9)

1.21 Definition: A C^*-algebra $\mathcal{B}$ is σ-unital if it contains a countable bounded approximate unit.

1.22. Definition: A positive element $h \in \mathcal{B}$ is called strictly positive if and only if $\phi(h) > 0$ for all states ϕ of $\mathcal{B}$.

1.23. Lemma: $\mathcal{B}$ is σ-unital $\Leftrightarrow$ $\mathcal{B}$ contains a strictly positive element.

Pf: Exercise.

1.24 Lemma: $h \in \mathcal{B}_+$ is strictly positive $\Leftrightarrow$ $\overline{h\mathcal{B}} = \mathcal{B}$.

Pf: Lemma 1.1.21 in Jensen/Thomsen.

1.25 Lemma Let E be a Hilbert $\mathcal{B}$-module, $T \in K(E)$ positive. Then T is strictly positive $\Leftrightarrow$ $\overline{T(E)} = E$.

Pf: Lemma 1.1.22 in Jensen/Thomsen.

1.26 Definition: A Hilbert $\mathcal{B}$-module E is called countably generated if there is a set $\{x_n : n \in \mathbb{N}\} \subseteq E$ such that the span of the set $\{x_n b : n \in \mathbb{N}, b \in \mathcal{B}\}$ is dense in E.

We will show: E countably generated $\Leftrightarrow$ $K(E)$ σ-unital.

This is a consequence of the following important theorem:

1.27 Theorem: <u>Kasparov's stabilization theorem</u>

If E is a countably generated Hilbert $\mathcal{B}$-module then

$$E \oplus \mathbb{H}_{\mathcal{B}} \cong \mathbb{H}_{\mathcal{B}}.$$

Pf: [Jensen/Thomsen page 11/12]

WLOG we can assume $\mathcal{B}$ to be unital.

We want to define a unitary $V: \mathbb{H}_{\mathcal{B}} \to E \oplus \mathbb{H}_{\mathcal{B}}$. *

Let ξ_k be the kth standard "basis" vector in $\mathbb{H}_{\mathcal{B}}$.

Let $(\eta_k)_{k\in\mathbb{N}}$ be a generating sequence of E. WLOG $\|\eta_k\| \leq 1$ for $k\in\mathbb{N}$ such that $\forall k\in\mathbb{N}: \{\ell\in\mathbb{N} \mid \eta_k = \eta_\ell\}$ is infinite.

Define $T := \sum_k 2^{-k} \theta_{(\eta_k, 2^{-k}\xi_k), \xi_k}$.

1) T has dense range: Let $k\in\mathbb{N}$. Then for every $\ell\in\mathbb{N}$ with $\eta_k = \eta_\ell$ we have $T(\xi_\ell) = 2^{-\ell}(\eta_k, 2^{-\ell}\xi_\ell)$, so

$$T(2^\ell \xi_\ell) = (\eta_k, 2^{-\ell}\xi_\ell) \to (\eta_k, 0) \text{ for } \ell\to\infty.$$

So $(\eta_k, 0) \in \overline{T(\mathbb{H}_{\mathcal{B}})}$. So also $2^k\left[(\eta_k, 2^{-k}\xi_k) - (\eta_k, 0)\right] = (0,\xi_k) \in \overline{T(\mathbb{H}_{\mathcal{B}})}$.

2) T^*T has dense range:

$$T^*T = \sum_{k,\ell} 2^{-k-\ell}\, \theta_{\xi_k(\langle\eta_k,\eta_\ell\rangle + \langle 2^{-k}\xi_k, 2^{-\ell}\xi_\ell\rangle), \xi_\ell}$$

$$= \underbrace{\sum_k 4^{-2k}\theta_{\xi_k,\xi_k}}_{=:S} + \Big(\sum_k 2^{-k}\theta_{(\eta_k,0),\xi_k}\Big)^*\Big(\sum_k 2^{-k}\theta_{(\eta_k,0),\xi_k}\Big) \geq S$$

* idea: Instead of defining V, we define $T \in \mathcal{L}(\mathbb{H}_{\mathcal{B}}, E\oplus\mathbb{H}_{\mathcal{B}})$ such that T and $|T| = (T^*T)^{\frac{1}{2}}$ have dense image. Then the isometry(!) V defined by $V(|T|(x)) := Tx$ can be extended to a unitary from $\mathbb{H}_{\mathcal{B}}$ to $E\oplus\mathbb{H}_{\mathcal{B}}$ (Range $V \supseteq$ Range T).

S is positive and has dense range, so it is strictly positive in $K(H|_B)$. So T^*T is strictly positive in $K(H|_B)$, so T^*T has dense range.

3) $|T|$ has dense range ✓ because $\text{Range}\,|T| \supseteq \text{Range}\,T^*T$.

1.28 <u>Corollary</u> $E_{\mathcal{B}}$ countably generated $\Leftrightarrow$ $K(E)$ σ-unital.

<u>Proof</u>: $\Rightarrow$

Case 1) $\mathcal{B}$ unital, $E = \mathbb{H}_{\mathcal{B}}$: Let $\xi_i = (0, 0, \dots, \underset{\in \mathcal{B}}{\overset{i^{\text{th}} \text{ position}}{1}}, 0, 0, \dots) \in \mathbb{H}_{\mathcal{B}}$

Then $h := \sum_{i=1}^{\infty} 2^{-i} \Theta_{\xi_i, \xi_i}$ is strictly positive since it has dense range.

Case 2) $\mathcal{B}$ unital, $E = P\,\mathbb{H}_{\mathcal{B}}$ for some $P \in L(\mathbb{H}_{\mathcal{B}})$ with $P = P^* = P^2$.

Then $PhP = \sum_{i=1}^{\infty} 2^{-i} \Theta_{P\xi_i, P\xi_i}$ is strictly positive in $K(E)$.

Note that $E = P\,\mathbb{H}_{\mathcal{B}}$ is not a restriction on E by 1.27.

Case 3) general case: $E_{\mathcal{B}}$ countably generated $\Leftrightarrow$ $E_{\mathcal{B}^+}$ countably gen. $\overset{\text{case 2)}}{\Leftrightarrow}$ $K_{\mathcal{B}^+}(E)$ σ-unital $\underset{K_{\mathcal{B}^+}(E) = K_{\mathcal{B}}(E)}{\Leftrightarrow}$ $K(E)$ σ-unital.

1.29 <u>Definition</u> Let B, C be C^*-algebras. Let E_B and F_C be Hilbert modules and $\phi: B \to L(F)$ be a $*$-hom.

On $E \otimes_{alg} F \times E \otimes_{alg} F$ define $\langle e\otimes f, e'\otimes f'\rangle := \langle f, \phi(\langle e,e'\rangle) f'\rangle \in C$.

This defines a C-valued bilinear map. Define $N := \{ t \in E \otimes_{alg} F \mid \langle t,t\rangle = 0\}$.

Then $\langle\ ,\ \rangle$ defines an inner product on $E \otimes_{alg} F / N$ which turns it into a pre-Hilbert C-module.

The completion is called the <u>inner tensor product</u> of E and F and is denoted by $E \otimes_B F$ or $E \otimes_\phi F$.

1.30 <u>Lemma</u> Let E_{1B}, E_{2B} and F_C be Hilbert modules and $\phi: B \to L(F)$ be a $*$-homomorphism. Let $T \in L(E_1, E_2)$.

Then $e_1 \otimes f \mapsto T(e_1) \otimes f$ defines a map $T \otimes 1 \in L(E_1 \otimes_B F, E_2 \otimes_B F)$ such that $(T\otimes 1)^* = T^* \otimes 1$ and $\|T\otimes 1\| \leq \|T\|$.

If $\phi(B) \subseteq K(F)$ then $T \in K(E_1, E_2)$ implies $T\otimes 1 \in K(E_1 \otimes F, E_2 \otimes F)$.

<u>Proof</u>: Only the last assertion: The map $T \mapsto T\otimes 1$ is linear and contractive from $L(E_1, E_2)$ to $L(E_1 \otimes F, E_2 \otimes F)$. So it suffices to consider T of the form ϑ_{e_2, e_1} with $e_2 \in E_2$ and $e_1 \in E_1$.

Because $E_2 = E_2 \cdot B$ it suffices to consider $\vartheta_{e_2 b, e_1}$ with $b \in B$.

Now
$$\left(\vartheta_{e_2 b, e_1} \otimes 1\right)(e_1' \otimes f) = \vartheta_{e_2 b, e_1}(e_1') \otimes f = e_2 b \langle e_1, e_1'\rangle \otimes f$$
$$= e_2 \otimes \phi(b)\phi(\langle e_1, e_1'\rangle) f = \left(M_{e_2} \circ \phi(b) \circ N_{e_1}\right)(e_1' \otimes f) \quad \text{f.a. } e_1' \otimes f \in E_1 \otimes F$$

where $M_{e_2}: F \to E_2 \otimes_B F$, $f' \mapsto e_2 \otimes f'$ and $N_{e_1}: E_1 \otimes_B F \to F$, $e_1' \otimes f' \mapsto \phi(\langle e_1, e_1'\rangle) f'$.

Because $M_{e_2} \in L(F, E_2 \otimes F)$, $\phi(b) \in K(F)$ and $N_{e_1} \in L(E_1 \otimes_B F, F)$ we have $\vartheta_{e_2 b, e_1} \otimes 1 \in K(\ldots)$.

1.31. <u>Lemma</u>: Let $\mathbb{B}$ and C be C^*-algebras and $\phi: \mathbb{B} \to C$ be a $*$-hom. Define $\tilde{\phi}: \mathbb{B} \to L(C) = M(C)$, $b \mapsto [c \mapsto \phi(b) \cdot c]$. Then $\tilde{\phi}(\mathbb{B}) \subseteq K(C)$. ☒

1.32 <u>Definition</u>: Let $E_{\mathbb{B}}$ be a Hilbert module and $\phi: \mathbb{B} \to C$ be a $*$-hom. Define the push-forward $\phi_*(E)$ as $E \underset{\mathbb{B}}{\otimes} C = E \underset{\phi}{\otimes} C$.

1.33 <u>Lemma</u>: 1.) $(id_{\mathbb{B}})_*(E) = E \underset{\mathbb{B}}{\otimes} \mathbb{B} \cong E$, canonically.

2.) $\psi_*(\phi_*(E)) \cong (\psi \circ \phi)_*(E)$, naturally, where $\psi: C \to D$.

Pf: Ex.

1.34 <u>Definition</u>: $\phi: \mathbb{B} \to C$ and E_1, E_2 Hilbert $\mathbb{B}$-modules. $T \in L(E_1, E_2)$.

$$\phi_*(T) := T \otimes 1 \in L(E_1 \underset{\mathbb{B}}{\otimes} C, E_2 \underset{\mathbb{B}}{\otimes} C) = L(\phi_*(E_1), \phi_*(E_2)).$$

1.35 <u>Lemma</u>: $T \in K(E_1, E_2) \Rightarrow \phi_*(T) \in K(\phi_*(E_1), \phi_*(E_2))$. ☒

Moreover: $\phi_*(\theta_{e_2 b_2, e_1 b_1}) = \theta_{e_2 \otimes \phi(b_2), e_1 \otimes \phi(b_1)}$ f.a. $b_1, b_2 \in \mathbb{B}$, $e_1 \in E_1$, $e_2 \in E_2$.

1.36 <u>Remark</u>: 1.) The push-forward has the following universal property. If $\phi: \mathbb{B} \to C$ and if $E_{\mathbb{B}}$ is a Hilbert module, then there is a natural homomorphism $\underline{\Phi}_\phi: E_{\mathbb{B}} \cong E \underset{\mathbb{B}}{\otimes} \mathbb{B} \to E \underset{\mathbb{B}}{\otimes} C = \phi_*(E)$ (defined by $\underline{\Phi}(e \otimes b) := e \otimes \phi(b)$).

If $\underline{\Psi}_\phi: E_{\mathbb{B}} \to F_C$ is any homomorphism with coefficient map ϕ, then there is a unique homomorphism $\tilde{\underline{\Psi}}_{id_C}: \phi_*(E)_C \to F_C$ (defined by $\tilde{\underline{\Psi}}(e \otimes c) := \underline{\Psi}(e)c$) such that the following diagram commutes:

$$\begin{tikzcd}
E \cong E \underset{\mathbb{B}}{\otimes} \mathbb{B} \arrow[rr, "\underline{\Psi}_\phi"] \arrow[dr, "\underline{\Phi}_\phi"'] & & F \\
& \phi_*(E) \arrow[ur, "\tilde{\underline{\Psi}}_{id_C}"'] &
\end{tikzcd}$$

1.36 2) You can show that $K(\cdot)$ is a functor:

If $E_{\mathcal{B}} \xrightarrow{\Psi_\phi} F_{C}$ is a homomorphism with coefficient map then there is a unique $*$-homomorphism $\Theta : K(E) \to K(F)$ such that $\Theta(\vartheta_{e,e'}) = \vartheta_{\Psi(e),\Psi(e')} \in K(F)$.

Exercise: Prove this fact.

1.37 Definition Let $\mathbb{B}$ and $\mathbb{B}'$ be C^*-algebras and $\mathbb{B}\otimes\mathbb{B}'$ their tensor product (minimal, maximal ...). Let $E_{\mathbb{B}}$ and $E'_{\mathbb{B}'}$ be Hilbert modules. Then define a bilinear map

$$\langle\, , \rangle : E\underset{alg}{\otimes} E' \times E\underset{alg}{\otimes} E' \to \mathbb{B}\otimes\mathbb{B}', \ (e_1\otimes e_1', e_2\otimes e_2') \mapsto \langle e_1, e_2\rangle \otimes \langle e_1', e_2'\rangle .$$

This defines an inner product on $E\underset{alg.}{\otimes}E'$. Its completion, denoted by $E\underset{\mathbb{C}}{\otimes}E' = E\otimes E'$ is a Hilbert $\mathbb{B}\otimes\mathbb{B}'$-module called the <u>outer tensor product</u> of $E_{\mathbb{B}}$ and $E'_{\mathbb{B}'}$.

or <u>external</u>

1.38 <u>Definition</u>: A <u>graded</u> C^*-algebra is a C^*-algebra $\mathcal{B}$ equipped with an order two $*$-automorphism $\beta_{\mathcal{B}}$ (called the grading (automorphism) of $\mathcal{B}$), i.e. $\beta_{\mathcal{B}}^2 = id_{\mathcal{B}}$.

A $*$-homomorphism ϕ from a graded algebra $(\mathcal{B}, \beta_{\mathcal{B}})$ to a graded algebra (C, β_C) is <u>graded</u> if
$$\beta_C \circ \phi = \phi \circ \beta_{\mathcal{B}}.$$

If $(\mathcal{B}, \beta_{\mathcal{B}})$ is graded, then $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ with $\mathcal{B}_0 = \{b \in \mathcal{B} \mid \beta_{\mathcal{B}}(b) = b\}$ and $\mathcal{B}_1 = \{b \in \mathcal{B} \mid \beta_{\mathcal{B}}(b) = -b\}$. The elements $b \in \mathcal{B}_0$ are called <u>even</u> (write $\deg b = 0$) and the elements $b \in \mathcal{B}_1$ are called <u>odd</u> (write $\deg b = 1$, $\deg 0$ is defined as you like). An element of $\mathcal{B}_0 \cup \mathcal{B}_1$ is called <u>homogeneous</u>.

1.39 <u>Remark</u>: Note that $\mathcal{B}_0 \cdot \mathcal{B}_0 \subseteq \mathcal{B}_0$, $\mathcal{B}_0 \cdot \mathcal{B}_1 \subseteq \mathcal{B}_1$
$\mathcal{B}_1 \cdot \mathcal{B}_0 \subseteq \mathcal{B}_1$, $\mathcal{B}_1 \cdot \mathcal{B}_1 \subseteq \mathcal{B}_0$.

Moreover, $\phi : \mathcal{B} \to C$ is graded iff $\phi(\mathcal{B}_i) \subseteq C_i$ for $i = 0, 1$.

1.40 Definition & Lemma: If $\mathcal{B}$ is graded then the <u>graded commutator</u> of $\mathcal{B}$ is defined on homogeneous elements by
$$[a, b] := ab - (-1)^{\deg a \cdot \deg b}\, ba.$$
It satisfies the following relations (exercise!):

1) $[a, b] = -(-1)^{\deg a \deg b} [b, a]$

2) $[a, bc] = [a, b]c + (-1)^{\deg a \deg b}\, b[a, c]$

3) $(-1)^{\deg a \deg c} [[a, b], c] + (-1)^{\deg a \deg b} [[b, c], a]$
$+ (-1)^{\deg b \deg c} [[c, a], b] = 0.$

1.41. Definition Let A and B be graded C^*-algebras. Define their graded tensor product as follows:

On $A \otimes_{alg} B$ define

$$(a_1 \hat{\otimes} b_1) \cdot (a_2 \hat{\otimes} b_2) := (-1)^{\deg b_1 \deg a_2} (a_1 a_2 \hat{\otimes} b_1 b_2)$$

and

$$(a_1 \hat{\otimes} b_1)^* := (-1)^{\deg a_1 \deg b_1} (a_1^* \hat{\otimes} b_1^*)$$

for all homogeneous elements $a_1, a_2 \in A$, $b_1, b_2 \in B$. Define a grading automorphism by $\beta_{A \hat{\otimes} B} := \beta_A \otimes \beta_B$.

[Just as in the ungraded case there are several feasible norms on $A \otimes_{alg} B$, and among them there is a maximal one. Completed for this norm the algebra $A \otimes_{alg} B$ becomes the maximal graded tensor product $A \hat{\otimes}_{max} B$. There is also a spacial graded tensor product $A \hat{\otimes} B$. In general, these completions can be different from their ungraded counterparts, but in the cases we are interested in, they agree. We will hence don't make a fuss about those norms. The interested reader might want to consult Blackadar's book, sections 14.4 and 14.5.]

1.42. Prop The (spacial) graded tensor product is associative ($A \hat{\otimes} (B \hat{\otimes} C) \cong (A \hat{\otimes} B) \hat{\otimes} C$) and commutative ($A \hat{\otimes} B \cong B \hat{\otimes} A$ via $a \hat{\otimes} b \mapsto (-1)^{\deg a \deg b}\, b \hat{\otimes} a$).

1.43 Examples

a) If A is an ungraded C^*-algebra, then id_A is a grading automorphism on A which we call the trivial grading. With this grading, A is called trivially graded.

b) If A is a C^*-algebra and $u \in M(A)$ satisfies $u^* = u = u^{-1}$ then one can define a grading on A by $a \mapsto uau^{-1}$.
Such a grading is called an inner grading.
[Remark: we will see later that inner gradings are the less interesting gradings.]

c) On $\mathbb{C}_{(1)} := \mathbb{C} \oplus \mathbb{C}$ define the following grading automorphism:

$$(a,b) \mapsto (b,a).$$

Then $(\mathbb{C}_{(1)})_0 = \{(a,a) \mid a \in \mathbb{C}\}$ and $(\mathbb{C}_{(1)})_1 = \{(a,-a) \mid a \in \mathbb{C}\}$.

This grading is called the standard odd grading.

d) More generally, define the odd grading also on $A_{(1)} := A \oplus A$ for any C^*-algebra A. Note that $A_{(1)} \cong A \hat{\otimes} \mathbb{C}_{(1)}$.

e) Alternatively, define $\mathbb{C}_1 := \mathbb{C} \oplus \mathbb{C}$ as follows:

The multiplication is given by $(1,0)\cdot(1,0) = (0,1)\cdot(0,1) = (1,0)$
and $(1,0)\cdot(0,1) = (0,1)\cdot(1,0) = (0,1)$.

The involution is given by $(a,b)^* = (\bar{a}, \bar{b})$.

The norm is given by $\|(a,b)\| = \max\{|a+b|, |a-b|\}$.

The grading is given by $(a,b) \mapsto (a,-b)$.

Then $\mathbb{C}_1$ is a graded C^*-algebra.

Exercises: 1) Show $\mathbb{C}_1 \cong \mathbb{C}_{(1)}$ as graded C^* algebras.

2) Let $\mathbb{C}_1$ act on the Hilbert space $\mathbb{C} \oplus \mathbb{C} (= \Lambda^* \mathbb{C})$ by $(a,b) \mapsto \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Show that this is a faithful rep. and calculate the norm on $\mathbb{C}_1$ explic

1.44. <u>Definition:</u> Let $n \in \mathbb{N}$. Let $\mathbb{C}_n$ be the (universal) unital $\mathbb{C}$-algebra defined in the following way; called the <u>nth complex Clifford algebra:</u>

1.) there is a $\mathbb{R}$-linear map $i : \mathbb{R}^n \to \mathbb{C}_n$ such that

$$i(v) \cdot i(v) = \langle v,v\rangle \cdot 1_{\mathbb{C}_n} \in \mathbb{C}_n \quad \text{f.a. } v \in \mathbb{R}^n$$

2.) if $\varphi : \mathbb{R} \to A$ is any $\mathbb{R}$-linear map from $\mathbb{C}^n$ to a unital $\mathbb{C}$-algebra satisfying 1.) then there is a unique unital $\mathbb{C}$-linear homomorphism $\hat{\varphi} : \mathbb{C}_n \to A$ such that $\varphi = \hat{\varphi} \circ i$.

Consider the complexified exterior algebra $\Lambda^*_{\mathbb{C}} \mathbb{R}^n$. It has a canonical Hilbert space structure. Let $\mathbb{C}_n$ act on $\Lambda^*_{\mathbb{C}} \mathbb{R}^n$ as follows: If $v \in \mathbb{R}^n$ then define $\mu(v) := \mathrm{ext}(v) + \mathrm{ext}(v)^* \in L(\Lambda^*_{\mathbb{C}} \mathbb{R}^n)$. From the universal property of the Clifford algebra we obtain a homomorphism from $\mathbb{C}_n$ to $L(\Lambda^*_{\mathbb{C}} \mathbb{R}^n)$.

On $\mathbb{C}_n$ we have an involution induced by the map

$$(v_1 \cdot v_2 \cdot \dots \cdot v_k)^* := v_k \cdot v_{k-1} \cdot \dots \cdot v_2 \cdot v_1 \quad \text{for all } v_1, \dots, v_k \in \mathbb{R}^n.$$

With this involution, $\mathbb{C}_n$ is a $*$-algebra and $\mu : \mathbb{C}_n \to L(\Lambda^*_{\mathbb{C}} \mathbb{R}^n)$ a $*$-homomorphism. It defines a <u>C^*-algebra structure on $\mathbb{C}_n$</u>.

1.45 <u>Examples:</u> a) $\mathbb{C}_1$ is the two-dimensional algebra defined above

b) $\mathbb{C}_2$ is the four-dimensional algebra with the basis

$$1, \; e_1, \; e_2, \; e_1 \cdot e_2 \quad \text{such that } e_1^2 = e_2^2 = 1 \text{ and } e_1 e_2 = -e_2 e_1.$$

1.46. <u>Definition:</u> The unitary map $v \mapsto -v$ on $\mathbb{R}^n$ lifts to an isomorphism $\beta_n : \mathbb{C}_n \to \mathbb{C}_n$ such that $(\beta_n)^2 = 1$. <u>It is a grading on $\mathbb{C}_n$</u>.

1.47 <u>Exercise:</u> Show that $\mathbb{C}_2$ is isomorphic to $M(2 \times 2)$ with the inner grading given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

1.48 Proposition We have $\mathbb{C}_{m+n} \cong \mathbb{C}_m \hat{\otimes} \mathbb{C}_n$ for all $m,n \in \mathbb{N}$.

Pf: Define $V := \mathbb{R}^m$ and $W := \mathbb{R}^n$. Let $i_m : V \to \mathbb{C}_m$, $i_m : W \to \mathbb{C}_n$ and $i_{m+n} : V \oplus W \to \mathbb{C}_{m+n}$ be the canonical injections. Let $\pi_V : V \oplus W \to V$ and $\pi_W : V \oplus W \to W$ be the canonical projections. Then

$$i := (i_V \hat{\otimes} 1) \circ \pi_V \oplus (1 \hat{\otimes} i_W) \circ \pi_W : V \oplus W \to \mathbb{C}_m \hat{\otimes} \mathbb{C}_n$$

satisfies $i(x)i(x) = \langle x,x \rangle 1_{\mathbb{C}_m \hat{\otimes} \mathbb{C}_n}$, so there is a unital $\mathbb{C}$-linear homomorphism $\hat{i} : \mathbb{C}_{m+n} \to \mathbb{C}_m \hat{\otimes} \mathbb{C}_n$ such that $i = \hat{i} \circ i_{m+n}$.
Similarly, one can construct homomorphisms $\mathbb{C}_m \to \mathbb{C}_{m+n}$ and $\mathbb{C}_n \to \mathbb{C}_{m+n}$ which gradedly commute, so there is a hom. $\mathbb{C}_m \hat{\otimes} \mathbb{C}_n \to \mathbb{C}_{m+n}$. It is an inverse of $\hat{i}$. ∎

1.49. Proposition: If $\underset{2m}{\overset{\shortparallel}{n}} \in \mathbb{N}$ is even then $\mathbb{C}_n \cong M(2^m)$ with an inner grading, if $n = 2m+1$ is odd, then $\mathbb{C}_n \cong M(2^m) \oplus M(2^m)$ with standard odd grading.

Pf: [Blackadar 14.5.1] ∎ Remark: You have done the case $n=2$ by hand (1.47).

1.50) <u>Definition</u> Let $(\mathcal{B}, \beta_{\mathcal{B}})$ be a graded C^*-algebra and $E_{\mathcal{B}}$ be a Hilbert module. A <u>grading automorphism</u> $\sigma_E : E \to E$ is a homomorphism with coefficient map $\beta_{\mathcal{B}}$ such that $\sigma_E^2 = id_E$, i.e. $\langle \sigma_E(e), \sigma_E(f) \rangle = \beta_E(\langle e,f \rangle)$ and $\sigma_E(eb) = \sigma_E(e)\beta_B(b)$ f.a. $e,f \in E$ and $b \in \mathcal{B}$.

1.51) <u>Remark</u>: With $E_0 := \{e \in E \mid \sigma_E(e) = e\}$ and $E_1 := \{e \in E \mid \sigma_E(e) = -e\}$ we have:

$$\langle E_i, E_j \rangle \subseteq \mathcal{B}_{i+j} \quad \text{and}$$

$$E_i \mathcal{B}_j \subseteq E_{i+j}.$$

If $\mathcal{B}$ is trivially graded then it still makes sense to consider graded Hilbert $\mathcal{B}$-modules; they are just orthogonal direct sums of two Hilbert $\mathcal{B}$-modules.

1.52) <u>Definition & Lemma</u>: If E and F are graded Hilbert modules over the graded C^*-algebra $\mathcal{B}$, then define

$$\sigma_{L(E,F)}(T) := \sigma_F \circ T \circ \sigma_E \quad \text{for all } T \in L(E,F).$$

This map satisfies:

1.) $\sigma^2_{L(E,F)}(T) = T$ f.a. $T \in L(E,F)$

2.) $\sigma_{L(F,E)}(T^*) = [\sigma_{L(E,F)}(T)]^*$ f.a. $T \in L(E,F)$

3.) $\sigma_{L(E,G)}(T \circ S) = \sigma_{L(F,G)}(T) \circ \sigma_{L(E,F)}(S)$ f.a. $T \in L(F,G)$ and $S \in L(E,F)$ where $G_{\mathcal{B}}$ is a H.M.

and $\sigma_{L(E)}(id_E) = id_E$.

4.) $\sigma_{L(E,F)}(K(E,F)) \subseteq K(E,F)$ with $\sigma_{L(E,F)}(\theta_{f,e}) = \theta_{\sigma_F(f), \sigma_E(e)}$ f.a. $e \in E$, $f \in F$.

<u>Pf</u>: <u>Ex.</u>

1.53) <u>Corollary</u>: If E is a graded Hilbert $\mathbb{B}$-module, then $L(E)$ and $K(E)$ are graded C^*-algebras.

1.54) <u>Definition</u>: The elements of $L(E,F)_0$ are called <u>even</u>, written $L(E,F)^{\text{even}}$, the elements of $L(E,F)_1$ are called <u>odd</u>.

1.55) <u>Remark</u>: An even element of $L(E,F)$ maps E_0 to F_0 and E_1 to F_1, an odd element maps E_0 to F_1 and E_1 to F_0.

1.56) <u>Remark</u>: The following concepts and results can easily be adapted from the trivially graded case to the general graded case. The details are left to the reader:

a) graded homomorphism with graded coefficient maps.

b) Kasparov's stabilisation theorem
caveat: $\mathbb{H}_{\mathbb{B}}$ has to be replaced by $\hat{\mathbb{H}}_{\mathbb{B}} := \mathbb{H}_{\mathbb{B}} \oplus \mathbb{H}_{\mathbb{B}}$ with grading $S := (\beta_{\mathbb{B}}, \beta_{\mathbb{B}}, \dots)$ on the first summand and $-S$ on the second summand.

c) the inner tensor product of graded Hilbert modules

d) the exterior tensor product of graded Hilbert modules
Caveat: the inner product is defined by
$$\langle e_1 \hat{\otimes} f_1, e_2 \hat{\otimes} f_2 \rangle = (-1)^{\deg f_1 \cdot (\deg e_1 + \deg e_2)} \langle e_1, e_2 \rangle \hat{\otimes} \langle f_1, f_2 \rangle$$

e) the push-forward along graded $*$-homomorphisms