2. The definition of KK-theory

All C^*-algebras $A, B, C, \ldots$ in this section will be σ-unital.
Let A, B be graded C^*-algebras.

2.1 Definition: A Kasparov A-B module is a triple $\mathcal{E} = (E, \phi, T)$ where E is a countable generated graded Hilbert B-module, $\phi: A \to L(E)$ is a graded $*$-homomorphism and $T \in L(E)$ is an odd operator such that

1) $\forall a \in A: \quad [\phi(a), T] \in K(E)$;
2) $\forall a \in A: \quad \phi(a)(T^2 - id_E) \in K(E)$;
3) $\forall a \in A: \quad \phi(a)(T - T^*) \in K(E)$.

(The commutator in 1) is graded.)
The class of all Kasparov A-B-modules will be denoted by $\mathbb{E}(A, B)$.
Sometimes we denote elements of $\mathbb{E}(A, B)$ also as pairs (E, T) without making reference to the action ϕ.

2.2 Remark: We are not going to discuss many examples at this point. They will occur later in the talks dedicated to applications of KK-theory.

Lab2!
2.3 Definition: a) Let $\mathbb{D}(A, B) \subseteq \mathbb{E}(A, B)$ be the class of degenerate Kasparov A-B-modules; i.e. $(E, \phi, T) \in \mathbb{D}(A, B)$ iff

1) $\forall a \in A: \quad [\phi(a), T] = 0$;
2) $\forall a \in A: \quad \phi(a)(T^2 - id) = 0$;
3) $\forall a \in A: \quad \phi(a)(T - T^*) = 0$.

b) Let $\mathbb{M}(A, B) \subseteq \mathbb{E}(A, B)$ be the class of what I'd call Morita cycles from A to B, i.e. $(E, \phi, T) \in \mathbb{M}(A, B)$ if $T = 0$. Note that $(E, \phi, 0) \in \mathbb{E}(A, B)$ iff $\phi(A) \subseteq K(E)$.

2.4) Definition and Lemma:

a) If $\mathcal{E}_1=(E_1,\phi_1,T_1)$ and $\mathcal{E}_2=(E_2,\phi_2,T_2)$ are elements of $\mathbb{E}(A,B)$, then $\mathcal{E}_1\oplus\mathcal{E}_2 := (E_1\oplus E_2,\ \phi_1\oplus\phi_2,\ T_1\oplus T_2)\in\mathbb{E}(A,B)$.

b) If C is another graded C^*-algebra and $\psi: B\to C$ is an even $*$-homomorphism and $\mathcal{E}=(E,\phi,T)\in\mathbb{E}(A,B)$ then

$$\psi_*(\mathcal{E}) := (\psi_*(E), \phi\hat{\otimes}1, \psi_*(T)=T\hat{\otimes}1)\in\mathbb{E}(A,C).$$

c) If C is another graded C^*-algebra, $\varphi: A\to B$ is an even $*$-hom. and $\mathcal{E}=(E,\phi,T)\in\mathbb{E}(B,C)$, then

$$\varphi^*(\mathcal{E}) := (E, \phi\circ\varphi, T)\in\mathbb{E}(A,C)$$

d) If $\mathcal{E}=(E,\phi,T)\in\mathbb{E}(A,B)$ then

$$-\mathcal{E} := (-E, \phi_-, -T)\in\mathbb{E}(A,B),$$

where $-E$ is the same Hilbert B-module as E but with the grading $\sigma_{-E} := -\sigma_E$, and $\phi_- := \phi\circ\beta_A$ where β_A is the grading on A.

Pf: We only show parts of b) (rest: exercise).

Let $a\in A$. Then

$$(\phi\hat{\otimes}1)(a)\left((T\hat{\otimes}1)^2-\mathrm{id}_{E\otimes_\psi C}\right) = (\phi(a)\hat{\otimes}\mathrm{id}_C)(T^2\hat{\otimes}\mathrm{id}_C-\mathrm{id}_E\hat{\otimes}\mathrm{id}_C)$$

$$= \left(\phi(a)(T^2-\mathrm{id}_E)\right)\hat{\otimes}\mathrm{id}_C = \psi_*\Big(\underbrace{\phi(a)(T^2-\mathrm{id}_E)}_{\in K(E)}\Big)\in K(\psi_*(E)).$$

The other conditions follow similarly.

2.5 <u>Definition</u> Let $\varphi : A \to A'$ and $\psi : \mathbb{B} \to \mathbb{B}'$ be $*$-homomorphisms and let $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, \mathbb{B})$ and $\mathcal{E}' = (E', \phi', T') \in \mathbb{E}(A', \mathbb{B}')$. A homomorphism from $\mathcal{E}$ to $\mathcal{E}'$ with coefficient maps φ and ψ is a homomorphism Φ_ψ from $E_{\mathbb{B}}$ to $E'_{\mathbb{B}'}$ such that

1.) $\forall a \in A \ \forall e \in E: \quad \Phi(\phi(a)e) = \phi'(\varphi(a))\Phi(e)$
i.e. Φ has coefficient map φ on the left.

2.) $\Phi \circ T = T' \circ \Phi$.

The most important case is the case that Φ is bijective and $\varphi = id_A$, $\psi = id_{\mathbb{B}}$. Then $\mathcal{E}$ and $\mathcal{E}'$ are called <u>isomorphic</u>.

Lemma
2.6. We have up to isomorphism (for all $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathbb{E}(A, \mathbb{B})$):

1.) $(\mathcal{E}_1 \oplus \mathcal{E}_2) \oplus \mathcal{E}_3 \cong \mathcal{E}_1 \oplus (\mathcal{E}_2 \oplus \mathcal{E}_3)$

2.) $\mathcal{E}_1 \oplus \mathcal{E}_2 \cong \mathcal{E}_2 \oplus \mathcal{E}_1$

3.) $\mathcal{E} \oplus (0,0,0) \cong \mathcal{E}$

4.) If $\psi : \mathbb{B} \to C$ and $\psi' : C \to D$ then
$\psi'_*(\psi_*(\mathcal{E})) \cong (\psi' \circ \psi)_*(\mathcal{E})$

5.) $(id_{\mathbb{B}})_*(\mathcal{E}) \cong \mathcal{E}$

6.) If $\varphi : A' \to A$ and $\varphi' : A'' \to A$ then
$\varphi'^*(\varphi^*(\mathcal{E})) = (\varphi \circ \varphi')^*(\mathcal{E})$, $\quad id_A^*(\mathcal{E}) = \mathcal{E}$.

7.) $\psi_*(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \psi_*(\mathcal{E}_1) \oplus \psi_*(\mathcal{E}_2)$, $\quad \psi_*(-\mathcal{E}) = -\psi_*(\mathcal{E})$

8.) $\varphi^*(\mathcal{E}_1 \oplus \mathcal{E}_2) = \varphi^*(\mathcal{E}_1) \oplus \varphi^*(\mathcal{E}_2)$, $\quad \varphi^*(-\mathcal{E}) = -\varphi^*(\mathcal{E})$

9.) $\varphi^*(\psi_*(\mathcal{E})) = \psi_*(\varphi^*(\mathcal{E}))$

2.7 <u>Definition</u> Let C be a graded C^*-algebra and $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A,B)$. We now give definition of a cycle $\tau_C(\mathcal{E}) = \mathcal{E} \hat{\otimes} id_C \in \mathbb{E}(A \hat{\otimes} C, B \hat{\otimes} C)$:
The module is $E_B \hat{\otimes} C_C$, the action of $A \hat{\otimes} C$ is $\phi \hat{\otimes} id_C$ and the operator is $T \hat{\otimes} id_C$.

2.8 <u>Example</u> If $C = \mathcal{C}([0,1]) = \{ f : [0,1] \to \mathbb{C},\ f \text{ continuous} \}$, then
$A \hat{\otimes} C \cong A[0,1] = \{ f : [0,1] \to A,\ f \text{ cont.} \}$ and $B \hat{\otimes} C \cong B[0,1]$.
Similarly $E_B \hat{\otimes} C_C \cong E[0,1]$ if $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A,B)$.
Now $\tau_{C[0,1]}(\mathcal{E}) \cong (E[0,1], \phi[0,1], T[0,1]) \in \mathbb{E}(A[0,1], B[0,1])$
under these identifications.

2.9 <u>Definition</u> (notions of homotopy):

Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be in $\mathbb{E}(A,B)$

1.) An operator homotopy from $\mathcal{E}_0$ to $\mathcal{E}_1$ is a norm-continuous path $(T_t)_{t \in [0,1]}$ in $L(E)$ for some graded Hilbert B-module E equipped with a graded left action $\phi : A \to L(E)$ such that

a) $\forall t \in [0,1]$: $(E, \phi, T_t) \in \mathbb{E}(A,B)$

b) $\mathcal{E}_0 \cong (E, \phi, T_0)$, $\mathcal{E}_1 \cong (E, \phi, T_1)$.

2.) A homotopy from $\mathcal{E}_0$ to $\mathcal{E}_1$ is an element $\mathcal{E} \in \mathbb{E}(A, B[0,1])$ such that $ev^B_{0,*}(\mathcal{E}) \cong \mathcal{E}_0$ and $ev^B_{1,*}(\mathcal{E}) \cong \mathcal{E}_1$, where $ev^B_t : B[0,1] \to B,\ \beta \mapsto \beta(t)$ for all $t \in [0,1]$.
We write $\mathcal{E}_0 \sim \mathcal{E}_1$ if such a homotopy exists.

2.10) <u>Lemma</u>: Homotopy is an equivalence relation on $\mathbb{E}(A,B)$.

<u>Pf</u>: 1) Reflexivity: Let $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A,B)$. Then
$i_A^*(\tau_{C[0,1]}(\mathcal{E})) \cong (E[0,1], \phi[0,1] \circ i_A, T[0,1])$ is a homotopy from
$\mathcal{E}$ to $\mathcal{E}$, where $i_A : A \to A[0,1]$.

2) Symmetry: Let $\mathcal{E} \in \mathbb{E}(A, B[0,1])$ and $\psi : B[0,1] \to B[0,1]$, $\beta \mapsto (t \mapsto \beta(1-t))$

Then $ev^B_{t,*}(\psi_*(\mathcal{E})) = \underbrace{(ev^B_t \circ \psi)}_{= ev^B_{1-t}}{}_*(\mathcal{E}) = (ev^B_{1-t})_*(\mathcal{E})$.

3) Transitivity: This is a non-trivial <u>exercise</u> recommended to the reader!

2.11. <u>Definition</u>: Define $KK(A,B) := \mathbb{E}(A,B)/_\sim$. If $\mathcal{E} \in \mathbb{E}(A,B)$ then we denote the corresponding element of $KK(A,B)$ by $[\mathcal{E}]$.

2.12. <u>Lemma</u>: $KK(A,B)$ is an abelian group when equipped with the (well-defined!) operation

$$[\mathcal{E}_1] \oplus [\mathcal{E}_2] := [\mathcal{E}_1 \oplus \mathcal{E}_2].$$

In particular, $KK(A,B)$ is a <u>set</u>.

We have

$$[\mathcal{E}] \oplus [-\mathcal{E}] = [(0,0,0)],$$ where $[(0,0,0)]$ is the zero-element of $KK(A,B)$.

Before we come to the proof of this important lemma, we define:

2.13 Definition: The class $\mathbb{D}(A,B) \subseteq \mathbb{E}(A,B)$ of *degenerate* Kasparov A-B-modules is the class of all elements (E,ϕ,T) such that $[\phi(a),T],\ \phi(a)(T^2-1),\ \phi(a)(T-T^*) = 0$ for all $a \in A$.

2.14 Lemma: If $\mathcal{E}=(E,\phi,T) \in \mathbb{D}(A,B)$ then $\mathcal{E} \sim 0$.

Pf: We construct a homotopy using a mapping cylinder, in this case for the rather trivial homomorphism $0 \xrightarrow{\sigma} E$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E[0,1]_{B[0,1]} \arrow[d, "ev_0^E"] \\
0_B \arrow[r, "\sigma"] & E_B
\end{tikzcd}$$

The pull-back $\mathcal{Z}$ in this diagram can be identified with the Hilbert $B[0,1]$-module $E]0,1] = \{\varepsilon : [0,1] \to E,\ \varepsilon \text{ cont } \& \ \varepsilon(0)=0\}$.

On $E]0,1]$ define an A-action by $(a \cdot \varepsilon)(t) := a(\varepsilon(t))$ f.a. $a \in A$, $\varepsilon \in E]0,1]$ and $t \in [0,1]$. Define $\tilde{T} \in L(E]0,1])$, $\varepsilon \mapsto T \circ \varepsilon$. Then $\tilde{\mathcal{E}} := (E]0,1], \tilde{T}) \in \mathbb{E}(A, B[0,1])$ and $ev^B_{0,*}(\tilde{\mathcal{E}}) \cong 0$ and $ev^B_{1,*}(\tilde{\mathcal{E}}) \cong \mathcal{E}$.

2.15 Proof of Lemma 2.12:

It is obvious that $KK(A,B)$ is a set because the class of isomorphism classes of (countably generated!) Kasparov A-B-modules is small. Moreover, the direct sum is well-defined from 2.6. 7) and [0] is a neutral element by 2.6. 3). Commutativity follows from 2.6. 2). What is left to show is $\mathcal{E} \oplus -\mathcal{E} \sim 0$ for $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A,B)$.

Define $G_t \in L(E \oplus -E)$ to be the element given by the matrix:

$$G_t = \begin{bmatrix} \cos t \cdot T & \sin t \, id_E \\ \sin t \, id_E & -\cos t \cdot T \end{bmatrix}.$$

Then $G_0 = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} = (T \oplus (-T))$, so $(E \oplus -E, \phi \oplus \phi_-, G_0) = (E \oplus E, \phi \oplus \phi_-, T \oplus -T)$

Also $G_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $(E \oplus -E, \phi \oplus \phi_-, G_1) \in \mathbb{D}(A,B)$.

That G_t is odd and $(E \oplus (-E), \phi \oplus \phi_-, G_t) \in \mathbb{E}(A,B)$ for all $t \in \mathbb{R}$ can be checked by direct calculations, see [Jensen/Thomsen, page 57] ☒

2.16 Lemma $KK(A,B)$ is a bifunctor from the category of graded (σ-unital) C^*-algebras and graded $*$-homomorphism to the category of abelian groups.

Pf: Let $\psi: B \to C$ be a graded $*$-homomorphism. Then $\mathcal{E} \mapsto \psi_*(\mathcal{E})$ lifts to a map $\psi_*: KK(A,B) \to KK(A,C)$ [use the diagram
$$\begin{array}{ccc} B[0,1] & \xrightarrow{\psi[0,1]} & C[0,1] \\ \downarrow ev_t^B & \circlearrowleft & \downarrow ev_t^C \\ B & \longrightarrow & C \end{array}$$
and 2.6. 4)]. It is a group homomorphism by 2.6.7) and the construction is functorial by 2.6.4) and 2.6.5).

Using 2.6.6), 2.6.8) and 2.6.9) one shows functoriality in the other component.

2.17 Definition Define $M(A,B) \subseteq \mathbb{E}(A,B)$ be the class of what I call Morita cycles from A to B by $(E,\phi,T) \in M(A,B)$ if $T=0$. Note that $(E,\phi,0) \in \mathbb{E}(A,B)$ iff $\phi(A) \subseteq K(E)$.

If $\psi: A \to B$ is a graded $*$-homomorphism, then we define $(\psi) = (B,\psi,0) \in M(A,B) \subseteq \mathbb{E}(A,B)$. We define $[\psi] = [(\psi)] \in KK(A,B)$.

If ${}_A E_B$ is a graded Morita equivalence, then $A \cong K(E)$, and if ϕ is the left action of A on E then $(E,\phi,0) \in M(A,B) \subseteq \mathbb{E}(A,B)$. We write (E) for $(E,\phi,0) \in \mathbb{E}(A,B)$ and $[E]$ for $[(E)] \in KK(A,B)$.

2.18 ~~Definition~~ & Lemma If $\mathcal{E} = (E,\phi,T) \in \mathbb{E}(A,B)$ and $\mathcal{F} = (F,\phi',0) \in M(B,C)$ then define $\mathcal{E} \hat\otimes_B \mathcal{F} := (E \hat\otimes_B F, \phi \hat\otimes 1, T \hat\otimes 1)$. Then $\mathcal{E} \hat\otimes_B \mathcal{F} \in \mathbb{E}(A,C)$.

This defines a group homomorphism

$$\hat\otimes_B \mathcal{F} : KK(A,B) \to KK(A,C)$$

such that 1) $\mathcal{E} \hat\otimes_B (\psi) = \psi_*(\mathcal{E})$ for all $\psi: B \to C$,

2) $(\mathcal{E} \hat\otimes_B \mathcal{F}) \hat\otimes_C \mathcal{F}' \cong \mathcal{E} \hat\otimes_B (\mathcal{F} \hat\otimes_C \mathcal{F}')$ for all $\mathcal{F}' \in M(C,D)$,

3) $\mathcal{E} \hat\otimes_B (\psi) \hat\otimes_C \mathcal{F}' \cong \psi_*(\mathcal{E}) \hat\otimes_C \mathcal{F}' \cong \mathcal{E} \hat\otimes_B \psi^*(\mathcal{F}')$.

Pf: a) $\hat{\otimes}_{\mathbb{B}} \mathcal{F}$ is well-defined on the level of KK:

If $\tilde{\mathcal{E}} \in \mathbb{E}(A, \mathbb{B}[0,1])$ then, because $\mathcal{F}[0,1] \in \mathbb{M}(\mathbb{B}[0,1], C[0,1])$,

$$ev^{C}_{t,*}\left(\tilde{\mathcal{E}} \hat{\otimes}_{\mathbb{B}[0,1]} \mathcal{F}[0,1]\right) \cong ev^{\mathbb{B}}_{t,*}(\tilde{\mathcal{E}}) \hat{\otimes}_{\mathbb{B}} \mathcal{F},$$

b) $\hat{\otimes}_{\mathbb{B}} \mathcal{F}$ is a group homomorphism: If $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{E}(A,\mathbb{B})$, then

$$(\mathcal{E}_1 \oplus \mathcal{E}_2) \hat{\otimes}_{\mathbb{B}} \mathcal{F} \cong \mathcal{E}_1 \hat{\otimes}_{\mathbb{B}} \mathcal{F} \oplus \mathcal{E}_2 \hat{\otimes}_{\mathbb{B}} \mathcal{F} \qquad \blacksquare$$

2.19 Corollary If $\mathbb{B}$ and $\mathbb{B}'$ are (gradedly) Morita equivalent with Morita equivalence ${}_{\mathbb{B}}E_{\mathbb{B}'}$, then $\otimes_{\mathbb{B}} E$ is an isomorphism

$$KK(A,\mathbb{B}) \cong KK(A,\mathbb{B}').$$

Pf: Let ${}_{\mathbb{B}'}\bar{E}_{\mathbb{B}}$ denote the flipped equivalence. Then

$${}_{\mathbb{B}}E \hat{\otimes}_{\mathbb{B}'} \bar{E}_{\mathbb{B}} \cong {}_{\mathbb{B}}\mathbb{B}_{\mathbb{B}} \quad \text{and} \quad {}_{\mathbb{B}'}\bar{E} \hat{\otimes}_{\mathbb{B}} E_{\mathbb{B}'} \cong {}_{\mathbb{B}'}\mathbb{B}'_{\mathbb{B}'},$$

so
$$\left(\mathcal{E} \hat{\otimes}_{\mathbb{B}} E\right) \hat{\otimes}_{\mathbb{B}'} \bar{E} \cong \mathcal{E} \hat{\otimes}_{\mathbb{B}} (E \hat{\otimes}_{\mathbb{B}'} \bar{E}) \cong \mathcal{E} \hat{\otimes}_{\mathbb{B}} \mathbb{B} = id_{\mathbb{B},*}(\mathcal{E}) \cong \mathcal{E}$$

and likewise
$$\mathcal{E}' \hat{\otimes}_{\mathbb{B}'} \bar{E} \otimes_{\mathbb{B}} E \cong \mathcal{E}'$$

for all $\mathcal{E} \in \mathbb{E}(A,\mathbb{B})$ and $\mathcal{E}' \in \mathbb{E}(A,\mathbb{B}')$.

2.20 Lemma (stability of KK)

Let $\mathbb{K}$ carry the grading given by $(1,-1)$ under an identification $\mathbb{K} \cong \mathbb{K} \otimes M_2(\mathbb{K})$.

a) $\tau_{\mathbb{K}}$ is an isomorphism $KK(A,B) \cong KK(A \hat{\otimes} \mathbb{K}, B \hat{\otimes} \mathbb{K})$.

b) We have $KK(A,B) \cong KK(A \hat{\otimes} \mathbb{K}, B) \cong KK(A, B \hat{\otimes} \mathbb{K})$.

Pf: Exercise (Hint: the inverse map for a) is given in the proof of 17.8.7 of [Blackadar]).

2.21. Lemma (homotopy invariance) Let $\varphi_0, \varphi_1 : B \to C$ be graded $*$-homomorphisms and $\varphi : B \to C[0,1]$ such that $\varphi_t = ev_t^C \circ \varphi$ for $t = 0,1$. Then $[\varphi_0] = [\varphi_1] \in KK(B,C)$ and (φ) is a homotopy from (φ_0) to (φ_1).

It follows that $\varphi_{0,*}(\mathcal{E}) \sim \varphi_{1,*}(\mathcal{E})$ for all $\mathcal{E} \in \mathbb{E}(A,B)$.

2.22. Corollary: If $A \sim 0$ is contractible then $KK(A,A) \cong KK(A,0) = 0$.