3.31 The general form of the product:

Let A_1, A_2, B_1, B_2 and D be graded σ-unital C^*-algebras and

$$x \in KK(A_1, B_1 \hat{\otimes} D), \quad y \in KK(D \hat{\otimes} A_2, B_2).$$

If A_1 and A_2 are separable, then we define

$$x \underset{D}{\otimes} y := \underbrace{(x \hat{\otimes} 1_{A_2})}_{\in KK(A_1 \hat{\otimes} A_2,\, B_1 \hat{\otimes} D \hat{\otimes} A_2)} \underset{B_1 \hat{\otimes} D \hat{\otimes} A_2}{\hat{\otimes}} \underbrace{(1_{B_1} \hat{\otimes} y)}_{\in KK(B_1 \hat{\otimes} D \hat{\otimes} A_2,\, B_1 \hat{\otimes} B_2)} \in KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

Check Blackadar, 18.9.1 for functorial properties of this general product.

If $\mathbb{C} = D$, then we obtain a product

$$\underset{\mathbb{C}}{\otimes} : KK(A_1, B_1) \otimes KK(A_2, B_2) \longrightarrow KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$$

It is commutative in the following sense. Let

$$\Sigma_{A_1, A_2} : A_1 \hat{\otimes} A_2 \to A_2 \hat{\otimes} A_1, \quad a_1 \hat{\otimes} a_2 \mapsto (-1)^{\partial a_1 \partial a_2} a_2 \hat{\otimes} a_1$$

and define Σ_{B_1, B_2} analogously. Then

$$x \underset{\mathbb{C}}{\otimes} y = \Sigma_{B_1, B_2}^{-1} \circ y \underset{\mathbb{C}}{\otimes} x \circ \Sigma_{A_1, A_2}$$

KK^1 and Bott periodicity

Let A, B be graded C^*-algebras.

3.32 Lemma: For all $n \in \mathbb{N}_0$ we have

$$KK(A,B) \cong KK(A \hat{\otimes} \mathbb{C}_n, B \hat{\otimes} \mathbb{C}_n).$$

Pf: The canonical homomorphism is $\tau_{\mathbb{C}_n}$. If we consider $\tau_{\mathbb{C}_n} \circ \tau_{\mathbb{C}_n}$ then this equivalent the homomorphism

$$\tau_{\mathbb{C}_n \hat{\otimes} \mathbb{C}_n} \cong \tau_{M_{2^n}(\mathbb{C})} : KK(A,B) \to KK(A \hat{\otimes} M_{2^n}(\mathbb{C}), B \hat{\otimes} M_{2^n}(\mathbb{C}))$$
$$\cong KK(M_{2^n}(A), M_{2^n}(B))$$

Now $M_{2^n}(A)$ is gradedly Morita equivalent to A and, likewise, $M_{2^n}(B) \sim_{\text{Morita}} B$. The standard Morita equivalence has the property that the induced homomorphism $KK(M_{2^n}(A), M_{2^n}(B)) \cong KK(A,B)$ is the inverse of $\tau_{M_{2^n}(\mathbb{C})}$. Hence $\tau_{\mathbb{C}_n}$ is (up to a canonical identification) a left inverse of $\tau_{\mathbb{C}_n}$. So $\tau_{\mathbb{C}_n}$ is an isomorphism.

3.33 Definition: for all $n \in \mathbb{N}_0$ define

$$KK^n(A,B) := KK(A, B \hat{\otimes} \mathbb{C}_n).$$

3.34 Remark: There are alternative descriptions of $KK^1(A,B)$, for example using trivially graded Hilbert modules (see [JT], section 3.3 or [Blackadar], 17.5.2). If A is separable and B is σ-unital (and stable) then $KK^1(A,B) \cong Ext^{-1}(A,B)$. ($A, B$ trivially graded).

3.35 Lemma (formal Bott periodicity)

$$KK^1(A,B) \cong KK(A \hat{\otimes} \mathbb{C}_1, B)$$

and

$$KK(A,B) \cong KK^1(A, B\hat{\otimes}\mathbb{C}_1) \cong KK^1(A\hat{\otimes}\mathbb{C}_1, B) \cong KK(A\hat{\otimes}\mathbb{C}_1, B\hat{\otimes}\mathbb{C}_1)$$

naturally. $\quad \cong KK^2(A,B)$

3.36 Definition: An exact sequence

$$0 \to J \xrightarrow{i} A \xrightarrow{q} A/J \to 0$$

of graded C^*-algebras is semisplit if there exists a completely positive, nonincreasing, grading-preserving cross-section for q.

3.37 Remark: If A is nuclear, then every ideal of A is semisplit [Blackadar, 15.8.3]

3.37 Theorem (long exact sequences in KK-theory)

Let $0 \to J \to A \to A/J \to 0$ be a semisplit short exact sequence of σ-unital C^*-algebras.

a) If D is a separable & graded, then the following six-term sequence is exact:

$$\begin{array}{ccccc} KK(D,J) & \xrightarrow{i_*} & KK(D,A) & \xrightarrow{q_*} & KK(D,A/J) \\ \uparrow \partial & & & & \downarrow \partial \\ KK^1(D,A/J) & \leftarrow & KK^1(D,A) & \leftarrow & KK^1(D,J). \end{array}$$

b) If A is separable and D is σ-unital & graded, then the following six-term sequence is exact:

$$\begin{array}{ccccc}
KK(J,D) & \xleftarrow{j^*} & KK(A,D) & \xleftarrow{q^*} & KK(A/J,D) \\
\downarrow \delta & & & & \uparrow \delta \\
KK^1(A/J,D) & \xrightarrow[q^*]{} & KK^1(A,D) & \xrightarrow[j^*]{} & KK^1(J,D)
\end{array}$$

In both cases, δ is given by multiplication with the element in $KK^1(A/J, J)$ corresponding to the extension.

Pf: [Blackadar, 19.5.7]

3.38 Definition: Define

$$S := C_0(\mathbb{R}) \quad \text{(as a trivially graded } C^*\text{-algebra)}$$

3.39 Theorem: $\mathbb{C}_1$ and S are KK-equivalent.

3.40 Corollaries (Bott periodicity)

a) $KK(A,B) \cong KK(A, B \hat{\otimes} S^2) \cong KK(S^2 \hat{\otimes} A, B)$

b) $KK^1(A,B) \cong KK(A, S \hat{\otimes} B) \cong KK(S \hat{\otimes} A, B)$

for A separable, B σ-unital.

3.41 Proof 3.39: