3. The Kasparov product

3.1) Theorem: Let A, B, C, D be graded σ-unital C^*-algebras. Let A be separable. Then there exists a map
$$\hat{\otimes}_B : KK(A,B) \times KK(B,C) \longrightarrow KK(A,C),$$
called the Kasparov product, that has the following properties:

a) biadditivity: $(x_1 \oplus x_2)\hat{\otimes}_B y = x_1 \hat{\otimes}_B y \oplus x_2 \hat{\otimes}_B y$
and $x \hat{\otimes}_B (y_1 \oplus y_2) = x \hat{\otimes}_B y_1 \oplus x \hat{\otimes}_B y_2$.

b) associativity: if B is separable as well, then
$$x \hat{\otimes}_B (y \hat{\otimes}_C z) = (x \hat{\otimes}_B y) \hat{\otimes}_C z$$
f.a. $x \in KK(A,B)$, $y \in KK(B,C)$ and $z \in KK(C,D)$.

c) unit elements: if we define $1_A := [id_A] \in KK(A,A)$ and $1_B = [id_B] \in KK(B,B)$ then f.a. $x \in KK(A,B)$:
$$1_A \hat{\otimes}_A x = x = x \hat{\otimes}_B 1_B .$$

d) functoriality: if $\varphi : A \to B$ and $\eta : B \to C$ are graded $*$-hom, then
$$x \hat{\otimes}_B [\eta] = \eta_*(x) \quad \text{and} \quad [\varphi] \hat{\otimes}_B y = \varphi^*(y)$$
f.a. $x \in KK(A,B)$ and $y \in KK(B,C)$.

e) it generalises the product of Morita cycles defined in ___

3.2. Remark: a) The separable graded C^*-algebras form an additive category when equipped with the KK-groups as morphism sets and the (flipped) Kasparov product as composition. The map $\eta \mapsto [\eta]$ is a functor from the category of sep. gr. C^*-algebras with gr. $*$-hom. in this category

b) isomorphisms in this category are also called KK-equivalences
c) 3.1e) and 3.1 c) imply that Morita equivalences give KK-equivalences. In particular, KK-theory is also Morita invariant in the first component.

3.3 Idea of proof: Let $(E_1, \phi_1, T_1) \in \mathbb{E}(A,B)$ and $(E_2, \phi_2, T_2) \in \mathbb{E}(B,C)$
As module for the product we can take $E_{12} := E_1 \hat{\otimes}_B E_2$ and as module action we can take $\phi_{12} := \phi_1 \hat{\otimes} 1$.

The problem is to find the operator

Very naive approach: $T_{12} := T_1 \hat{\otimes} 1 + 1 \hat{\otimes} T_2$.
$T_1 \hat{\otimes} 1$ is okay, but $1 \hat{\otimes} T_2$ does not make any sense as long as T_2 is not B-linear on the left. (Compare to the last lecture of Michael Joachim). If we neglect this problem then we calculate

$$T_{12}^2 = T_1^2 \hat{\otimes} 1 + 1 \hat{\otimes} T_2^2$$

So we end up with something which is rather 2, than 1 up to compact operators.
Idea: Find suitable „coefficient" operators $M, N \in L_C(E_{12})$ such that $M^2 + N^2 = 1$ and $M, N \geq 0$.
Define

$$T_{12} := M\, T_1 \hat{\otimes} 1 + N\, 1 \hat{\otimes} T_2 .$$

Then $T_{12}^2 \approx M^2 \underbrace{T_1^2 \hat{\otimes} 1}_{\approx 1} + N^2 \underbrace{1 \hat{\otimes} T_2^2}_{\approx 1} + \text{rest} \approx 1 + \text{rest}.$

The critical point is that we need a lemma which ensures the existence of such coefficients such that the calculations are justified and rest $= 0$ up to compact operators. This is the subject of „Kasparov's technical lemma"

To give a sense to an expression like $1\hat{\otimes}T_2$ is subject of the theory of connections. Such connections will only be unique up to „compact perturbations" and also the technical lemma involves some choices, so there is need for a condition when two operators are homotopic so that they give the same element in KK.

These are the three tools which we introduce before we come to the proof of the existence of the product.

3.4. Proposition (a sufficient condition for op. homotopy)
Let A, B be graded C^*-algebras, $\mathcal{E}=(E,\phi,T)$, $\mathcal{E}'=(E,\phi,T')\in\mathbb{E}(A,B)$.
If

$$\forall a\in A: \quad \phi(a)[T,T']\phi(a^*)\geq 0 \quad \mathrm{mod}\ K_B(E),$$

then $\mathcal{E}$ and $\mathcal{E}'$ are operator homotopic. $\Leftrightarrow$ $\exists k\in K_B(E): \dots + k\geq 0$.

Proof: My 2004 notes, Prop 2.5, or JT, Lemma 2.1.18.

3.5 Def: If (B,β) is a graded C^*-algebra and $A\subseteq B$ is a sub-C^*-algebra then A is called graded iff $\beta(A)\subseteq A$. [All subalgebras of graded algebras will be assumed graded]

3.6 Def B C^*-alg, $A\subseteq B$ subalg., $\mathcal{F}\subseteq B$ subset. We say that $\mathcal{F}$ derives A if $\forall a\in A\ \forall f\in\mathcal{F}: [f,a]\in A$. (graded comm.)

3.7 Thm: Let B be a graded σ-unital C^*-alg., Let A_1, A_2 be σ-unital sub C^*-algebras of $M(B)$ and let $\mathcal{F}$ be a separable, closed linear subspace of $M(B)$ such that $\beta_B(\mathcal{F})\subseteq\mathcal{F}$. Assume that

1) $A_1\cdot A_2\subseteq B$ $\quad [A_1\perp A_2 \ \mathrm{mod}\ B]$
2) $[\mathcal{F}, A_1]\subseteq A_1$ $\quad [\mathcal{F}$ derives $A_1]$

Then there exist elements $M, N\in M(B)$ of degree 0 such that $M+N=1$, $M,N\geq 0$, $MA_1\subseteq B$, $NA_2\subseteq B$, $[N,\mathcal{F}]\subseteq B$.

Proof: My 2004 notes, Thm 2.8 or JT, Thm 2.2.1.

3.8 Remark: 1) The larger A_1, A_2 and $\mathcal{F}$, the stronger the lemma.

2) we can always assume WLOG: $\mathcal{B} \subseteq A_1, A_2$.

Pf: We can replace A_i with $A_i + \mathcal{B} =: A_i'$. A_i' is a graded sub C^*-algebra that is σ-unital: If b is strictly positive in $\mathcal{B}$ and a_i is strictly pos. in A_i then $b + a_i$ is strictly pos. in A_i' because $b + a_i \geq 0$ and $(a_i + b)(A_i + \mathcal{B}) \supseteq a_i A + b\mathcal{B}$ (dense in A_i')

3) we will use the lemma in the case $\mathcal{B} = \mathcal{K}(E)$, $M(\mathcal{B}) = \mathcal{L}(E)$ for a countably gen. HM. E.

3.9 Exercise: Let X be a locally compact, σ-compact Hausdorff space and $\partial X := \beta X \setminus X$ its "corona space". Then ∂X is stonean, i.e. the closures of open sets are open [$\Leftrightarrow$ $\forall\, U, V \subseteq \partial X$ open, $U \cap V = \phi$ $\exists\, f: \partial X \to [0,1]$ cont s.t. $f|_U = 0$, $f|_V = 1$].

Connections

In this section let $\mathbb{B}, C$ be graded C^*-algebras, E_1 a Hilbert $\mathbb{B}$-module, E_2 a Hilbert C-module, $\phi : \mathbb{B} \to L_C(E_2)$ a graded $*$-hom

$E_{12} := E_1 \hat{\otimes}_{\mathbb{B}} E_2$.

3.10 <u>Remark</u> Let $T_2 \in L_C(E_2)$ and assume that

$(*)$ $\quad \forall b \in \mathbb{B} : \; [\phi(b), T_2] = 0.$

Define $1 \hat{\otimes} T_2 \in L_C(E_{12})$ on elementary tensors by

$$(1 \hat{\otimes} T_2)(e_1 \hat{\otimes} e_2) := (-1)^{\partial T_2 \partial e_1} e_1 \hat{\otimes} T_2(e_2)$$

(in the sense that you first split T_2 into odd and even part...)

If T_2 is just $\mathbb{B}$-linear up to compact operators, i.e. if

$(**)$ $\quad \forall b \in \mathbb{B} \;\; [\phi(b), T_2] \in K_C(E_2)$,

then this construction no longer works. We can, however, construct a substitute for $1 \hat{\otimes} T_2$ "up to compact operators".

3.11. <u>Definition</u> For any $x \in E_1$ define

$$T_x : E_2 \to E_{12}, \; e_2 \mapsto x \hat{\otimes} e_2 .$$

~~3.12~~ 3.13 <u>Lemma</u> If $T_2 \in L_C(E_2)$ satisfies $(*)$, then

$(***)_1$
$$\begin{tikzcd} E_2 \arrow[r, "T_2"] \arrow[d, "T_x"'] & E_2 \arrow[d, "T_x"] \\ E_{12} \arrow[r, "1 \hat{\otimes} T_2"'] & E_{12} \end{tikzcd}$$

gradedly commutes for all $x \in E_1$ (i.e. $T_x \circ T_2 = (1 \hat{\otimes} T_2) \circ T_x \cdot (-1)^{\partial x \partial T_2}$).

similarly

$(***)_2$
$$\begin{tikzcd} E_2 \arrow[r, "T_2"] & E_2 \\ E_{12} \arrow[u, "T_x^*"] \arrow[r, "1 \hat{\otimes} T_2"'] & E_{12} \arrow[u, "T_x^*"'] \end{tikzcd}$$

gradedly commutes

3.12 <u>Lemma</u>: For all $x \in E$, we have $T_x \in L_C(E_2, E_{12})$ with
$T_x^*: E_{12} \to E_2$, $e_1 \otimes e_2 \mapsto \phi(\langle x, e_1 \rangle) e_2$. ∎

3.14 <u>Definition</u>: Let $T_2 \in L_C(E_2)$. Then an operator $F_{12} \in L_C(E_{12})$ is called an <u>T_2-connection</u> for E_1/on E_{12} if, for all $x \in E_1$, the diagrams $(***)_1$ and $(***)_2$ <u>commute up to compact operators</u>.

3.15 <u>Proposition</u> (inheritance properties) Let $T_2, T_2' \in L_C(E_2)$, let T_{12} be a T_2-connection and T_{12}' be a T_2'-connection.

1.) T_{12}^* is a T_2^*-connection
2.) $T_{12}^{(i)}$ is a $T^{(i)}$-connection for $i = 0, 1$.
3.) $T_{12} + T_{12}'$ is a $(T_2 + T_2')$-connection.
4.) $T_{12} \cdot T_{12}'$ is a $(T_2 T_2')$-connection.
5.) If T_2 and T_{12} are normal, then $f(T_{12})$ is an $f(T_2)$-connection for every continuous function f such that the spectra of T_2 and T_{12} are contained in its domain of def.
6.) If E_3 is a Hilbert $\mathcal{D}$-module, $\psi: C \to L_{\mathcal{D}}(E_3)$ is a graded $*$-hom. and $T_3 \in L_{\mathcal{D}}(E_3)$ with $[T_3, \psi(C)] \subseteq K_{\mathcal{D}}(E_3)$, and if

T_{23} is a T_3-connection on $E_2 \hat{\otimes}_C E_3$ and if
T is a T_{23}-connection on $E = E_1 \hat{\otimes}_{\mathcal{B}} (E_2 \hat{\otimes}_C E_3)$, then
T is a F_3-connection on $E \cong (E_1 \hat{\otimes}_{\mathcal{B}} E_2) \hat{\otimes}_C E_3$.

7.) If $E_1 = E_1' \oplus E_1''$ and if we identify $E_1 \hat{\otimes}_{\mathcal{B}} E_2$ with $E_1' \hat{\otimes}_{\mathcal{B}} E_2 \oplus E_1'' \hat{\otimes}_{\mathcal{B}} E_2$ then T_2 has the form $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ and T_{12} has the form $\begin{pmatrix} A_{12} & B_{12} \\ C_{12} & D_2 \end{pmatrix}$ and A_{12} is an A_2-connection on $E_1' \hat{\otimes}_{\mathcal{B}} E_2$ and D_{12} is a D_2-connection on $E_1'' \hat{\otimes}_{\mathcal{B}} E_2$.
Conversely if $T_2 = \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix}$ and A_{12}/D_{12} is an A_2/D_2-connection, then $\begin{pmatrix} A_{12} & 0 \\ 0 & D_2 \end{pmatrix}$ is a T_2-connection.

Pf: <u>Exercise</u>.

3.16 Proposition Let $T_2 \in L_C(E_2)$ and let T_{12} be a T_2-connection.

a) $\forall k \in K_{\mathbb{B}}(E_1)$: $[T_{12}, k \hat{\otimes} 1] \in K_C(E_{12})$

b) T_{12} is a zero-connection on E_{12} iff

$$\forall k \in K_{\mathbb{B}}(E_1): \quad T_{12}(k \hat{\otimes} 1),\ (k \hat{\otimes} 1)T_{12} \in K_C(E_{12}).$$

Pf a) let $k \in K_{\mathbb{B}}(E_1)$ WLOG $k = \vartheta_{y,x}$ for $x, y \in E_1$.
WLOG x, y, T_2, T_{12} are homogeneous with $\partial T_2 = \partial T_{12}$. Then

$$\vartheta_{y,x} \hat{\otimes} 1 = T_y T_x^*$$

by definition of T_x, T_y. Hence

$$(\vartheta_{y,x} \hat{\otimes} 1) \circ T_{12} = T_y \circ T_x^* \circ T_{12} \equiv T_y \circ (-1)^{\partial x \partial T_2} T_2 \circ T_x^*$$

$$\equiv \underbrace{(-1)^{\partial x \partial T_2} (-1)^{\partial y \partial T_2}}_{=(-1)^{(\partial x + \partial y)\partial T_2}} T_{12} \circ T_y \circ T_x^* = (-1)^{\partial \vartheta_{y,x} \partial T_2} T_{12} \circ (\vartheta_{y,x} \hat{\otimes} 1)$$

mod $K_C(E_{12})$

i.e. $[k, T_{12}] \in K_C(E_{12})$.

b) "⇒" Let $k \in K_{\mathbb{B}}(E_1)$. As above, WLOG $k = \vartheta_{y,x}$ for $x, y \in E_1$,

T_{12} is a 0-connection $\Leftrightarrow$ $\forall z \in E_1$: $T_z^* T_{12}$, $T_{12} T_z$ compact.

We hence have $T_{12}(k \hat{\otimes} 1) = T_{12}(T_y T_x^*) = \underbrace{(T_{12} T_y)}_{\text{compact}} T_x^*$ compact iff T_{12} is a 0-conn.

This shows "⇒".

Conversely, if $T_{12}(k \hat{\otimes} 1)$ is compact for all k, then

$$\underbrace{T_{12}(\vartheta_{z,z} \hat{\otimes} 1)}_{\text{compact}} T_{12}^* = T_{12} T_z T_z^* T_{12}^* \text{ is compact for all } z \in E_1.$$

So $(T_{12} T_z)(T_{12} T_z)^* \in K_C(E_{12})$, hence by a Lemma from the first section: $T_{12} T_z \in K_C(E_2, E_{12})$. Similarly for $T_z^* T_{12}$. So T_{12} is a 0-connection

3.17 Lemma: Let $T_2, T_2' \in L_C(E_2)$ such that

$$\forall b \in B: \quad \phi(b)(T_2 - T_2'), \ (T_2 - T_2')\phi(b) \in K_C(E_2)$$

Then: T_{12} is a T_2-connection iff T_{12} is a T_2'-connection

Pf: Let T_{12} be a T_2-connection. Let $x \in E_1$. Find $\tilde{x} \in E_1, b \in B$ such that $x = \tilde{x}b$. Then $T_x = T_{\tilde{x}} \circ \phi(b)$

$$T_{12} \circ T_x \equiv (-1)^{\partial x \partial T_{12}} T_x \circ T_2 = (-1)^{\partial x \partial T_{12}} T_{\tilde{x}} \circ \phi(b) \circ T_2$$

$$\equiv (-1)^{\partial x \partial T_{12}} T_{\tilde{x}} \circ \phi(b) \circ T_2' = (-1)^{\partial x \partial T_{12}} T_x \circ T_2' \mod K_C(E_2, E_{12})$$

and similarly for $T_x^* \circ T_{12}$. ☒

3.18 Theorem (Existence of connections):

Let E_1 be a countably generated Hilbert B-module, E_2 a Hilbert C-module, $\phi: B \to L_C(E_2)$ a graded $*$-hom..

If $T_2 \in L_C(E_2)$ satisfies $\forall b \in B: [T_2, \phi(b)] \in K_C(E_2)$, then there exists an T_2-connection on $E_1 \hat{\otimes}_B E_2$.

Pf: Case 1) Assume $\forall b \in B: [T_2, \phi(b)] = 0$. Then $1 \hat{\otimes}_B T_2$ is a T_2-connection [in particular: 0 is a 0-connection, and if $B = \mathbb{C}$ and ϕ is unital, then the above result always applies.] ☒

Case 2) $\phi: B \to L_C(E_2)$ non-degenerate & $E_1 = B$.

Then $\Phi: B \hat{\otimes}_B E_2 \to E_2, \ b \otimes e_2 \mapsto be_2$ is an isomorphism

$\Rightarrow T_{12} := \Phi^* T_2 \Phi \in L_C(B \hat{\otimes}_B E_2)$ is a T_2-connection because

$\phi(b) = \Phi \circ T_b$ f.a. $b \in B$ and hence

$$T_{12} T_b = \Phi^* T_2 \Phi T_b = \Phi^* T_2 \phi(b) \equiv (-1)^{\partial b \partial T_2} \Phi^* \phi(b) T_2 = (-1)^{\partial b \partial T_2} T_b T_2 \mod K_C(E_2, E_{12})$$

and similarly for T_{12}^*. ☒

Case 3) $\mathbb{B}$ unital, ϕ unital and $E_1 = \hat{\mathbb{H}}_{\mathbb{B}}$.

Note that
$$\hat{\mathbb{H}}_{\mathbb{B}} \hat{\otimes}_{\mathbb{B}} E_2 \cong (\hat{\mathbb{H}} \hat{\otimes}_{\mathbb{C}} \mathbb{B}) \hat{\otimes}_{\mathbb{B}} E_2 \cong \hat{\mathbb{H}} \hat{\otimes}_{\mathbb{C}} (\mathbb{B} \hat{\otimes}_{\mathbb{B}} E_2).$$

From case 2) we know that there is a T_2-connection T_{23} on $\mathbb{B} \hat{\otimes}_{\mathbb{B}} E_2$. From case 1) we know that there is a T_{23}-connection T on $\hat{\mathbb{H}}_{\mathbb{B}} \hat{\otimes}_{\mathbb{B}} E_2$. From 3.15 6) we hence know that T is a T_2-connection on $\hat{\mathbb{H}}_{\mathbb{B}} \hat{\otimes}_{\mathbb{B}} E_2$.

Case 4) $\mathbb{B}$ unital, ϕ unital, E_1 arbitrary.

We have $E_1 \oplus \hat{\mathbb{H}}_{\mathbb{B}} \cong \hat{\mathbb{H}}_{\mathbb{B}}$. By case 3) there is a T_2-connection on $\hat{\mathbb{H}}_{\mathbb{B}} \hat{\otimes}_{\mathbb{B}} E_2$. By 3.15 7) there is also a T_2-connection on $E_1 \hat{\otimes}_{\mathbb{B}} E_2$.

Case 5) general case:

Let $\mathbb{B}^+$ be the unital algebra $\mathbb{B} \oplus \mathbb{C}$ and $\phi^+ : \mathbb{B}^+ \to L_C(E_2)$ be the unital extension of ϕ, e.g. Then E_1 is also a graded $\mathbb{B}^+$-HM. The notion of a T_2-connection does not depend on this change of coefficients and $E_1 \hat{\otimes}_{\mathbb{B}^+} E_2 = E_1 \hat{\otimes}_{\mathbb{B}} E_2$. Also $[T_2, \phi^+(b+\lambda 1)] \in k_C(E_2)$ for all $b + \lambda 1 \in \mathbb{B}^+$. So there is a T_2-connection on $E_1 \hat{\otimes}_{\mathbb{B}} E_2$ by 5).

3.19 Exercise. Show: For every $(E, \phi, T) \in \mathbb{E}(A, \mathbb{B})$ there is some $(E', \phi', T') \in \mathbb{E}(A, \mathbb{B})$ homotopic to (E, ϕ, T) with ϕ' non-degenerate. (actually, you can take $E' = A \cdot E$).

3.20 <u>Definition</u> (Kasparov product) $\mathcal{E}_{12} = (E_{12}, \phi_{12}, T_{12})$ is called a <u>Kasparov product</u> for (E_1, ϕ_1, T_1) and (E_2, ϕ_2, T_2) if

1.) $(E_{12}, \phi_{12}, T_{12}) \in \mathbb{E}(A, C)$,

2.) T_{12} is a T_2-connection on E_{12},

3.) $\forall a \in A$: $\phi_{12}(a)\,[T_1 \hat{\otimes} 1, T_{12}]\,\phi_{12}(a)^* \geq 0$ mod $K_C(T_{12})$.

The set of all operators T_{12} on E_{12} such that $(E_{12}, \phi_{12}, T_{12})$ is a K.p. is denoted by $T_1 \# T_2$.

3.21 <u>Theorem</u> Assume that A is separable. Then there exists a Kasparov product $\mathcal{E}_{12}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. It is unique up to operator homotopy, and T_{12} can be chosen self-adjoint if T_1 and T_2 are self-adjoint.

[It remains to show that the product is well-defined on the level of KK.]

3.22 <u>Examples</u>

a) Assume $T_2 = 0$, i.e. $(E_2, \phi_2, 0) \in \mathbb{M}(B, C)$. Then $T_{12} := T_1 \hat{\otimes} 1$ is a Kasparov product of T_1 and 0:

1.) $(E_{12}, \phi_{12}, T_1 \hat{\otimes} 1) \in \mathbb{E}(A, C)$ as stated above

2.) $T_1 \hat{\otimes} 1$ is a 0-connection because $(k \hat{\otimes} 1)(T_1 \hat{\otimes} 1) = (k T_1) \hat{\otimes} 1 \in K_C(E_{12})$ because $\phi_2(B) \subseteq K_C(E_2)$. (also $T_1 k \hat{\otimes} 1 \in K_C(E_{12})$) for all $k \in K_B(E_1)$.

3.) Let $a \in A$. Then $\phi_{12}(a)\,[T_1 \hat{\otimes} 1, T_1 \hat{\otimes} 1]\,\phi_{12}(a)^* = \phi_{12}(a)\, 2 \cdot T_1^2 \hat{\otimes} 1\, \phi_{12}(a)^*$

$\overset{\text{mod comp.}}{\equiv} 2\,\phi_{12}(a)\,\phi_{12}(a)^* \geq 0$

So the multiplication between $\mathbb{E}(A, B)$ and $\mathbb{M}(B, C)$ defined earlier agrees with the Kasparov product.

b) In particular, the pushforward along a $*$-hom. is a Kasparov product