2.22 Proposition: If $\mathbb{B}$ is σ-unital, then it suffices in the def. of $KK(A,\mathbb{B})$ to consider only those triples (E,ϕ,T) where $E = \hat{\mathbb{H}}_{\mathbb{B}}$.

Pf: $(\hat{\mathbb{H}}_{\mathbb{B}}, 0, 0) \in \mathbb{D}(A,\mathbb{B})$ and hence $(E,\phi,T) \sim (\underbrace{E \oplus \hat{\mathbb{H}}_{\mathbb{B}}}_{\cong \hat{\mathbb{H}}_{\mathbb{B}}}, \phi\oplus 0, T\oplus 0)$.

(and $ev_{t,*}^{\mathbb{B}}(\hat{\mathbb{H}}_{\mathbb{B}[0,1]}) \cong \hat{\mathbb{H}}_{\mathbb{B}}$ for all $t \in [0,1]$) ☒

2.23 Definition: Let $\mathcal{E} = (E,\phi,T) \in \mathbb{E}(A,\mathbb{B})$. Then a „compact perturbation" of T (or of $\mathcal{E}$) is an operator T' (or the cycle (E,ϕ,T')) such that

$$\forall a \in A: \quad \phi(a)\,(T - T') \in K_{\mathbb{B}}(E).$$

2.24 Lemma: In this case: $\mathcal{E}' := (E,\phi,T') \in \mathbb{E}(A,\mathbb{B})$ and $\mathcal{E} \sim \mathcal{E}'$.

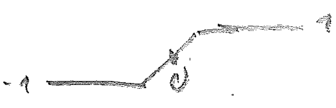
Pf: Consider the straight line segment. ☒

2.25 Proposition: If $(E,\phi,T) \in \mathbb{E}(A,\mathbb{B})$, then there is a „compact perturbation" (E,ϕ,S) of (E,ϕ,T) such that $S^* = S$; so in the definition of $KK(A,\mathbb{B})$ it suffices to consider only those triples with self-adjoint operator; and „compact perturbations", homotopies and operator homotopies may be taken within this class.

Pf: Replace T with $(T + T^*)/2$. ☒

2.26 Proposition If $(E,\phi,T) \in \mathbb{E}(A,\mathbb{B})$, then there is a „compact perturbation" $(E,\phi,S) \in \mathbb{E}(A,\mathbb{B})$ of (E,ϕ,T) with $S = S^*$ and $\|S\| \le 1$. If A is unital we may in addition obtain an S with $S^2 - 1 \in \mathbb{K}(E)$. „Compact perturbations", homotopies and operator homotopies may be taken within this class.

Pf WLOG $T^* = T$. Use functional calculus



If A is unital: $P := \phi(1)$. Replace S with $PSP + \underbrace{(1-P)S(1-P)}_{\text{or id}}$

2.27 The Fredholm picture of $KK(A,B)$

Let A be unital (the σ-unital case is more complicated).

In the definition of KK-theory it suffices to consider only those triples (E, ϕ, T) with ϕ unital (replace E with PE and T with PTP).

If there exists a unital graded $*$-hom. from A to $L_B(\hat{H}_B)$ then WLOG $E = \hat{H}_B$.

If A and B are <u>trivially graded</u>:

Identify $L(\hat{H}_B)$ with $M_2(L(H_B))$ with grading given by $\mathrm{diag}(1,-1)$. $\phi = \mathrm{diag}(\phi_0, \phi_1)$ with $\phi_i : A \to L_B(H_B)$ unital.

$T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ for some $S \in L_B(H_B)$ with $\|S\| \leq 1$.

The intertwining conditions become $S^*S - 1, SS^* - 1 \in K_B(H_B)$, $S\phi_1(a) - \phi_0(a)S \in K_B(H_B)$ f.a. $a \in A$.

homotopy becomes homotopy of triples (ϕ_0, ϕ_1, S) (with strong continuity)

In this picture modules are denoted by

$(E_0 \oplus E_1, \phi_0 \oplus \phi_1, S)$ where $S \in L_B(E_0, E_1)$.

In particular, if $A = \mathbb{C}$ then

$KK(\mathbb{C}, B) \cong \{[T] : T \in L_B(H_B), T^*T - 1, TT^* - 1 \in K_B(H_B)\}$.

2.28 $KK(\mathbb{C},\mathbb{B}) \cong K_0(\mathbb{B})$ for $\mathbb{B}$ trivially graded & σ-unital

Three methods of proof:

1) Assuming $KK(\mathbb{C},\mathbb{B})$ can be described as the set of all triples $(\hat{\mathbb{H}}_{\mathbb{B}}, \phi, T)$ where ϕ is unital, $T=T^*$, $\|T\| \leq 1$ and $T^2-1 \in \mathbb{K}(\hat{\mathbb{H}}_{\mathbb{B}})$ modulo the equivalence relation generated by

a) operator homotopy

b) addition of degenerate cycles (with unital $\mathbb{C}$-action).

I.e. we assume $KK(\mathbb{C},\mathbb{B}) = \hat{KK}(\mathbb{C},\mathbb{B})$.

Then for all such triples T has the form $T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$.

The condition on T is equivalent to $\pi(S)$ being unitary in $Q := L_{\mathbb{B}}(\mathbb{H}_{\mathbb{B}}) / K_{\mathbb{B}}(\mathbb{H}_{\mathbb{B}})$, where $\pi: L_{\mathbb{B}}(\mathbb{H}_{\mathbb{B}}) \to Q$ is the canonical projection.

So every cycle $\mathcal{E}$ for $KK(\mathbb{C},\mathbb{B})$ gives an element $\Psi(\mathcal{E})$ in $K_1(Q)$.

The exact sequence $0 \to \underset{K}{K_{\mathbb{B}}(\mathbb{H}_{\mathbb{B}})} \to \underset{L}{L_{\mathbb{B}}(\mathbb{H}_{\mathbb{B}})} \to Q \to 0$ gives a long exact sequence in K-theory

$$\begin{array}{ccccc} K_0(K) & \to & \overset{0}{K_0(L)} & \to & K_0(Q) \\ \uparrow \text{index} & & & & \downarrow \\ K_1(Q) & \leftarrow & \underset{0}{K_1(L)} & \leftarrow & K_1(K) \end{array}$$

So $K_1(Q) \cong K_0(K) = K_0(\mathbb{K} \otimes \mathbb{B}) \cong K_0(\mathbb{B})$.

So we obtain a map from $KK(\mathbb{C},\mathbb{B})$ to $K_0(\mathbb{B})$ after observing that the K_1-elements are invariant under the elementary „moves" a) and b).

By a general lifting argument you can lift homotopies from Q to L, so Ψ is injective. It is clearly surjective and a homomorphism.

Let $\mathbb{B}$ be unital.

2) Let $(\hat{H}_{\mathbb{B}}, \phi, T)$ be a cycle as above, so $T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$.

We try to define an "index" of $S: H_{\mathbb{B}} \to H_{\mathbb{B}}$ as an element of $K_0(\mathbb{B})$. <u>Problem</u>: The image of S does not have to be closed and $\ker S$, $\operatorname{coker} S$ don't have to be finitely generated and projective.

<u>Solution</u>: One can show that there is an $S' \in L_{\mathbb{B}}(H_{\mathbb{B}})$ such that

$$S - S' \in K_{\mathbb{B}}(H_{\mathbb{B}}) \text{ and}$$

$\ker S'$, $\operatorname{coker} S'^*$ are f.g.p..

<u>Definition</u>: $\operatorname{index}(S) := [\ker S'] - [\ker S'^*] \in K_0(\mathbb{B})$

<u>Problems</u>: 1) Is this well-defined and a homomorphism?
2) Is this invariant under homotopy (again, use $\widetilde{KK}$ instead of KK)?
3) Is it a bijection on the level of $KK(\mathbb{C}, \mathbb{B})$?

For details, see chapter 17 of Wegge-Olsen or Section 4.3 of Bondia, Várilly, Figueroa

3) We define a map from $K_0(\mathbb{B}) \to KK(\mathbb{C}, \mathbb{B})$ for $\mathbb{B}$ unital.

Start with a finitely generated projective $\mathbb{B}$-module E. Find a $\mathbb{B}$-valued inner product on E (one can show that there is an essentially unique one). Define $\Phi([E]) := \left(E \underset{0}{\overset{0}{\rightleftarrows}} 0\right) \in \mathbb{E}(\mathbb{C}, \mathbb{B})$. Moreover, define $\Phi([-E]) := \left(0 \underset{0}{\overset{0}{\rightleftarrows}} E\right)$. Then $\Phi([E] \oplus [-E]) = \left(E \underset{0}{\overset{0}{\rightleftarrows}} E\right)$ $\sim \left(E \underset{id}{\overset{id}{\rightleftarrows}} E\right) \sim 0$ because $id_E \in K_{\mathbb{B}}(E)$ (which one has to show).

So Φ is well-defined as a map from $K_0(\mathbb{B})$ to $KK(\mathbb{C}, \mathbb{B})$.

We indicate how to show that it is surjective:

Let $\mathcal{E} = \left(E_0 \underset{g}{\overset{f}{\rightleftarrows}} E_1\right) \in \mathbb{E}(\mathbb{C}, \mathbb{B})$. Find an $n \in \mathbb{N}$, $R \in K_{\mathbb{B}}(\mathbb{B}^n, E_1)$, $S \in K_{\mathbb{B}}(E_1, \mathbb{B}^n)$ such that

$$\| \underbrace{1 - fg}_{\text{compact}} - RS \| < \tfrac{1}{2}$$

(„every compact operator almost factors through some $\mathbb{B}^n$")

$\Rightarrow$ $fg + RS$ is invertible in $L_{\mathbb{B}}(E_1)$. Define $w := (fg + RS)^{-1}$

note that $w \in 1 + K_{\mathbb{B}}(E_1)$ (exercise)

Now:

$$\left(E_0 \underset{g}{\overset{f}{\rightleftarrows}} E_1\right) \oplus \left(\mathbb{B}^n \underset{0}{\overset{0}{\rightleftarrows}} 0\right) = \left(E_0 \oplus \mathbb{B}^n \underset{\binom{g}{0}}{\overset{(f,0)}{\rightleftarrows}} E_1\right)$$

$$\sim \left(E_0 \oplus \mathbb{B}^n \underset{\check{g} := \binom{g}{S} w}{\overset{\check{f} := (f, R)}{\rightleftarrows}} E_1\right) =: (*).$$

Observe:

$$\check{f}\check{g} = fgw + RSw = (fg + RS)w = \mathrm{id}_{E_1}$$

Hence $\check{p} := \check{g}\check{f} \in L_{\mathbb{B}}(E_0 \oplus \mathbb{B}^n)$ is idempotent.

Let us assume (with some handwaving) that $\check{p} = \check{p}^*$.

Then $E_0 \oplus \mathbb{B}^n \cong \mathrm{Im}\, \check{p} \oplus \mathrm{Im}(1 - \check{p})$.

$$\Rightarrow (*) = \underbrace{\left(\mathrm{Im}\, \check{p} \underset{\check{g}}{\overset{\check{f}}{\rightleftarrows}} E_1\right)}_{\sim 0 \text{ in } KK(\mathbb{C}, \mathbb{B})} \oplus \left(\mathrm{Im}(1 - \check{p}) \underset{0}{\overset{0}{\rightleftarrows}} 0\right)$$

Observe $\check{f}\check{p} = \check{f}$ and $\check{p}\check{g} = \check{g}$. Note

$$1 - \check{p} \in K_{\mathbb{B}}(E_0 \oplus \mathbb{B}^n) \qquad \text{(exercise)}$$

$\Rightarrow$ $\mathrm{Im}(1-\check{p})$ has compact identity $\Rightarrow$ $\mathrm{Im}(1-\check{p})$ is f.g. & p.

$$\Rightarrow \quad [\mathcal{E}] = \underbrace{[\mathrm{Im}\,1-\check{p}]}_{\in \Phi(K_0(\mathbb{B}))} - \underbrace{[\mathbb{B}^n]}_{\in \Phi(K_0(\mathbb{B}))} \qquad \in \Phi(K_0(\mathbb{B}))$$

injectivity: similar.