto F_2^t$ is a norm-continuous path of operators, being B-linear up to compact operators, then there is a norm-continuous path F_{12}^t, where F_{12}^t is an F_2^t-connection for E_1. If all the F_2^t are homogeneous of degree n or self-adjoint then all the F_{12}^t may be chosen homogeneous of degree n or self-adjoint, respectively.

2.3.4 A first application

Proposition 2.36. Let A and B be graded C^*-algebras. For every $(E, \phi, F) \in \mathbb{E}(A, B)$, there is some $(E', \phi', F') \in \mathbb{E}(A, B)$ with ϕ' non-degenerate and $(E, \phi, F) \sim (E', \phi', F')$.

Proof. Let $E_0 := AE$. Define $\overline{E} := E[0, 1]$ and the sub-$B[0, 1]$-module $\overline{E}_0 := \{f \in E[0, 1] : f(1) \in E_0\}$ as in the proof of lemma 1.7. As above, let $\gamma\colon \mathrm{L}_B(E) \longrightarrow \mathrm{L}_{B[0,1]}(E[0, 1]) \cong \mathrm{L}_B(E)[0, 1]$ be the embedding of $\mathrm{L}_B(E)$ as constant functions. Then $\gamma \circ \phi\colon A \longrightarrow \mathrm{L}_B(E[0, 1])$ is a graded $*$-homomorphism. Again, $\overline{E}_0$ is $\gamma(\phi(A))$-invariant so that $\overline{E}_0$ is a Hilbert A-B-module and

$$\psi_t\colon \overline{E}_0 \otimes_{\mathrm{ev}_t} B \longrightarrow E, f \otimes b \mapsto f(t)b$$

is an isometric left A-linear and right B-linear map respecting the inner product for every $t \in [0, 1]$. It is surjective for every $t < 1$ and has image E_0 for $t = 1$. The only problem is to find an appropriate operator G on $\overline{E_0}$ such that $\overline{E}_0, \gamma \circ \phi, G)$ is in $\mathbb{E}(A, B[0, 1])$. To construct it, we use the existence of connections. Firstly, we rewrite $\overline{E}_0$ as a tensor product of some suitably chosen module and $\overline{E}$:

Define $J := \Big\{f \in \tilde{A}[0, 1] \mid f(1) \in A\Big\}$; J is an ideal in $\tilde{A}[0, 1]$. Let $\omega\colon A \longrightarrow J$ be the embedding as constant functions. If J is regarded as graded Hilbert $\tilde{A}[0, 1]$-module, then $\overline{E}_0 \cong J \otimes_{\tilde{\phi}\otimes 1} \overline{E}$, where $\tilde{\phi}\colon \tilde{A} \longrightarrow \mathrm{L}_B(E)$ is the unital extension of ϕ. Note that the left action of A on $\overline{E}_0$ translates to the action $a \mapsto \omega(a) \otimes 1$.

Let $\tilde{F} := F \otimes 1 \in \mathrm{L}_{B[0,1]}(E[0, 1])$. We check that $\tilde{F}$ is $\tilde{A}[0, 1]$-linear up to compact operators: Let $\lambda \in \mathbb{C}$, $a \in A$ and $f \in \mathcal{C}[0, 1]$. Then

$$\begin{aligned}\Big[\tilde{F}, (\tilde{\phi} \otimes 1)((\lambda 1 + a) \otimes f)\Big] &= \Big[F \otimes 1, \tilde{\phi}(\lambda 1 + a) \otimes f\Big] \\ &= [F, \lambda 1 + \phi(a)] \otimes f = \underbrace{[F, \phi(a)]}_{\in \mathrm{K}_B(E)} \otimes f \in \mathrm{K}_{B[0,1]}(E[0, 1]).\end{aligned}$$

Now we can find an $\tilde{F}$-connection G on $\overline{E}_0$. Then $(\overline{E}_0, \gamma \circ \phi, G) \in \mathbb{E}(A, B[0, 1])$. To see this note that $\omega(a) \in \mathrm{K}_{\tilde{A}[0,1]}(J)$, and therefore, by proposition 2.29, 4:

$$[G, \omega(a) \otimes 1] \in \mathrm{K}_{B[0,1]}(\overline{E_0}).$$

Moreover, $(G^2 - 1)$ is a $(\tilde{F}^2 - 1)$-connection. So by 2.29, 5, $(G^2 - 1)$ is a 0-connection, too. The same holds for $G^* - G$. Hence:

$$\forall a \in A:\ (G^2 - 1)(\omega(a) \otimes 1),\ (G^* - G)(\omega(a) \otimes 1)\ \in\ \mathrm{K}_{B[0,1]}(\overline{E_0})$$

by proposition 2.28.

The restriction to zero gives a Kasparov (A, B)-module of the form $(\tilde{A} \otimes_{\tilde{A}} E, j \otimes 1, G_0)$, where $j\colon A \longrightarrow \tilde{A}$ is the inclusion. When $\tilde{A} \otimes_{\tilde{A}} E$ is identified with E, this triple becomes (E, ϕ, F_0), where F_0 is a compact perturbation of F. Similarly, the restriction to one gives a Kasparov (A, B)-module of the form $(A \otimes_{\tilde{\phi}} E, \mathrm{Id}_A \otimes 1, G_1)$; under the isomorphism $A \otimes_A E \cong E_0$ this triple becomes (E_0, ϕ, H) for some operator H. □

3 Definition of the product

In this section, let A, B, C be graded C^*-algebras, $\mathcal{E}_1 = (E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. Let $E_{12} := E_1 \otimes_B E_2$ be the graded tensor product of E_1 and E_2 and ϕ_{12} the action of A on E_{12}.

Definition 3.1 (Kasparov product)**.** $\mathcal{E}_{12} = (E_{12}, \phi_{12}, F_{12})$ is called a Kasparov product for (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) if

1. $(E_{12}, \phi_{12}, F_{12})$ is a Kasparov (A, C)-bimodule,
2. F_{12} is an F_2-connection on E_{12}, and
3. $\forall a \in A: \ \phi_{12}(a)[F_1 \otimes 1, F_{12}]\phi_{12}(a)^* \geq 0 \mod \mathrm{K}_C(E_{12})$.

The set of all operators F_{12} on E_{12} such that $(E_{12}, \phi_{12}, F_{12})$ is a Kasparov product of F_1 abd F_2 is denoted by $F_1 \sharp F_2$.

We are going to prove the following theorem:

Theorem 3.2. *Assume that A is separable. Then there exists a Kasparov product $\mathcal{E}_{12}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. It is unique up to operator homotopy, and the operator F_{12} can be chosen to be self-adjoint if F_1 and F_2 are self-adjoint.*

This theorem ensures that the map sending $\mathcal{E}_1$ and $\mathcal{E}_2$ to $[\mathcal{E}_{12}] \in \mathrm{KK}(A, C)$ exists and is well-defined. We then have to show that homotopic modules have homotopic Kasparov products in order to be able to pass to a map on the level of KK-elements.

3.1 Existence in some special cases

Example 3.3. Assume that $f\colon A \longrightarrow B$ is a homomorphism and $(E_1, \phi_1, F_1) = (B, f, 0) = (f)$. Assume moreover that ϕ_2 is non-degenerate (by proposition 2.36 we can always arrange this within any given KK-equivalence class). Then there is an isomorphism $\Phi\colon B \otimes_B E_2 \simeq E_2$. As in example 2.32, $\Phi^* F_2 \Phi$ is an F_2-connection on $B \otimes E_2$, and $\Phi^* F_2 \Phi \in 0 \sharp F_2$. In other words, $(B \otimes_B E_2, f \otimes 1, \Phi^* F_2 \Phi)$ is a Kasparov product for $(B, f, 0)$ and (E_2, ϕ_2, F_2). It is obviously isomorphic to $(E_2, \phi_2 \circ f, F_2)$. But this cycle is actually $f^*(\mathcal{E}_2)$. So if we have proved that the Kasparov product is well-defined on the level of KK-elements, we can conclude that

$$\forall y \in \mathrm{KK}(B, C): \ [f] \otimes_B y = f^*(y).$$

Example 3.4. Assume that $F_2 = 0$. Then $F_1 \otimes 1$ is a 0-connection on E_{12} because for every $T \in \mathrm{K}_B(E_1)$ we have $(T \otimes 1)(F_1 \otimes 1) = (TF_1) \otimes 1 \in \mathrm{K}_B(E_1) \otimes 1 \subseteq \mathrm{K}_C(E_{12})$, where the last inclusion follows from the fact that $\phi_2(B) \subseteq \mathrm{K}_C(E_2)$. Similarly, one shows that $(F_1 \otimes 1)(T \otimes 1) \in \mathrm{K}_C(E_{12})$. Hence $(E_{12}, \phi_{12}, F_1 \otimes 1)$ is a Kasparov product because the third condition of the definition is trivially satisfied, and the second follows from the fact that $\mathrm{K}_B(E_1) \otimes 1 \subseteq \mathrm{K}_C(E_{12})$.

Example 3.5. As a special case of the preceding example consider $F_1 = F_2 = 0$. Then $(E_1 \otimes_B E_2, \phi_1 \otimes 1, 0)$ is a Kasparov product of $(E_1, \phi_1, 0)$ and $(E_2, \phi_2, 0)$. So we have proved

$$[(E_1, \phi_1, 0)] \otimes_B [(E_2, \phi_2, 0)] = [(E_1, \phi_1, 0) \otimes_B (E_2, \phi_2, 0)],$$

if the Kasparov product is well-defined on KK-level.

Example 3.6. As another special case of 3.4, let $g\colon B \longrightarrow C$ be a homomorphism and $(E_2, \phi_2, F_2) = (C, g, 0) = (g)$. Then $(E_1 \otimes_g C, \phi_1 \otimes 1, F_1 \otimes 1)$ is a Kasparov product. But this is the cycle $g_*(\mathcal{E}_1)$. So we have proved that if the Kasparov product exists, then

$$\forall x \in \mathrm{KK}(A, B): \ g_*(x) = x \otimes_B [g].$$

Remark 3.7. As $\mathrm{Id}^*(x) = \mathrm{Id}_*(x) = x$ for every $x \in \mathrm{KK}(A, B)$ the above examples show in particular that

$$\forall x \in \mathrm{KK}(A, B): \ 1_A \otimes_A x = x \otimes_B 1_B = x.$$

3.2 Existence in general and uniqueness

Proof of theorem 3.2. Let's proof *existence* of an element in $F_1 \sharp F_2$. Let G be an F_2-connection for E_2 of degree 1. Using the technical lemma, we are going to find suitable $M, N \in \mathrm{L}_C(E_{12})$, such that $M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}} G \in F_1 \sharp F_2$. To find candidates for the algebras A_1, A_2 and the space $\mathcal{F} \subseteq \mathrm{L}_C(E_{12})$ to which we will apply the technical lemma we define

$$F_M := M^{\frac{1}{2}}(F_1 \otimes_B 1) + (1 - M)^{\frac{1}{2}} G$$

for every degree zero operator $M \in \mathrm{L}_C(E_{12})$, $0 \leq M \leq 1$. Now we give conditions on M, i.e. conditions on A_1, A_2 and $\mathcal{F}$, ensuring that F_M is in $F_1 \sharp F_2$. We will then check that the resulting candidates fulfill the conditions of the technical lemma.

So let $M \in \mathrm{L}_C(E_{12})$ be of degree zero, $0 \leq M \leq 1$. Define $N := 1 - M$.

- A first sensible condition would be that $M^{\frac{1}{2}}$ commutes with $F_1 \otimes_B 1$ and $N^{\frac{1}{2}}$ commutes with G modulo compact operators, because this will come in handy when we compute the square of F_M. This condition is obviously equivalent to the condition that N commutes with $F_1 \otimes_B 1$ and G $\mod \mathrm{K}_C(E_{12})$. In other words, we would like to have that

  (7) $$F_1 \otimes_B 1,\ G\ \in \mathcal{F}.$$

- We also want to have that $M^{\frac{1}{2}}(F_1 \otimes_B 1)$ is a 0-connection and $N^{\frac{1}{2}} G$ is an F_2-connection, because by proposition 2.29, this ensures that F_M is an F_2-connection. As G is already an F_2-connection it would suffice for the second property that $N^{\frac{1}{2}}$ is a 1-connection. But by proposition 2.29, 2., this is the case precisely if N is a 1-connection, which in turn is equivalent to M being a 0-connection. So a good condition on M would be that M is a 0-connection, or in other words:

  (8) $$\mathrm{K}_B(E_1) \otimes 1 \subseteq A_1.$$

  If this is the case, then also $M^{\frac{1}{2}}$ is a 0-connection by proposition 2.29, 2. The product $M^{\frac{1}{2}}(F_1 \otimes_B 1)$ is also a 0-connection because it is compact when multiplied with elements of $\mathrm{K}_B(E_1) \otimes_B 1$ from the left and from the right.

- We have to make sure that (E_{12}, ϕ_{12}, F) is indeed a Kasparov triple:

  – Note that by (7)

  $$\begin{aligned} F_M^2 - 1 &= M(F_1^2 \otimes_B 1) + NG^2 + M^{\frac{1}{2}} N^{\frac{1}{2}} \left(G(F_1 \otimes_B 1) + (F_1 \otimes_B 1)G\right) - 1 \\ &= M((F_1^2 - 1) \otimes_B 1) + N(G^2 - 1) + M^{\frac{1}{2}} N^{\frac{1}{2}} \left[G, F_1 \otimes_B 1\right] \mod \mathrm{K}_C(E_{12}). \end{aligned}$$

  Thus for every $a \in A$:

  $$\begin{aligned} (F_M^2 - 1)\phi_{12}(a) &= M((F_1^2 - 1)\phi_1(a) \otimes_B 1) + N(G^2 - 1)\phi_{12}(a) \\ &\quad + M^{\frac{1}{2}} N^{\frac{1}{2}} \left[G, F_1 \otimes_B 1\right] \phi_{12}(a) \mod \mathrm{K}_C(E_{12}). \end{aligned}$$

  Because $(F_1^2 - 1)\phi_1(a)$ is in $\mathrm{K}_B(E_1)$ it follows by (8) that the first term is $0 \mod \mathrm{K}_C(E_{12})$. The second term will be compact if $(G^2 - 1)\phi_{12}(a) \in A_2$. For the third term it suffices to ask for $N \left[G, F_1 \otimes_B 1\right] \phi_{12}(a)$ being compact (here, we could also use M instead, but as we will have more to check for A_1 than for A_2 in the technical lemma, we prefer to express everything in terms of N). So one possible condition on A_2 is

  (9) $$\forall a \in A:\ \left[G, F_1 \otimes_B 1\right] \phi_{12}(a),\ (G^2 - 1)\phi_{12}(a)\ \in A_2.$$

  – For every $a \in A$ we have (under condition (7))

  $$(F_M - F_M^*)\phi_{12}(a) = M^{\frac{1}{2}}((F_1 - F_1^*)\phi_1(a) \otimes 1) + N^{\frac{1}{2}}(G - G^*)\phi_{12}(a) \mod \mathrm{K}_C(E_{12}).$$

Because $(F_1 - F_1^*)\phi(a) \in \mathrm{K}_B(E_1)$ follows from the fact that $\mathcal{E}_1$ is a Kasparov cycle, and $M^{\frac{1}{2}}$ is a zero-connection by (8), the first term is compact. So the following condition seems natural in order to get rid of the second term:

$$(10) \qquad \forall a \in A: \ (G - G^*)\phi_{12}(a) \ \in A_2.$$

- For every $a \in A$ we have

$$\begin{aligned} [\phi_{12}(a), F_M] &= \left[\phi_{12}(a), M^{\frac{1}{2}}(F_1 \otimes_B 1)\right] + \left[\phi_{12}(a), N^{\frac{1}{2}}G\right] \\ &= \left[\phi_{12}(a), M^{\frac{1}{2}}\right](F_1 \otimes_B 1) + M^{\frac{1}{2}}\left[\phi_{12}(a), (F_1 \otimes_B 1)\right] \\ &\quad + \left[\phi_{12}(a), N^{\frac{1}{2}}\right] G + N^{\frac{1}{2}}\left[\phi_{12}(a), G\right]. \end{aligned}$$

Note that for any $T \in \mathrm{L}_C(E_{12})$ the following statements are equivalent: T commutes with $M^{\frac{1}{2}}$ up to compact operators, T commutes with M up to c.o. (use functional calculus), T commutes with N up to c.o., T commutes with $N^{\frac{1}{2}}$ up to compact operators. To make the first and the third term compact it suffices to have

$$(11) \qquad \forall a \in A: \ \phi_{12}(a) \ \in \mathcal{F}.$$

The second term is equal to $M^{\frac{1}{2}}([\phi_{12}(a), F_1] \otimes_B 1)$, so this is compact using condition (8). The last term is compact if we have the following:

$$(12) \qquad \forall a \in A: \ [G, \phi_{12}(a)] \ \in A_2.$$

- Let $a \in A$. Under which circumstances is $\phi_{12}(a)[F_1 \otimes_B 1, F_M]\phi_{12}(a)^* \geq 0 \mod \mathrm{K}_C(E_{12})$? We have

$$\begin{aligned} [F_1 \otimes_B 1, F_M] &= \left[F_1 \otimes_B 1, M^{\frac{1}{2}}(F_1 \otimes_B 1)\right] + \left[F_1 \otimes_B 1, N^{\frac{1}{2}}G\right] \\ &= \left[F_1 \otimes_B 1, M^{\frac{1}{2}}\right](F_1 \otimes_B 1) + M^{\frac{1}{2}}\left[F_1 \otimes_B 1, F_1 \otimes_B 1\right] \\ &\quad + \left[F_1 \otimes_B 1, N^{\frac{1}{2}}\right] G + N^{\frac{1}{2}}\left[F_1 \otimes_B 1, G\right]. \end{aligned}$$

From (7) it follows as above that the first and the third term is compact. If we multiply with $\phi_{12}(a)^*$ from the right, it follows from (9) that the fourth term becomes compact. So

$$\begin{aligned} [F_1 \otimes_B 1, F_M]\phi_{12}(a)^* &= M^{\frac{1}{2}}\left[F_1 \otimes_B 1, F_1 \otimes_B 1\right]\phi_{12}(a)^* \\ &= 2M^{\frac{1}{2}}\left(F_1^2\phi_1(a)^* \otimes_B 1\right) \mod \mathrm{K}_C(E_{12}). \end{aligned}$$

Because $M^{\frac{1}{2}}$ is a 0-connection if condition (8) holds, it follows, using $(1 - F_2^2)\phi_1(a) \in \mathrm{K}_B(E_1)$:

$$[F_1 \otimes_B 1, F_M]\phi_{12}(a)^* = M^{\frac{1}{2}}\phi_{12}(a)^* \mod \mathrm{K}_C(E_{12}).$$

If we multiply this by $\phi_{12}(a)$ from the left, the right-hand side is obviously positive. So without any extra condition we have positivity of the left-hand-side $\mod \mathrm{K}_C(E_{12})$.

If all of these conditions are satisfied, we know that F_M is in $F_1 \sharp F_2$. So let us define

$$A_1 := \mathrm{K}_C(E_{12}) + \mathrm{K}_B(E_1) \otimes_B 1,$$

$$A_2 := C^*\left([G, F_1 \otimes_B 1]\phi_{12}(A), (G^2 - 1)\phi_{12}(A), (G - G^*)\phi_{12}(A), [G, \phi_{12}(A)]\right),$$

$$\mathcal{F} := \overline{\langle F_1 \otimes_B 1,\ G,\ \phi_{12}(A)\rangle_{\mathbb{C}}}.$$

Note that our A_1 also contains the compact operators as this will ensure that $\mathcal{F}$ derives A_1.

What is left to check is that these data satisfy the conditions of the technical lemma. Obviously, all of the three spaces are invariant under the grading. That A_1 is σ-unital was already shown directly after the statement of the technical lemma. A_2 and $\mathcal{F}$ are separable because they are each generated by a separable set. So the size conditions are satisfied.

In order to show $A_1A_2 \subseteq \mathrm{K}_C(E_{12})$ take $k \in \mathrm{K}_B(E_1)$. Note that $G^2 - 1$ is an $(F_2^2 - 1)$-connection, and this operator is a compact perturbation of 0. So $G^2 - 1$ is a 0-connection and hence

$$(k \otimes_B 1)(G^2 - 1)\phi_{12}(a) \in \mathrm{K}_C(E_{12}).$$

For the same reason we have

$$(k \otimes_B 1)(G - G^*)\phi_{12}(a) \in \mathrm{K}_C(E_{12}).$$

Moreover,

$$(-1)^{\deg k}(k \otimes_B 1)[G, F_1 \otimes_B 1] \quad = \quad [G, (kF_1) \otimes_B 1] - [G, k \otimes_B 1](F_1 \otimes_B 1).$$

The first terms on the right-hand side are compact proposition 2.29, 4. So the left-hand side is compact (and stays compact when multiplied with $\phi_{12}(a)$. Similarly,

$$(-1)^{\deg k}(k \otimes_B 1)[G, \phi_1(a) \otimes_B 1] \quad = \quad [G, (k\phi_1(a)) \otimes_B 1] - [G, k \otimes_B 1](\phi_1(a) \otimes_B 1)$$

is compact. So we have shown that A_1A_2 is contained in $\mathrm{K}_C(E_{12})$.

The last thing that remains to be checked is $[\mathcal{F}, A_1] \subseteq A_1$. It is obvious that $F_1 \otimes_B 1$ and $\phi_{12}(A)$ derive A_1. G derives A_1 by proposition 2.29, 4.

So we have shown that A_1, A_2 and $\mathcal{F}$ satisfy the conditions of the technical lemma, so we can find appropriate M and N such that

$$F := M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}}G$$

is in $F_1 \sharp F_2$.

Note that the F_M we have just constructed is a compact perturbation of

$$\hat{F}_M := M^{\frac{1}{4}}(F_1 \otimes_B 1)M^{\frac{1}{4}} + N^{\frac{1}{4}}GN^{\frac{1}{4}},$$

because M and N commute with $F_1 \otimes_B 1$ and G modulo compacts, respectively. So $\hat{F}_M \in F_1 \sharp F_2$, as well. Now, if F_1 and F_2 are self-adjoint, we can take G self-adjoint and then $\hat{F}_M$ will be self-adjoint.

To prove *uniqueness* note that if the operator G is already in $F_1 \sharp F_2$, then F_M is operator homotopic to G. To see this note that

$$\begin{aligned}
[G, F_M] &= [G, M^{\frac{1}{2}}(F_1 \otimes 1)] + [G, N^{\frac{1}{2}}G] \\
&= [G, M^{\frac{1}{2}}](F_1 \otimes 1) + M^{\frac{1}{2}}[G, F_1 \otimes 1] + [G, N^{\frac{1}{2}}]G + N^{\frac{1}{2}}[G, G] \\
&= 0 + M^{\frac{1}{4}}[G, F_1 \otimes 1]M^{\frac{1}{4}} + 0 + 2N^{\frac{1}{4}}G^2N^{\frac{1}{4}} \mod \mathrm{K}_C(E_{12}).
\end{aligned}$$

Hence we have for every $a \in A$, because $\phi_{12}(A)$ commutes with M and N modulo compacts:

$$\begin{aligned}
\phi_{12}(a)[G, F_M]\phi_{12}(a)^* &= \phi_{12}(a)M^{\frac{1}{4}}[G, F_1 \otimes 1]M^{\frac{1}{4}}\phi_{12}(a)^* + 2\phi_{12}(a)N^{\frac{1}{4}}G^2N^{\frac{1}{4}}\phi_{12}(a)^* \\
&= M^{\frac{1}{4}}\phi_{12}(a)[G, F_1 \otimes 1]\phi_{12}(a)^*M^{\frac{1}{4}} + 2N^{\frac{1}{4}}\phi_{12}(a)G^2\phi_{12}(a)^*N^{\frac{1}{4}} \mod \mathrm{K}_C(E_{12}).
\end{aligned}$$

The first term is positive because $G \in F_1 \sharp F_2$. The second term is positive because $\phi_{12}(a)G^2\phi_{12}(a)^* = \phi_{12}(a)\phi_{12}(a)^* \mod \mathrm{K}_C(E_{12})$. Thus we see that $\phi_{12}(a)[G, F_M]\phi_{12}(a)^* \geq 0 \mod \mathrm{K}_C(E_{12})$ for every $a \in A$. This shows that G and F_M are operator homotopic.

So let G and G' be in $F_1 \sharp F_2$. Now the trick is to modify the above proof to produce M and N such that $M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}}G$ as well as $M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}}G'$ are in $F_1 \sharp F_2$ and differ by a compact operator. The difference is $N^{\frac{1}{2}}(G - G')$. After everything we have done, it is obvious that a good choice would be

$$A_1 := \mathrm{K}_C(E_{12}) + \mathrm{K}_B(E_1) \otimes_B 1,$$

$$A_2 := C^*\left([G, F_1 \otimes_B 1]\phi_{12}(A), [G', F_1 \otimes_B 1]\phi_{12}(A), (G - G')\phi_{12}(A), \mathrm{K}_C(E_{12})\right),$$
$$\mathcal{F} := \overline{\langle F_1 \otimes_B 1,\ G, G',\ \phi_{12}(A)\rangle_{\mathbb{C}}}.$$

Note that the algebra A_2 is now defined in a way that ensures that it contains the one defined above because it follows from $G, G' \in F_1 \sharp F_2$ that the operators $(G^2 - 1)\phi_{12}(a)$ etc. are contained in the algebra $\mathrm{K}_C(E_{12}) \subseteq A_2$.

The size conditions are obviously met. When proving $A_1 A_2 \subseteq \mathrm{K}_C(E_{12})$ just note that $G - G'$ is a 0-connection. The rest is trivial. Note that the "old" A_2 for G and G' is contained in the new one. This ensures that $M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}}G$ is in $F_1 \sharp F_2$ and operator homotopic to G, $M^{\frac{1}{2}}(F_1 \otimes_B 1) + N^{\frac{1}{2}}G'$ is in $F_1 \sharp F_2$ and operator homotopic to G', and $N^{\frac{1}{2}}(G - G')\phi_{12}(A) \subseteq \mathrm{K}_C(E_{12})$. □

3.3 The Kasparov product on the level of KK-theory

For the rest of this section, let A be separable.

The proof of the following lemma is straightforward but tedious and will be left to the reader.

Lemma 3.8 (Poor man's associativity)**.** *Let D be another C^*-algebra, $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$, $h\colon C \longrightarrow D$ be $*$-homomorphisms. Let $\mathcal{E}_1 = (E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$, $\mathcal{E}_2 = (E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$, and $\mathcal{E}_3 = (E_3, \phi_3, F_3) \in \mathbb{E}(C, D)$.*

1. *If $\mathcal{E}_{23}$ is a Kasparov product for $\mathcal{E}_2$ and $\mathcal{E}_3$, then $f^*(\mathcal{E}_{23})$ is a Kasparov product for $f^*(\mathcal{E}_2)$ and $\mathcal{E}_3$.*
2. *Define*
$$\Psi\colon (E_1 \otimes_g C) \otimes_{\phi_3} E_3 \longrightarrow E_1 \otimes_{\phi_3 \circ g} E_3,\ (e_1 \otimes c) \otimes e_3 \mapsto e_1 \otimes ce_3.$$
Then Ψ is an isomorphism of Hilbert A-D-bimodules. Let $F \in \mathrm{L}_D\left(E_1 \otimes_{\phi_3 \circ g} E_3\right)$. Define $F' := \Psi^{-1} \circ F \circ \Psi$. Then $(E_1 \otimes E_3, \phi_1 \otimes 1, F)$ is a Kasparov product for $\mathcal{E}_1$ and $g^*(\mathcal{E}_3)$ if and only if $((E_1 \otimes C) \otimes E_3, (\phi_1 \otimes 1) \otimes 1, F')$ is a Kasparov product for $g_*(\mathcal{E}_1)$ and $\mathcal{E}_3$.
3. *If $\mathcal{E}_{12}$ is a Kasparov product for $\mathcal{E}_1$ and $\mathcal{E}_2$, then $h_*(\mathcal{E}_{12})$ is canonically isomorphic to a Kasparov product of $\mathcal{E}_1$ and $h_*(\mathcal{E}_2)$, the isomorphism being*
$$\Phi\colon (E_1 \otimes_{\phi_2} E_2) \otimes_h D \longrightarrow E_1 \otimes_{\phi_2 \otimes 1} (E_2 \otimes_h D),\ (e_1 \otimes e_2) \otimes d \mapsto e_1 \otimes (e_2 \otimes d).$$

Remark 3.9. We will use the preceding lemma to show that the Kasparov product is well-defined on the level of KK-elements. If this is achieved, the lemma yields the following corollary: Let f, g, h be as above, $x \in \mathrm{KK}(A, B)$, $y \in \mathrm{KK}(B, C)$ and $z \in \mathrm{KK}(C, D)$. Then

1. $$([f] \otimes_B y) \otimes_C z = f^*(y) \otimes_C z = f^*\left(y \otimes_C z\right) = [f] \otimes_B (y \otimes_C z),$$
2. $$(x \otimes_B [g]) \otimes_C z = g_*(x) \otimes_C z = x \otimes_B g^*(z) = x \otimes_B ([g] \otimes_C z),$$
3. $$(x \otimes_B y) \otimes_C [h] = h_*\left(x \otimes_B y\right) = x \otimes_B h_*(y) = x \otimes_B (y \otimes_C [h]).$$

This explains the name of the lemma. Note that the lemma gives some additional information as it does not involve homotopies but gives proper isomorphisms.

The following lemma is a direct consequence of lemma 3.8, 3:

Lemma 3.10 (Homotopy invariance in the second variable)**.** *Let $\mathcal{E}_1 = (E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2) \in \mathbb{E}(B, C[0, 1])$ (this a homotopy!). Let $\mathcal{E}_{12} \in \mathbb{E}(A, C[0, 1])$ be a Kasparov product for $\mathcal{E}_1$ and $\mathcal{E}_2$ (this is again a homotopy!). Then for every $t \in [0, 1]$ we have that $\pi_{t,*}(\mathcal{E}_{12})$ is isomorphic to a Kasparov product of $\mathcal{E}_1$ and $\pi_{t,*}(\mathcal{E}_2)$.*

In particular, $\mathcal{E}_{12}$ is a homotopy from a Kasparov product for $\mathcal{E}_1$ and $\pi_{0,*}(\mathcal{E}_2)$ to a Kasparov product for $\mathcal{E}_1$ and $\pi_{1,*}(\mathcal{E}_2)$.

For the moment, we need the following definition just for notational convenience but the notation will be made clearer in the subsequent talks:

Definition 3.11. Suppose that $\mathcal{E}_2 \in \mathbb{E}(B, C)$. Then we define

$$\tau_{\mathbb{C}[0,1]}(\mathcal{E}_2) := (E_2[0,1], \phi_2 \otimes 1, F_2 \otimes 1) \in \mathbb{E}(B[0,1], C[0,1]).$$

Note that we have for every $t \in [0,1]$:

$$\pi_{t,*}\left(\tau_{\mathbb{C}[0,1]}(\mathcal{E}_2)\right) \cong \pi_t^*(\mathcal{E}_2).$$

Lemma 3.12 (Homotopy invariance in the first variable)**.** *Let $\mathcal{E}_1 = (E_1, \phi_1, F_1) \in \mathbb{E}(A, B[0,1])$ (yet another homotopy!) and $\mathcal{E}_2 = (E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$.*

Let $\mathcal{E}_{12} \in \mathbb{E}(A, C[0,1])$ be a Kasparov product for $\mathcal{E}_1$ and $\tau_{\mathbb{C}[0,1]}(\mathcal{E}_2)$ (homotopy!). Then for every $t \in [0,1]$ we have that $\pi_{t,*}(\mathcal{E}_{12})$ is isomorphic to a Kasparov product of $\pi_{t,*}(\mathcal{E}_1)$ and $\mathcal{E}_2$.

In particular, $\mathcal{E}_{12}$ is a homotopy from a Kasparov product for $\pi_{0,*}(\mathcal{E}_1)$ and $\mathcal{E}_2$ to a Kasparov product for $\pi_{1,*}(\mathcal{E}_1)$ and $\mathcal{E}_2$.

Proof. By lemma 3.8, 3, we know that $\pi_{t,*}(\mathcal{E}_{12})$ is isomorphic to a Kasparov product of the cycles $\mathcal{E}_1$ and $\pi_{t,*}\left(\tau_{\mathbb{C}[0,1]}(\mathcal{E}_2)\right)$, where the latter is isomorphic to $\pi_t^*(\mathcal{E}_2)$. But by lemma 3.8, 2, this is in turn isomorphic to a Kasparov product of $\pi_{t,*}(\mathcal{E}_1)$ and $\mathcal{E}_2$. □

Lemma 3.13. *Let $\mathcal{E}_1 = (E_1, \phi_1, F_1)$, $\mathcal{E}_1' = (E_1', \phi_1', F_1') \in \mathbb{E}(A, B)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2)$, $\mathcal{E}_2' = (E_2', \phi_2', F_2') \in \mathbb{E}(B, C)$.*

1. *If $\mathcal{E}_{12}$ is a Kasparov product of $\mathcal{E}_1$ by $\mathcal{E}_2$ and $\mathcal{E}_{12}'$ is a Kasparov product of $\mathcal{E}_1'$ by $\mathcal{E}_2$ then $\mathcal{E}_{12} \oplus \mathcal{E}_{12}'$ is isomorphic to a Kasparov product of $\mathcal{E}_1 \oplus \mathcal{E}_1'$ by $\mathcal{E}_2$.*

2. *If $\mathcal{E}_{12}$ is a Kasparov product of $\mathcal{E}_1$ by $\mathcal{E}_2$ and $\mathcal{E}_{12}'$ is a Kasparov product of $\mathcal{E}_1$ by $\mathcal{E}_2'$ then $\mathcal{E}_{12} \oplus \mathcal{E}_{12}'$ is isomorphic to a Kasparov product of $\mathcal{E}_1$ by $\mathcal{E}_2 \oplus \mathcal{E}_2'$.*

Proof. 1. Let

$$\Phi\colon (E_1 \otimes_B E_2) \oplus (E_1' \otimes_B E_2) \longrightarrow \underbrace{(E_1 \oplus E_1') \otimes_B E_2}_{=:E}$$

be the obvious isomorphism. Note that $\Phi^{-1} = \Phi^*$. Let $F := \Phi \otimes (F_{12} \oplus F_{12}') \circ \Phi^{-1}$ and $\phi(a) := \Phi \circ (\phi_{12}(a) \oplus \phi_{12}'(a)) \circ \Phi^{-1}$. Then $\mathcal{E} := (E, \phi, F)$ is in $\mathbb{E}(A, C)$ by definition. Now note that F is an F_2-connection by proposition 2.30. Moreover, we have

$$(F_1 \oplus F_1') \otimes_B 1 = \Phi \circ (F_1 \otimes_B 1 \oplus F_1' \otimes_B 1) \circ \Phi^{-1}.$$

Let $a \in A$. Then we have

$$\begin{aligned} & \phi(a)[(F_1 \oplus F_1') \otimes_B 1, F]\phi(a^*) \\ = \ & \Phi(\phi_{12}(a) \oplus \phi_{12}'(a)) \left[(F_1 \otimes_B 1 \oplus F_1' \otimes_B 1), (F_{12} \oplus F_{12}')\right] (\phi_{12}(a^*) \oplus \phi_{12}'(a^*))\Phi^* \\ = \ & \Phi\left((\phi_{12}(a)[F_1 \otimes_B 1, F_{12}]\phi_{12}(a^*)) \oplus (\phi_{12}'(a)[F_1' \otimes_B 1, F_{12}']\phi_{12}'(a^*))\right)\Phi^*. \end{aligned}$$

As the direct sum of two positive operators is positive and the direct sum of two compact operators is compact, we conclude that we have indeed constructed an Kasparov product.

2. Let

$$\Phi\colon (E_1 \otimes_B E_2) \oplus (E_1 \otimes_B E_2') \longrightarrow \underbrace{E_1 \otimes_B (E_2 \oplus E_2')}_{=:E}$$

be the obvious isomorphism. Let $F := \Phi \otimes (F_{12} \oplus F_{12}') \circ \Phi^{-1}$ and $\phi(a) := \Phi \circ (\phi_{12}(a) \oplus \phi_{12}'(a)) \circ \Phi^{-1}$. Then $\mathcal{E} := (E, \phi, F)$ is in $\mathbb{E}(A, C)$ by definition. A short calculation similar to the one in proposition 2.30 shows that F is an $F_2 \oplus F_2'$-connection. As above we can conclude that (E, ϕ, F) is a Kasparov product for $\mathcal{E}_1$ and $\mathcal{E}_2 \oplus \mathcal{E}_2'$.

□

From what we have done we can now derive the following theorem:

Theorem 3.14. *The Kasparov product is a well-defined bi-additive map on the level* KK*-elements.*

Proof. That the Kasparov product is invariant up to homotopy under homotopies in the first variable follows from lemma 3.10. In the second this follows from 3.12. Biadditivity is a consequence of the preceding lemma. □

Theorem 3.15. *The Kasparov product is associative.*

This last theorem of this section will not be proved here as the proof is rather technical and the main techniques used in it, e.g. the technical lemma, as well as the way they are applied have already been presented in this talk. A proof of the associativity of the Kasparov product can be found in any introduction to KK-theory, for example in [JT91].

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