AN INTRODUCTION TO KK-THEORY

These are the lecture notes of Walther Paravicini in the Focused Semester 2009 in Münster; the notes were taken by Lin Shan.
In these notes, all C^*-algebras are complex algebras.

1. Hilbert modules and adjointable operators

Let B be a C^*-algebra.

DEFINITION 1. A (right) pre-Hilbert module E over B is a complex vector space E which is at the same time a (right) B-module compatible with the vector space structure of E and is equipped with a map

$$\langle \cdot, \cdot \rangle : E \times E \to B,$$

such that

1. $\langle \cdot, \cdot \rangle$ is sesquilinear (linear in the right component);
2. $\forall b \in B$ and $\forall e, f \in E$, $\langle e, fb \rangle = \langle e, f \rangle b$;
3. $\forall e, f \in E$, $\langle e, f \rangle^* = \langle f, e \rangle \in B$;
4. $\forall e \in E$, $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0$ if and only if $e = 0$.

Define $\|e\| = \sqrt{\langle e, e \rangle}$ for all $e \in E$. If E is complete with respect to this norm, then we call E a Hilbert B-module. E is called full if $\overline{\langle E, E \rangle} = B$.

Exercise 2. Show that $\|\cdot\|$ defines a norm on E.

Example 3.

1. If $B = \mathbb{C}$, then a Hilbert module over B is the same as a Hilbert space;

2. B itself is a B-module with the module action

   $$e \cdot b = eb \quad \forall e, b \in B$$

   and the inner product

   $$\langle e, f \rangle = e^* f \in B \quad \forall e, f \in B;$$

3. More generally, any closed right ideal $I \leq B$ is a right Hilbert B-module;
4. Let $(E_i)_{i \in I}$ be a family of pre-Hilbert B-modules. Then the direct sum $\oplus_{i \in I} E_i$ is a pre-Hilbert B-module with the inner product

   $$\langle (e_i), (f_i) \rangle = \sum_{i \in I} \langle e_i, f_i \rangle_{E_i}.$$

   Because the completion of a pre-Hilbert B-module is a Hilbert B-module, we can form the completion of $\oplus_{i \in I} E_i$, and also call it $\oplus_{i \in I} E_i$;
5. In the above example, let $I = \mathbb{N}$ and $E_i = B$. Define $\mathbb{H}_B = \oplus_{i \in \mathbb{N}} B$ to be the Hilbert B-module.

Example 4. Define

$$\ell^2(\mathbb{N}, B) = \left\{ (b_i)_{i\in\mathbb{N}} | b_i \in B \ \forall\ i \in \mathbb{N} \text{ and } \sum_{i\in\mathbb{N}} \|b_i\|^2 < \infty \right\}.$$

Show that $\ell^2(\mathbb{N}, B) \subset \mathbb{H}_B$ and find an example such that $\ell^2(\mathbb{N}, B) \neq \mathbb{H}_B$.

LEMMA 5. *If E is a pre-Hilbert B-module, then for all $e, f \in E$*

$$\|e\| \, \|f\| \geq \|\langle e, f\rangle\| \, .$$

Proof. If $f \neq 0$, define $b = \dfrac{-\langle f, e\rangle}{\|f\|^2}$. Then the inequality follows from $\langle e + fb, e + fb\rangle \geq 0$. □

REMARK 6. Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then T^* is the unique operator such that

$$\langle Tx, y\rangle = \langle x, T^* y\rangle$$

for all $x, y \in H$. Such T^* alsways exists and this star operator turns $\mathcal{L}(H)$ into a iC^*-algebra.

DEFINITION 7. Let E_B and F_B be Hilbert B-modules. Let T be a map from E to F. Then $T^* : F \to E$ is called the adjoint of T if for all $e \in E, f \in F$

$$\langle Te, f\rangle = \langle e, T^* f\rangle.$$

If such T^* exists, we call T adjointable. The set of all such operator is denoted by $\mathcal{L}(E, F)$.

Exercise 8. Find an example such that a continuous linear map $T : E \to F$ is not adjointable.

PROPOSITION 9. *Let E, F be Hilbert B-modules, and let T be an adjointable map from E to F. Then*

1. *T^* is unique, and T^* is also adjointable and $(T^*)^* = T$,*
2. *T is linear, B-linear and continuous,*
3. $\|T\|^2 = \|T^*\|^2 = \|TT^*\| = \|T^*T\|$.

PROPOSITION 10. *Let E, F be Hilbert B-modules, then $\mathcal{L}(E) = \mathcal{L}(E, E)$ is a C^*-algebra and $\mathcal{L}(E, F)$ is a Banach space.*

DEFINITION 11. Let E, F be Hilbert B-modules. For all $e \in E, f \in F$, define

$$\theta_{f,e} : E \to F$$

by

$$\theta_{f,e}(e') = f\langle e, e'\rangle_E.$$

PROPOSITION 12. *In the above situation, we have*

1. *$\theta_{f,e} \in \mathcal{L}(E, F)$ and $\theta_{f,e}^* = \theta_{e,f}$,*
2. *for all $T \in \mathcal{L}(F)$ and $S \in \mathcal{L}(E)$, we have*

$$T \circ \theta_{f,e} = \theta_{Tf,e}, \quad \theta_{f,e} \circ S = \theta_{f,S^*e}.$$

DEFINITION 13. Define $\mathcal{K}(E, F) = \mathcal{K}_B(E, F)$ to be the closed linear span of $\{\theta_{f,e} | e \in E, f \in F\}$. Elements in $\mathcal{K}(E, F)$ is called compact operators.

PROPOSITION 14.

$$\mathcal{L}(F)\mathcal{K}(F,E) = \mathcal{K}(F,E);$$
$$\mathcal{K}(E,F)\mathcal{L}(F) = \mathcal{K}(E,F);$$
$$\mathcal{K}(E,F)^* = \mathcal{K}(F,E).$$

In particular, $\mathcal{K}(E) = \mathcal{K}(E,E)$ *is a closed,* $*$*-closed two-sided ideal of* $\mathcal{L}(E)$.

LEMMA 15. *Let* E, F *be Hilbert* B*-modules. Then*

$$\mathcal{K}(E,F) = \{T \in \mathcal{L}(E,F) | TT^* \in \mathcal{K}(F)\}.$$

Proof. "$\subset$" is obvious.

"$\supset$": Let $(U_\lambda)_\lambda$ be a bounded approximate unit for $\mathcal{K}(F)$. Then using $U_\lambda = U_\lambda^*$,

$$\|U_\lambda T - T\|^2 = \|U_\lambda TT^*U_\lambda - U_\lambda TT^* - TT^*U_\lambda + TT^*\|.$$

Since $TT^* \in \mathcal{K}(F)$ implies $U_\lambda T \to T \in \mathcal{L}(E,F)$ and $U_\lambda T \in \mathcal{K}(E,F)$, we have $T \in \mathcal{K}(E,F)$. □

Example 16.

(1) Let $B = \mathbb{C}$, and let H be a Hilbert space. Then $\mathcal{K}(H)$ is the usual algebra of compact operators,

(2) If B is arbitrary, and if you regard B as a Hilbert B-module, then $\mathcal{K}(B) = B$.

Proof. Define $\Phi : B \to \mathcal{L}(B)$ by $b(b') = bb'$ for all $b' \in B$. Then Φ is a $*$-homomorphism and $\Phi(b^*c) = \theta_{b,c}$ for all $b, c \in B$. So $\Phi(B \cdot B) \subset \mathcal{K}(B)$. But $B \cdot B = B$. □

(3) If $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$, then

$$\mathcal{K}(E,F) = \bigoplus_{i=1,2} \bigoplus_{j=1,2} \mathcal{K}(E_i, F_j),$$

and every $T \in \mathcal{K}(E,F)$ can be expressed as a matrix

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

(4) As a consquence of above, we have $\mathcal{K}(B^m, B^n) = M_{m\times n}(B)$.

DEFINITION 17. If B is a C^*-algebra, then we define

$$M(B) = \mathcal{L}(B).$$

$M(B)$ is called the multiplier algebra of B. For example $M(C_0(X)) = C_b(X)$ if X is a locally compact space.

PROPOSITION 18. *If* E *is a Hilbert* B*-module, then*

$$M(\mathcal{K}(E)) = \mathcal{L}(E).$$

Sketch of proof. If $T \in \mathcal{L}(E)$, then $S \to TS$ defines an element $T\cdot \in M(\mathcal{K}(E)) = \mathcal{L}(\mathcal{K}(E))$. This defines a $*$-homomorphism $\Psi : \mathcal{L}(E) \to M(\mathcal{K}(E))$.
For $T \in \ker(\Psi)$: Let $e \in E$.

$$0 = \langle \Psi(T)(\theta_{e,Te})(Te), \Psi(T)(\theta_{e,Te})(Te)\rangle = \langle (T\theta_{e,Te})(Te), (T\theta_{e,Te})(Te)\rangle = \langle Te, Te\rangle^3$$

So $Te = 0$ for all $e \in E$. Hence $T = 0$ and Ψ is injective.

If $m \in M(\mathcal{K}(E))$ and $e \in E$, we define

$$T(e) = \lim_{\epsilon \to 0} m(\theta_{e,e})(e)(\langle e, e\rangle + \epsilon)^{-1}.$$

Then this is a well-defined element of $\mathcal{L}(E)$ and $\Psi(T) = m$. So Ψ is surjective. □

DEFINITION 19. Let B, B' be C^*-algebras, and let $\psi : B \to B'$ be a $*$-homomorphism. Let E_B is a Hilbert B-module and $E'_{B'}$ is a Hilbert B'-module. A homomorphism with coefficient map ψ from E_B to $E'_{B'}$ is a map $\Phi : E_B \to E'_{B'}$ such that

(1) Φ is $\mathbb{C}$-linear,
(2) $\Phi(eb) = \Phi(e)\psi(b)$ for all $e \in E_B$ and $b \in B$,
(3) $\langle \Phi(e), \Phi(f)\rangle = \phi(\langle e, f\rangle) \in B'$ for all $e, f \in E_B$.

We denote such a map also by Φ_ψ by emphsizing ψ.

REMARK 20. From the definition, it follows that $\|\Phi(e)\| \leq \|e\|$ for all $e \in E_B$ and equality holds when ψ is injective.

REMARK 21. There is an obvious composition of homomorphisms with coefficient maps: for $\Phi_\psi : E_B \to E'_{B'}$ and $\Psi_\chi : E'_{B'} \to E''_{B''}$, we have a homomorphism

$$(\Psi \circ \Phi)_{\chi \circ \psi} : E_B \to E''_{B''}.$$

Also $(\mathrm{Id}_E)_{\mathrm{Id}_B} : E_B \to E_B$ is a homomorphism.

DEFINITION 22. Two Hilbert B-modules E_B and $E_{B'}$ are called isomorphic if there is a homomorphism $\Phi_{\mathrm{Id}_B} : E_B \to E'_B$ which is bijective. Then $\Phi^{-1}_{\mathrm{Id}_B} : E'_B \to E_B$. Write $E_B \cong E'_B$. Note that in this case, $\Phi_{\mathrm{Id}_B} \in \mathcal{L}(E_B, E'_B)$ and $\Phi^*_{\mathrm{Id}_B} = \Phi^{-1}_{\mathrm{Id}_B}$.

DEFINITION 23. A C^*-algebra B is called σ-unital if there exists a countable bounded approximate unit.

DEFINITION 24. A positive element $h \in B$ is called strictly positive if $\phi(h) > 0$ for all states ϕ of B.

LEMMA 25. *B is σ-unital if and only if B contains a strictly positive element.*

LEMMA 26. *A positive element $h \in B$ is strictly positive if and only if $\overline{hB} = B$.*

LEMMA 27. *Let E be a Hilbert B-module, and let $T \in \mathcal{L}(E)$ be positive. Then T is strictly positive if and only if $\overline{T(E)} = E$.*

DEFINITION 28. A Hilbert B-module E is called countably generated if there is a set $\{x_n : x_n \in E, \forall\, n \in \mathbb{N}\}$ such that the span of the set $\{x_n b : x_n \in E\ b \in B, \forall\, n \in \mathbb{N}\}$ is dense in E.

We will show that E is countably generated if and only if $\mathcal{K}(E)$ is σ-unital. This is a consquence of the following important theorem.

THEOREM 29 (Kasparov's Stabilization Theorem)**.** *If E is a countably generated Hilbert B-module, then*

$$E \oplus \mathbb{H}_B \cong \mathbb{H}_B.$$

Proof. Without loss of generality, we assume that B is unital. We want to define a unitary $V : \mathbb{H}_B \to E \oplus \mathbb{H}_B$.
Instead of defining V directly, we define $T \in \mathcal{L}(\mathbb{H}_B, E \oplus \mathbb{H}_B)$ such that T and $|T| = (T^*T)^{\frac{1}{2}}$ have dense range. Then the isometry V defined by $V(|T|(x)) = T(x)$

can be extended to an isometry from $\mathbb{H}_B$ to $E \oplus \mathbb{H}_B$ with Range$(V) \supset$ Range(T) (which is dense, so V is a unitary).

Let ξ_n be the n-th standard basis vector in $\mathbb{H}_B$, and let (η_n) be a generating sequence of E such that for all $n \in \mathbb{N}$, $\{l \in \mathbb{N} | \eta_n = \eta_l\}$ is an infinite set. WLOG, we assume that $\|\eta_n\| \leq 1$ for all $n \in \mathbb{N}$. Define

$$T = \sum_k 2^{-k} \theta_{(\eta_k, 2^{-k}\xi_k), \xi_k}.$$

(1) T has a dense range: Let $k \in \mathbb{N}$. Then for any $l \in \mathbb{N}$ with $\eta_k = \eta_l$, we have that $T(\xi_l) = 2^{-l}(\eta_k, 2^{-l}\xi_l)$, so
$$T(2^l \xi_l) = (\eta_k, 2^{-l}) \to (\eta_k, 0)$$
as $l \to \infty$. Hence $(\eta_k, 0) \in \overline{T(\mathbb{H}_B)}$, and also $2^l((\eta_k, 2^{-l}\xi_l) - (\eta_k, 0)) = (0, \xi_l) \in \overline{T(\mathbb{H}_B)}$;

(2) T^*T has dense range:
$$\begin{aligned} T^*T = \sum_{k,l} &= 2^{-k-l} \theta_{\xi_k(\langle \eta_k, \eta_l \rangle + \langle 2^{-k}\xi_k, 2^{-l}\xi_l \rangle), \xi_l} \\ &= \sum_k 4^{-2k} \theta_{\xi_k, \xi_k} + \left(\sum_k 2^{-k} \theta_{(\eta_k, 0), \xi_k} \right)^* \left(\sum_k 2^{-k} \theta_{(\eta_k, 0), \xi_k} \right) \\ &\geq \sum_k 4^{-2k} \theta_{\xi_k, \xi_k} (\overset{def}{=} S). \end{aligned}$$
S is positive and has dense range, so it is strictly positive in $\mathcal{K}(\mathbb{H}_B)$. Hence T^*T is stricly positive in $\mathcal{K}(H)$ and has dense range;

(3) $|T|$ has dense range because Range$(|T|) \supset$ Range(T^*T).

□

COROLLARY 30. *E_B is countably generated if and only if $\mathcal{K}(E)$ is σ-unital.*

Proof.

(1) If B is unital and $E = \mathbb{H}_B$. Let ξ_i be the standard i-th basis vector in $\mathbb{H}_B$. Then
$$h = \sum_i 2^{-i} \theta_{\xi_i, \xi_i}$$
is strictly positive in $\mathcal{K}(E)$ since it has dense range;

(2) If B is unital and $E = P\mathbb{H}_B$ for some $P \in \mathcal{L}(\mathbb{H}_B)$ with $P^* = P = P^2$. (This is almost generic my the above theorem.) Then
$$PhP = \sum_i 2^{-i} \theta_{P\xi_i, P\xi_i}$$
is strictly positive in $\mathcal{K}(E)$;

(3) B is countable generated if and only if B^+ is countably generated. So $\mathcal{K}_{B^+}(E)$ is σ-unital if and only if $\mathcal{K}_B(E)$ is σ-unital since $\mathcal{K}_{B^+}(E) = \mathcal{K}_B(E)$.

□

DEFINITION 31. Let B, C be C^*-algebras, and let E_B and F_C be Hilbert B, C-modules respectively and let $\phi : B \to \mathcal{L}(F_C)$ be a $*$-homomorphism. On $E \otimes_{alg} F \times E \otimes_{alg} F$, define

$$\langle e \otimes f, e' \otimes f' \rangle = \langle f, \phi(\langle e, e' \rangle) f' \rangle \in C.$$

This defines a C-valued bilinear map. Define $N = \{t \in E \otimes_{alg} F | \langle t, t \rangle = 0\}$. Then $\langle \cdot, \cdot \rangle$ defines an inner product on $E \otimes_{alg} F/N$ which turns it to be a pre-Hilbert C-module.
The completion is called the inner tensor product of E and F and is denoted by $E \otimes_B F$ or $E \otimes_\phi F$.

LEMMA 32. *Let E_{1B}, E_{2B} and F_C be Hilbert B, C module respectively, and let $\phi : B \to \mathcal{L}(F)$ be a $*$-homomorphism. Let $T \in \mathcal{L}(E_1, E_2)$. Then $e_1 \otimes f \to T(e_1) \otimes f$ defines a map $T \otimes 1 \in \mathcal{L}(E_1 \otimes_B F, E_2 \otimes_B F)$ such that $(T \otimes 1)^* = T^* \otimes 1$ and $\|T \otimes 1\| \leq \|T\|$. If $\phi(B) \subset \mathcal{K}(F)$, then $T \in \mathcal{K}(E_1, E_2)$ implies $T \otimes 1 \in \mathcal{K}(E_1 \otimes F, E_2 \otimes F)$.*

Proof. We only prove the last assertion here. The map $T \to T \otimes 1$ is linear and contractive from $\mathcal{L}(E_1, E_2)$ to $\mathcal{L}(E_1 \otimes F, E_2 \otimes F)$. So it suffices to consider T of the form θ_{e_2, e_1} with $e_1 \in E_1$ and $e_2 \in E_2$. Because $E_2 = E_2 \cdot B$, it suffices to consider $\theta_{e_2 b, e_1}$ with $b \in B$. Now for all $e_1' \otimes f \in E_1 \otimes F$,

$$\begin{aligned}(\theta_{e_2 b, e_1} \otimes 1)(e_1' \otimes f) &= \theta_{e_2 b, e_1}(e_1') \otimes f \\ &= e_2 b(e_1, e_1') \otimes f \\ &= e_2 \otimes \phi(b)\phi(\langle e_1, e_1' \rangle) f \\ &= (M_{e_2} \circ \phi(b) \circ N_{e_1})(e_1' \otimes f),\end{aligned}$$

where $M_{e_2} : F \to E_2 \otimes_B F$ by $f' \to e_2 \otimes f'$ and $N_{e_1} : E_1 \otimes_B F \to F$ by $e_1' \otimes f' \to \phi(\langle e_1, e_1' \rangle) f'$. Because $M_{e_2} \in \mathcal{L}(F, E_2 \otimes_B F)$, $N_{e_1} \in \mathcal{L}(E_1 \otimes_B F, F)$ and $\phi(b) \in \mathcal{K}(F)$, we have $\theta_{e_2 b, e_1} \otimes 1 \in \mathcal{K}(E_1 \otimes F, E_2 \otimes F)$. ☐

LEMMA 33. *Let B and C be C^*-algebras, and let $\phi : B \to C$ be a $*$-homomorphism. Define $\tilde{\phi} : B \to \mathcal{L}(C) = M(C)$ by $b \to (c \to \phi(b)c)$. Then $\tilde{\phi}(B) \subset \mathcal{K}(C)$.*

DEFINITION 34. Let E_B be a Hilbert B-module, and let $\phi : B \to C$ be a $*$-homomorphism. Define the push-forward $\phi_*(E)$ as $E \otimes_B C = E \otimes_\phi C$.

LEMMA 35.

(1) $(\mathrm{id}_B)_*(E) = E \otimes_B B \cong E$ *canonically;*
(2) $\psi_*(\phi_*(E)) \cong (\psi \circ \phi)_*(E)$ *naturally, where $\psi : C \to D$ is a $*$-homomorphism.*

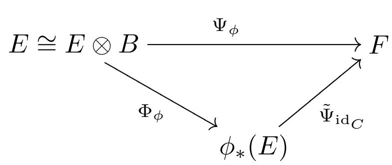
LEMMA 36. *$T \in \mathcal{K}(E_1, E_2)$ implies $\phi_*(T) \in \mathcal{K}(\phi_*(E_1), \phi_*(E_2))$. Moreover,*

$$\phi_*(\theta_{e_2 b_2, e_1 b_1}) = \theta_{e_2 \otimes \phi(b_2), e_1 \otimes \phi(b_1)}$$

for all $b_1, b_2 \in B$, $e_1 \in E_1$ and $e_2 \in E_2$.

REMARK 37.

(1) The push-forward has the following universal property. If $\phi : B \to C$ and if E_B is a Hilbert B-module, then there is a natural homomorphism $\Phi_\phi : E_B \cong E_B \otimes B \to E \otimes_B C = \phi_*(E)$ defined by $\Phi(e \otimes b) = e \otimes \phi(b)$. If $\Psi_\phi : E_B \to F_C$ is any homomorphism with coefficient map ϕ, there is a unique homomorphism $\Phi_{\mathrm{id}_C} : \phi_*(E)_C \to F_C$ defined by $\tilde{\Psi}(e \otimes c) = \Psi(e)c$ such that the following diagram commutes



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(2) You can show that $\mathcal{K}(\cdot)$ is a functor. If $\Phi_\phi : E_B \to F_C$ is a homomorphism with coefficient map ϕ, then there is a unique $*$-homomorphism $\Theta : \mathcal{K}(E) \to \mathcal{K}(F)$ such that $\Theta(\theta_{e,e'}) = \theta_{\phi(e),\phi(e')} \in \mathcal{K}(F)$ for all $e, e' \in E$.

DEFINITION 38. Let B, B' be C^*-algebras, and let $E_B, E'_{B'}$ be Hilberts B, B' modules respectively. Then define a bilinear map

$$\langle \cdot, \cdot \rangle : E \otimes_{alg} E' \times E \otimes_{alg} E' \to B \otimes B'$$

by

$$\langle e_1 \otimes e'_1, e_2 \otimes e'_2 \rangle = \langle e_1, e_2 \rangle \otimes \langle e_2, e'_2 \rangle.$$

This defines an inner product on $E \otimes_{\mathbb{C}} E'$. Its completion, denoted by $E \otimes E'$, is a Hilbert $B \otimes B'$-module, called the external tensor product of E and E'.

DEFINITION 39. A graded C^*-algebra is a C^*-algebra B equipped with an order two $*$-homomorphism β_B, called the grading automorphism of B, i.e. $\beta_B^2 = \beta_B$. A $*$-homomorphism ϕ from a graded algebra (B, β_B) to a graded algebra (C, β_C) is graded if $\beta_C \circ \phi = \phi \circ \beta_B$.
If (B, β_B) is graded, then $B = B_0 \oplus B_1$ with $B_0 = \{b \in B | \beta_B(b) = b\}$ and $B_1 = \{b \in B | \beta_B(b) = -b\}$. The element $b \in B_0$ is called even with $\deg(b) = 0$ and the element $b \in B_1$ is called odd with $\deg(b) = 1$. An element of $B_0 \cup B_1$ is called homogeneous.

REMARK 40. Note we have

$$B_0 \cdot B_1 \subset B_1 \quad B_1 \cdot B_0 \subset B_1$$
$$B_0 \cdot B_0 \subset B_0 \quad B_1 \cdot B_1 \subset B_0.$$

Moreover, $\phi : B \to C$ is graded if and only if $\phi(B_i) \subset C_i$ for $i = 0, 1$.

DEFINITION 41 (Definition and lemma)**.** If B is graded, then the graded commutator of B is defined on homogeneous elements a, b, c by

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba.$$

It satisfies the following properties.

(1) $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$;
(2) $[a, bc] = [a, b]c + (-1)^{\deg(a)\deg(b)} b[a, c]$;
(3) $(-1)^{\deg(a)\deg(c)}[[a, b], c] + (-1)^{\deg(a)\deg(b)}[[b, c], a] + (-1)^{\deg(b)\deg(c)}[[c, a], b] = 0$.

DEFINITION 42. Let A and B be graded C^*-algebras. Define their graded tensor product as follows. On $A \otimes_{alg} B$, define

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\deg(a_1)\deg(b_1)}(a_1 a_2 \hat{\otimes} b_1 b_2)$$

and

$$(a_1 \hat{\otimes} b_1)^* = (-1)^{\deg(a_1)\deg(b_1)}(a_1^* \hat{\otimes} b_1^*)$$

for all homogeneous element $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Define a grading automorphism by $\beta_{A \hat{\otimes} B} = \beta_A \otimes \beta_B$.
Just as in the ungraded case, there are several feasible norms on $A \otimes_{alg} B$ and among them there is a maximal one. Completed for this norm the algebra $A \otimes_{alg} B$ becomes the maximal graded tensor product $A \hat{\otimes}_{max} B$. There is also a spacial graded tensor product $A \hat{\otimes} B$. In general these completions can be different from there ungraded counterparts, but in the cases we are interested in, they agree. Hence we will not make a fuss about these norms.

PROPOSITION 43. *The spatial graded tensor product $A\hat{\otimes}B$ is associative ($A\hat{\otimes}(B\hat{\otimes}C) = (A\hat{\otimes}B)\hat{\otimes}C$) and commutative ($A\hat{\otimes}B \cong B\hat{\otimes}A$ via $a\hat{\otimes}b \to (-1)^{\deg(a)\deg(b)}b\hat{\otimes}a$).*

Example 44.

(1) If A is an ungraded C^*-algebra, then id_A is a grading automorphism on A which we call the trivial grading. With this grading, A is called trivially graded;

(2) If A is a C^*-algebra and $u \in M(A)$ satisfies $u = u^* = u^{-1}$, then one can define a grading on A by $a \to uau$. Such a grading is called an inner grading. We will see later that inner gradings are the less interesting gradings.

(3) On $\mathbb{C}_{(1)} = \mathbb{C} \oplus \mathbb{C}$, define the following grading automorphism:
$$(a, b) \to (b, a).$$
Then $(\mathbb{C}_{(1)})_0 = \{(a, a)|a \in \mathbb{C}\}$ and $(\mathbb{C}_{(1)})_1 = \{(a, -a)|a \in \mathbb{C}\}$. This grading is called the standard odd grading;

(4) More generally, define the odd grading also on $A_{(1)} = A \oplus A$ for any C^*-algebra A. Note that $A_{(1)} \cong A\hat{\otimes}\mathbb{C}_{(1)}$;

(5) Alternatively, define $\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C}$ as follows.
The multiplication is given by
$$(1, 0)(1, 0) = (0, 1)(0, 1) = (1, 0);$$
$$(1, 0)(0, 1) = (0, 1)(1, 0) = (0, 1).$$
The involution is given by $(a, b)^* = (\bar{a}, \bar{b})$.
The norm is given by $\|(a, b)\| = \max\{|a + b|, |a - b|\}$.
The grading is given by $(a, b) \to (a, -b)$.
Then $\mathbb{C}_1$ is a graded C^*-algebra.
Also $\mathbb{C}_1 \cong \mathbb{C}_{(1)}$ as a graded C^*-algebra. Let $\mathbb{C}_1$ act on $\mathbb{C} \oplus \mathbb{C}$ by
$$(a, b) \to \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$
This is a faithful representation.

DEFINITION 45. Let $n \in \mathbb{N}$. Let $\mathbb{C}_n$ be the universal unital $\mathbb{C}$-algebra defined in the following way, called the n-th complex Clifford algebra:

(1) there is an $\mathbb{R}$-linear map $i : \mathbb{R}^n \to \mathbb{C}_n$ such that
$$i(v) \cdot i(v) = \langle v, v\rangle \cdot 1_{\mathbb{C}_n} \in \mathbb{C}_n$$
for all $v \in \mathbb{R}^n$;

(2) if $\phi : \mathbb{R}^n \to A$ is any $\mathbb{R}$-linear map from $\mathbb{R}^n$ to a unital $\mathbb{C}$-algebra satisfying the above condition, then there is a unique unital $\mathbb{C}$-linear homomorphism $\hat{\phi} : \mathbb{C}_n \to A$ such that $\phi = \hat{\phi} \circ i$.

Consider the complexified exterior algebra $\Lambda^*_{\mathbb{C}}\mathbb{R}^n$. It has a canonical Hilbert space structure. Let $\mathbb{C}_n$ act on $\Lambda^*_{\mathbb{C}}\mathbb{R}^n$ as follows: if $v \in \mathbb{R}^n$ then define $\mu(v) = ext(v) + ext(v)^* \in \mathcal{L}(\Lambda^*_{\mathbb{C}}\mathbb{R}^n)$. From the universal property of the Clifford algebra we obtain a homomorphism from $\mathbb{C}_n$ to $\mathcal{L}(\Lambda^*_{\mathbb{C}}\mathbb{R}^n)$.
On $\mathbb{C}_n$ we have an involution induced by the map
$$(v_1 \cdot v_2 \cdots v_k)^* = v_k \cdot v_{k-1} \cdots v_1$$
for all $v_1, \cdots, v_k \in \mathbb{R}^n$. With this involution, $\mathbb{C}_n$ is a $*$-algebra and $\mu : \mathbb{C}_n \to \mathcal{L}(\Lambda^*_{\mathbb{C}}\mathbb{R}^n)$ a $*$-homomorphism. It defines a C^*-algebra structure on $\mathbb{C}_n$.

Example 46.

(1) $\mathbb{C}_1$ is the two-dimensional algebra defined above;
(2) $\mathbb{C}_2$ is the four-dimensional algebra with the basis $1, e_1, e_2, e_1e_2$ such that $e_1^2 = e_2^2 = 1$ and $e_1e_2 = -e_2e_1$.

DEFINITION 47. The unitary map $v \to -v$ in $\mathbb{R}^n$ lifts to an isomorphism $\beta_n : \mathbb{C}_n \to \mathbb{C}_n$ such that $(\beta_n)^2 = 1$. It is a grading on $\mathbb{C}_n$.

Exercise 48. Show that $\mathbb{C}_2$ is isomorphic to $\mathbb{M}_{2\times 2}(\mathbb{C})$ with the inner grading given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

PROPOSITION 49. *We have $\mathbb{C}_{m+n} \cong \mathbb{C}_m \hat{\otimes} \mathbb{C}_n$ for all $m, n \in \mathbb{N}$.*

Proof. Define $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$. Let $i_m : V \to \mathbb{C}_m$, $i_n : W \to \mathbb{C}_n$ and $i_{m+n} : V \oplus W \to \mathbb{C}_{m+n}$ be the canonical injections. Let $\pi_V : V \oplus W \to V$ and $\pi_W : V \oplus W \to W$ be the canonical projections. Then

$$i = (i_V \hat{\otimes} 1) \circ \pi_V \oplus (1 \hat{\otimes} i_W) \circ \pi_W : V \oplus W \to \mathbb{C}_m \hat{\otimes} \mathbb{C}_n$$

satisfies $i(x)i(x) = \langle x, x\rangle 1_{\mathbb{C}_m \hat{\otimes} \mathbb{C}_n}$, so there is a unital $\mathbb{C}$-linear homomorphism $\hat{i} : \mathbb{C}_{m+n} \to \mathbb{C}_m \hat{\otimes} \mathbb{C}_n$ such that $i = \hat{i} \circ i_{m+n}$. Similarly, one can construct homomorphisms $\mathbb{C}_m \to \mathbb{C}_{m+n}$ and $\mathbb{C}_n \to \mathbb{C}_{m+n}$ which gradedly commute, so there is a homomophism $\mathbb{C}_m \hat{\otimes} \mathbb{C}_n \to \mathbb{C}_{m+n}$. It is an inverse of $\hat{i}$. □

PROPOSITION 50. *If $n \in \mathbb{N}$ is even, then $\mathbb{C}_n \cong \mathbb{M}_{2^m \times 2^m}(\mathbb{C})$ with an inner grading. If $n = 2m + 1$ is odd, then $\mathbb{C}_n \cong \mathbb{M}_{2^m \times 2^m}(\mathbb{C}) \oplus \mathbb{M}_{2^m \times 2^m}(\mathbb{C})$ with standard odd grading.*

DEFINITION 51. Let (B, β_B) be a graded C^*-algebra and E_B be a Hilbert B-module. A grading automorphism $\sigma_E : E \to E$ is a homomorphism with coefficient map β_B such that $\sigma_E^2 = \mathrm{id}_E$, i.e.

$$\langle \sigma_E(e), \sigma_E(f)\rangle = \beta_E(\langle e, f\rangle)$$

and $\sigma_E(eb) = \sigma_E(e)\beta_B(b)$ for all $e, f \in E$ and $b \in B$.

REMARK 52. With $E_0 = \{e \in E | \sigma_E(e) = e\}$ and $E_1 = \{e \in E | \sigma_E(e) = -e\}$, we have

$$\langle E_i, E_j\rangle \subset B_{i+j}$$

and

$$E_i B_j \subset E_{i+j}.$$

If B is trivially graded, then it still makes sense to consider graded Hilbert B-modules; they are just orthogonal direct sums of two Hilbert B-modules.

DEFINITION 53 (Definition and Lemma)**.** If E and F are graded Hilbert modules over the graded C^*-algebra B, then define

$$\sigma_{\mathcal{L}(E,F)}(T) = \sigma_F \circ T \circ \sigma_E$$

for all $T \in \mathcal{L}(E, F)$.
This map satisfies:

(1) $\sigma^2_{\mathcal{L}(E,F)}(T) = T$ for all $T \in \mathcal{L}(E, F)$;
(2) $\sigma_{\mathcal{L}(F,E)}(T^*) = [\sigma_{\mathcal{L}(E,F)}(T)]^*$ for all $T \in \mathcal{L}(E, F)$;
(3) $\sigma_{\mathcal{L}(E,G)}(T \circ S) = \sigma_{\mathcal{L}(F,G)}(T) \circ \sigma_{\mathcal{L}(E,F)}(S)$ for all $T \in \mathcal{L}(F, G)$ and $S \in \mathcal{L}(E, F)$ where G_B is another Hilbert B-module;

(4) $\sigma_{\mathcal{L}(E,F)}(\mathcal{K}(E,F)) \subset \mathcal{K}(E,F)$ with $\sigma_{\mathcal{L}(E,F)}(\theta_{f,e}) = \theta_{\sigma_F(f),\sigma_E(e)}$ for all $e \in E$ and $f \in F$.

COROLLARY 54. *If E is a graded Hilbert B-module, then $\mathcal{L}(E)$ and $\mathcal{K}(E)$ are graded C^*-algebras.*

DEFINITION 55. The elements of $\mathcal{L}(E,F)_0$ are called even, written $\mathcal{L}(E,F)^{even}$, the elements of $\mathcal{L}(E,F)_1$ are called off, written $\mathcal{L}(E,F)^{odd}$.

REMARK 56. An even element of $\mathcal{L}(E,F)$ maps E_0 to F_0 and E_1 to F_1, and an odd element maps E_0 to F_1 and E_1 to F_0.

REMARK 57. The following concepts and results can easily be adapted from the trivially graded case to the general graded case.

(1) graded homomorphism with graded coefficient maps;
(2) Kasparov stabilization theorem: $\mathbb{H}_B$ has to be replaced by $\hat{\mathbb{H}_B} = \mathbb{H}_B \oplus \mathbb{H}_B$ with grading $S = (\beta_B, \beta_B, \cdots)$ on the first summand and $-S$ on the second summand;
(3) the interior tensor product of graded Hilbert modules;
(4) the exterior tensor product of graded Hilbert modules. The inner product is defined by
$$\langle e_1 \hat{\otimes} f_1, e_2 \hat{\otimes} f_2 \rangle = (-1)^{\deg(f_1)(\deg(e_1)+\deg(e_2))} \langle e_1, e_2 \rangle \hat{\otimes} \langle f_1, f_2 \rangle.$$
(5) the push-forward along graded $*$-homomorphisms.

2. The definition of KK-theory

All C^*-algebras $A, B, C, \cdots$ in this section will be σ-unital. Let A, B be graded C^*-algebras.

DEFINITION 58. A Kasparov A-B-module or a Kasparov A-B-cycle is a triple $\mathcal{E} = (E, \phi, T)$ where E is a countably generated graded Hilbert B-module, $\phi : A \to \mathcal{L}(E)$ is a graded $*$-homomorphism and $T \in \mathcal{L}(E)$ is an odd operator such that

(1) $\forall\, a \in A : \ [\phi(a), T] \in \mathcal{K}(E)$;
(2) $\forall\, a \in A : \ \phi(a)(T^2 - \mathrm{id}_E) \in \mathcal{K}(E)$;
(3) $\forall\, a \in A : \ \phi(a)(T - T^*) \in \mathcal{K}(E)$.

Note that the commutator in 1) is graded. The class of all Kasparov A-B-modules will be denoted by $\mathbb{E}(A,B)$. Sometimes we denote elements of $\mathbb{E}(A,B)$ also as pairs (E,T) without making reference to the action ϕ.

REMARK 59. We are not going to discuss many examples at this point. They will occur later in the talks dedicated to applications of KK-theory.

DEFINITION 60 (Definition and Lemma)**.**

(1) If $\mathcal{E}_1 = (E_1, \phi_1, T_1)$ and $\mathcal{E}_2 = (E_2, \phi_2, T_2)$ are elements of $\mathbb{E}(A,B)$, then $\mathcal{E}_1 \oplus \mathcal{E}_2 := (E_1 \oplus E_2, \phi_1 \oplus \phi_2, T_1 \oplus T_2 \in \mathbb{E}(A,B)$;
(2) If C is another graded C^*-algebra and $\psi : B \to C$ is an even $*$-homomorphism and $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A,B)$ then
$$\psi_*(\mathcal{E}) := (\psi_*(E), \phi \hat{\otimes} 1, \psi_*(T) = T \hat{\otimes} 1) \in \mathbb{E}(A,C).$$
(3) If C is another graded C^*-algebra, $\varphi : A \to B$ is an even $*$-homomorphism and $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(B,C)$, then
$$\phi^*(\mathcal{E}) := (E, \phi \circ \varphi, T) \in \mathbb{E}(A,C);$$

(4) If $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$ then
$$-\mathcal{E} := (-E, \phi_-, -T) \in \mathbb{E}(A, B),$$
where $-E$ is the same Hilbert B-module as E but with the grading $\sigma_{-E} := -\sigma_E$, and $\phi_- := \phi \circ \beta_A$ where β_A is the grading on A.

Proof. We only show parts of (2). Let $a \in A$. Then
$$\begin{aligned}(\phi\hat{\otimes}1)(a)((T\hat{\otimes}1)^2 - \mathrm{id}_{E\otimes_\psi C} &= (\phi(a)\hat{\otimes}\,\mathrm{id}_C)(T^2\hat{\otimes}\,\mathrm{id}_C - \mathrm{id}_E\hat{\otimes}\,\mathrm{id}_C)\\ &= (\phi(a)(T^2 - \mathrm{id}_E)) \otimes \mathrm{id}_C\\ &= \psi_*(\phi(a)(T^2 - \mathrm{id}_E)) \in \mathcal{K}(\psi_*(E)).\end{aligned}$$
Here we use that $\phi(a)(T^2 - \mathrm{id}_E) \in \mathcal{K}(E)$. The other conditions follow similarly. $\square$

DEFINITION 61. Let $\varphi : A \to A'$ and $\psi : B \to B'$ be $*$-homomorphisms and let $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$ and $\mathcal{E}' \in \mathbb{E}(A', B')$. A homomorphism from $\mathcal{E}$ to $\mathcal{E}'$ with coefficient maps φ and ψ is a homomorphism Φ_ψ from E_B to E'_B such that

(1) $\forall\ a \in A \forall\ e \in E, \Phi(\phi(a)e) = \phi'(\varphi(a))\Phi(e)$ i.e. Φ has coefficient map φ on the left;
(2) $\Phi \circ T = T' \circ \Phi$;

The most important case is the case that Φ is bijective and $\varphi = \mathrm{id}_A, \psi = \mathrm{id}_B$. Then $\mathcal{E}$ and $\mathcal{E}'$ are called isomorphic.

LEMMA 62. *We have up to isomorphism (for all $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathbb{E}(A, B)$):*

(1) $(\mathcal{E}_1 \oplus \mathcal{E}_2) \oplus \mathcal{E}_3 \cong \mathcal{E}_1 \oplus (\mathcal{E}_2 \oplus \mathcal{E}_3)$;
(2) $\mathcal{E}_1 \oplus \mathcal{E}_2 \cong \mathcal{E}_2 \oplus \mathcal{E}_1$;
(3) $\mathcal{E} \oplus (0, 0, 0) \cong \mathcal{E}$;
(4) *If $\psi : B \to C$ and $\psi' : C \to C'$ then*
$$\psi'_*(\psi_*(\mathcal{E})) \cong (\psi' \circ \psi)_*(\mathcal{E});$$
(5) $(\mathrm{id}_B)_*(\mathcal{E}) \cong \mathcal{E}$;
(6) *If $\phi : A' \to A$ and $\phi' : A'' \to A$ then*
$$\phi'^*(\phi^*(\mathcal{E})) = (\phi \circ \phi')^*(\mathcal{E}),\ \mathrm{id}_A^*(\mathcal{E}) = \mathcal{E};$$
(7) $\psi_*(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \psi_*(\mathcal{E}_1) \oplus \psi_*(\mathcal{E}_2),\ \psi_*(-\mathcal{E}) = -\psi_*(\mathcal{E})$;
(8) $\phi^*(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \phi^*(\mathcal{E}_1) \oplus \phi^*(\mathcal{E}_2),\ \phi^*(-\mathcal{E}) = -\phi^*(\mathcal{E})$;
(9) $\phi^*(\psi_*(\mathcal{E})) = \psi_*(\phi^*(\mathcal{E}))$.

DEFINITION 63. Let C be a graded C^*-algebra and $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$. We now give the definition of a cycle $\tau_C(\mathcal{E}) = \mathcal{E}\hat{\otimes}\,\mathrm{id}_C \in \mathbb{E}(A\hat{\otimes}C, B\hat{\otimes}C)$: the module is $E_B\hat{\otimes}C_C$, the action of $A\hat{\otimes}C$ is $\phi\hat{\otimes}\,\mathrm{id}_C$ and the operator is $T\hat{\otimes}\,\mathrm{id}_C$.

Example 64. If $C = \mathcal{C}([0, 1]) = \{f : [0, 1] \to \mathbb{C},\ f \text{ continuous}\}$, then $A\hat{\otimes}C \cong A[0, 1] = \{f : [0, 1] \to A,\ f \text{ continuous}\}$ and $B\hat{\otimes}C \cong B[0, 1]$. Similarly $E_B\hat{\otimes}C_C \cong E[0, 1]$ if $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$. Now $\tau_{C[0,1]}(\mathcal{E}) \cong (E[0, 1], \phi[0, 1], T[0, 1]) \in \mathbb{E}(A[0, 1], B[0, 1])$ under this identifications.

DEFINITION 65 (Notions of homotopy)**.** Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be in $\mathbb{E}(A, B)$:

(1) An operator homotopy from $\mathcal{E}_0$ to $\mathcal{E}_1$ is a norm-continuous path $(T_t)_{t\in[0,1]}$ in $\mathcal{L}(E)$ for some graded Hilbert B-module E equipped with a graded left action $\phi : A \to \mathcal{L}(E)$ such that
   (a) $\forall\ t \in [0, 1] :\ (E, \phi, T_t) \in \mathbb{E}(A, B)$;

(b) $\mathcal{E}_0 \cong (E, \phi, T_0),\ \mathcal{E}_1 \cong (E, \phi, T_1)$.

(2) A homotopy from $\mathcal{E}_0$ to $\mathcal{E}_1$ is an element $\mathcal{E} \in \mathbb{E}(A, B[0,1])$ such that $ev_{0,*}^B(\mathcal{E}) \cong \mathcal{E}_0$ and $ev_{1,*}^B(\mathcal{E}) \cong \mathcal{E}_1$, where $ev_t^B : B[0,1] \to B, \beta \to \beta(t)$ for all $t \in [0,1]$. We write $\mathcal{E}_0 \sim \mathcal{E}_1$ if such that a homotopy exists.

LEMMA 66. *Homotopy is an equivalence relation on* $\mathbb{E}(A, B)$.

Proof.

(1) Reflexivity: let $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$. Then $i_A^*(\tau_{C[0,1]}(\mathcal{E})) \cong (E[0,1], \phi[0,1] \circ i_A, T[0,1])$ is a homotopy from $\mathcal{E}$ to $\mathcal{E}$, where $i_A : A \to A[0,1]$ is the inclusion as constant functions.

(2) Symmetry: let $\mathcal{E} \in \mathbb{E}(A, B[0,1])$ and $\psi : B[0,1] \to B[0,1]$, $\beta \to (t \to \beta(1-t))$. Then $ev_{t,*}^B(\psi_*(\mathcal{E})) = (ev_t^B \circ \psi)_(\mathcal{E}) = (ev_{1-t,*}^B(\mathcal{E})$, where $ev_t^B \circ \psi = ev_{1-t}^B$.

(3) Transitivity: this is a non-trivial exercise.

□

DEFINITION 67. Define $KK(A, B) := \mathbb{E}(A, B)/\sim$. If $\mathcal{E} \in \mathbb{E}(A, B)$ then we denote the corresponding element of $KK(A, B)$ by $[\mathcal{E}]$.

LEMMA 68. $KK(A, B)$ *is an abelian group when equipped with the well-defined operation*

$$[\mathcal{E}_1] \oplus [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2].$$

In particular, $KK(A, B)$ *is a set. We have*

$$[\mathcal{E}] \oplus [-\mathcal{E}] = [0, 0, 0],$$

where $[0, 0, 0]$ *is the zero element of* $KK(A, B)$.

Before we come to the proof of this important lemma, we define:

DEFINITION 69. The class $\mathbb{D}(A, B) \subset \mathbb{E}(A, B)$ of degenerate Kasparov $A-B$-modules is the class of all elements (E, ϕ, T) such that $[\phi(a), T], \phi(a)(T^2 - 1), \phi(a)(T - T^*) = 0$ for all $a \in A$.

LEMMA 70. *If* $\mathcal{E} = (E, \phi, T) \in \mathbb{D}(A, B)$*, then* $\mathcal{E} \sim 0$.

Proof. We construct a homotopy using a mapping cylinder, in this case for the rather trivial homomorphism $0 \xrightarrow{\sigma} E$. Consider the following diagram

$$\begin{array}{ccc} Z & \longrightarrow & E[0,1]_{B[0,1]} \\ \downarrow & & \downarrow{\scriptstyle ev_0^E} \\ 0_B & \xrightarrow{\ \sigma\ } & E_B \end{array}$$

The pull-back Z in this diagram can be identitfied with the Hilbert $B[0,1]$-module $E(0,1] = \{\epsilon : [0,1] \to E,\ \epsilon \text{ continuous and } \epsilon(0) = 0\}$. On $E(0,1]$ define an A-action by $(a \cdot \epsilon)(t) = a(\epsilon(t))$ for all $a \in A$, $\epsilon \in \mathbb{E}(0,1]$ and $t \in [0,1]$. Define $\tilde{T} \in \mathcal{L}(E(0,1])$, $\epsilon \to T \circ \epsilon$. Then $\tilde{\mathcal{E}} = (E(0,1], \tilde{T}) \in \mathbb{E}(A, B[0,1])$ and $ev_{0,*}^B(\tilde{\mathcal{E}}) \cong 0$ and $ev_{1,*}^B(\tilde{\mathcal{E}}) \cong \mathcal{E}$. □

Proof of the important lemma. It is obvious that $KK(A, B)$ is a set because the class of isomorphism classes of countable generated Kasparov $A-B$-modules is small. Moreover, the direct sum is well-defined and $[0]$ is the zero element. The addition is commutative. What is left to show is that $\mathcal{E} \oplus -\mathcal{E} \sim 0$ for $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$. Define $G_t \in \mathcal{L}(E \oplus -E)$ to be the element given by the matrix:

$$G_t = \begin{pmatrix} \cos t \cdot T & \sin t \, \mathrm{id}_E \\ \sin t \, \mathrm{id}_E & -\cos t T \end{pmatrix}.$$

Then $G_0 = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} = (T \oplus (-T))$, so $(E \oplus -E, \phi \oplus \phi_-, G_0) = (E \oplus -E, \phi \oplus \phi_-, T \oplus -T)$. Also $G_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $(E \oplus -E, \phi \oplus \phi_-, G_1) \in \mathbb{D}(A, B)$. That G_t is odd and $(E \oplus -E, \phi \oplus \phi_-, G_t) \in \mathbb{E}(A, B)$ for all $t \in \mathbb{R}$ can be checked by direct calculations. □

LEMMA 71. *$KK(A, B)$ is a bifunctor from the category of graded (σ-unital) C^*-algebras and graded $*$-homomorphism to the category of abelian groups.*

Proof. Let $\psi : B \to C$ be a graded $*$-homomorphism. Then $\mathcal{E} \to \psi_*(\mathcal{E})$ lifts to a map $\psi_* : KK(A, B) \to KK(A, C)$. Here using the diagram

$$\begin{array}{ccc} B[0,1] & \xrightarrow{\psi[0,1]} & C[0,1] \\ \downarrow{\scriptstyle ev_*^B} & & \downarrow{\scriptstyle ev_*^C} \\ B & \longrightarrow & C \end{array}$$

It is a group homomorphism and the constructoin is functorial. □

DEFINITION 72. Define $\mathbb{M}(A, B) \subset \mathbb{E}(A, B)$ be the class of what I call Morita cycles from A to B by $(E, \phi, T) \in \mathbb{M}(A, B)$ if $T = 0$. Note that $(E, \phi, 0) \in \mathbb{E}(A, B)$ if and only if $\phi(A) \subset \mathcal{K}(E)$. If $\psi : A \to B$ is a graded $*$-homomorphism, then we define $(\psi) = (B, \psi, 0) \in \mathbb{M}(A, B) \subset \mathbb{E}(A, B)$. We define $[\psi] = [(\psi)] \in KK(A, B)$. If ${}_A E_B$ is a graded Morita equivalence, then $A \cong \mathcal{K}(E)$, and if ϕ is the left action of A on E then $(E, \phi, 0) \in \mathbb{M}(A, B) \subset \mathbb{E}(A, B)$, we write (E) for $(E, \phi, 0) \in \mathbb{E}(A, B)$ and $[E]$ for $[(E)] \in KK(A, B)$.

DEFINITION 73 (Definition and lemma)**.** If $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A, B)$ and $\mathcal{F} = (F, \phi', 0) \in \mathbb{M}(B, C)$ then define $\mathcal{E} \hat{\otimes}_B \mathcal{F} = (E \hat{\otimes}_B F, \phi \hat{\otimes} 1, T \hat{\otimes} 1)$. Then $\mathcal{E} \hat{\otimes}_B \mathcal{F} \in \mathbb{E}(A, C)$. This defines a group homomorphism

$$\hat{\otimes}_B \mathcal{F} : KK(A, B) \to KK(A, C)$$

such that

(1) $\mathcal{E} \hat{\otimes}_B (\psi) = \psi_*(\mathcal{E})$ for all $\psi : B \to C$;
(2) $(\mathcal{E} \hat{\otimes}_B \mathcal{F}) \hat{\otimes}_C \mathcal{F}' \cong \mathcal{E} \hat{\otimes}_B (\mathcal{F} \hat{\otimes}_C \mathcal{F}')$ for all $\mathcal{F}' \in \mathbb{M}(C, D)$;
(3) $\mathcal{E} \hat{\otimes}_B (\psi)_C \hat{\otimes} \mathcal{F}' \cong \psi_*(\mathcal{E}) \hat{\otimes}_C \mathcal{F}' \cong \mathcal{E} \hat{\otimes}_B \psi^*(\mathcal{F}')$.

Proof. (1) $\hat{\otimes}_B \mathcal{F}$ is well-defined on the level of KK. If $\tilde{\mathcal{E}} \in \mathbb{E}(A, B[0, 1])$ then, because $\mathcal{F}[0, 1] \in \mathbb{M}(B[0, 1], C[0, 1])$,

$$ev_{t,*}^C (\tilde{\mathcal{E}} \hat{\otimes}_{B[0,1]} \mathcal{F}[0, 1]) \cong ev_{t,*}^B (\tilde{\mathcal{E}}) \hat{\otimes} \mathcal{F}.$$

(2) $\hat{\otimes}_B \mathcal{F}$ is a group homomorphism. If $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{E}(A, B)$, then

$$(\mathcal{E}_1 \oplus \mathcal{E}_2) \hat{\otimes}_B \mathcal{F} \cong \mathcal{E}_1 \hat{\otimes}_B \mathcal{F} \oplus \mathcal{E}_2 \hat{\otimes} \mathcal{F}.$$

□

COROLLARY 74. *If B and B' are (gradedly) Morita equivalent with Morita equivalence ${}_BE_{E'}$, then $\otimes_B E$ is an isomorphism.*

$$KK(A,B) \cong KK(A,B').$$

Proof. Let ${}_{B'}\bar{E}_B$ denote the flipped equivalence. Then

$${}_BE\hat{\otimes}_{B'}\bar{E}_B \cong {}_BB_B \quad \text{and} \quad {}_{B'}\bar{E}\hat{\otimes}_B E_{B'} \cong {}_{B'}B'_{B'},$$

so

$$(\mathcal{E}\hat{\otimes}_B E)\hat{\otimes}_{B'}\bar{E} \cong \mathcal{E}\hat{\otimes}_B(E\hat{\otimes}_{B'}\bar{E}) \cong \mathcal{E}\hat{\otimes}_B B = \mathrm{id}_{B,*}(\mathcal{E}) \cong \mathcal{E}$$

and likewise

$$\mathcal{E}'\hat{\otimes}_{B'}\bar{E}\hat{\otimes}_B E \cong \mathcal{E}'$$

for all $\mathcal{E} \in \mathbb{E}(A,B)$ and $\mathcal{E}' \in \mathbb{E}(A,B')$. □

LEMMA 75 (Stability of KK-theory)**.** *Let $\mathbb{K}$ carry the grading given by $(1,-1)$ under an identification $\mathbb{K} \cong M_2(\mathbb{K})$.*

1. *$\tau_{\mathbb{K}}$ is an isomorphism $KK(A,B) \cong KK(A\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathbb{K})$.*
2. *We have $KK(A,B) \cong KK(A\hat{\otimes}\mathbb{K}, B) \cong KK(A, B\hat{\otimes}\mathbb{K})$.*

LEMMA 76 (Homotopy invariance)**.** *Let $\psi_0, \psi_1 : B \to C$ be graded $*$-homomorphisms and $\psi : B \to C[0,1]$ such that $\psi_t = ev_t^C \circ \psi$ for $t = 0,1$. Then $[\psi_0] = [\psi_1] \in KK(B,C)$ and (ψ) is a homotopy from (ψ_0) to (ψ_1).*
It follows that $\psi_{0,*}(\mathcal{E}) \sim \psi_{1,*}(\mathcal{E})$ for all $\mathcal{E} \in \mathbb{E}(A,B)$.

COROLLARY 77. *If $A \sim 0$ is contractible, then $KK(A,A) \cong KK(A,0) \cong 0$.*

PROPOSITION 78. *If B is σ-unital, then it suffices in the definition of $KK(A,B)$ to consider only those triples (E,ϕ,T) where $E = \hat{\mathbb{H}}_B$.*

Proof. $(\hat{\mathbb{H}}_B, 0, 0) \in \mathbb{D}(A,B)$ and hence $(E,\phi,T) \sim (E \oplus \hat{\mathbb{H}}_B, \phi \oplus 0, T \oplus 0)$. (and $ev_{t,*}^B(\hat{\mathbb{H}}_{B[0,1]}) \cong \hat{\mathbb{H}}_B$ for all $t \in [0,1]$.) □

DEFINITION 79. Let $\mathcal{E} = (E,\phi,T) \in \mathbb{E}(A,B)$. Then a "compact perturbation" of T (or of $\mathcal{E}$) is an operator T' (or the cycle (E,ϕ,T')) such that

$$\forall\, a \in A: \quad \phi(a)(T - T') \in \mathcal{K}_B(E).$$

LEMMA 80. *In this case: $\mathcal{E}' = (E,\phi,T') \in \mathbb{E}(A,B)$ and $\mathcal{E} \sim \mathcal{E}'$.*

Proof. Consider the straight line segment. □

PROPOSITION 81. *If $(E,\phi,T) \in \mathbb{E}(A,B)$, then there is a compact perturbation (E,ϕ,S) of (E,ϕ,T) such that $S^* = S$, so in the definition of $KK(A,B)$ it suffices to consider only those triples with self-adjoint operator; and compact perturbations, homotopies and operator homotopies may be taken within this class.*

Proof. Replace T with $\frac{T-T^*}{2}$. □

PROPOSITION 82. *If $(E,\phi,T) \in \mathbb{E}(A,B)$, then there is a compact perturbation $(E,\phi,S) \in \mathbb{E}(A,B)$ of (E,ϕ,T) with $S = S^*$ and $\|S\| \leq 1$. If A is unital we may in addition obtain an S with $S^2 - 1 \in \mathbb{K}(E)$, compact perturbations, homotopies and operator homotopies may be taken within this class.*

Proof. WLOG, $T^* = T$, use functional calculus for

$$f(x) = \begin{cases} 1, & x > 1 \\ x, & -1 \leq x \leq 1 \\ -1, & x < -1. \end{cases}$$

□

REMARK 83 (The Fredholm picture of $KK(A,B)$.)**.** If A is unital: $P = \phi(1)$. Replace S with $PSP + (1-P)S(1-P)$. Let A be unital (the σ-unital case is more complicated). In the definition of KK-theory it suffices to consider only those triples (E,ϕ,T) with ϕ unital (replace E with PE and T with PTP). If there exists a unital graded $*$-homomorphism from A to $\mathcal{L}_B(\hat{\mathbb{H}}_B)$, then WLOG $E = \hat{\mathbb{H}}_B$. If A and B are trivially graded: Identity $\mathcal{L}(\hat{\mathbb{H}}_B)$ with $M_2(\mathcal{L}(\mathbb{H}_B))$ with grading given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix}$ with $\phi_i : A \to \mathcal{L}_B(\mathbb{H}_B)$ unital. $T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ for some $S \in \mathcal{L}_B(\mathbb{H}_B)$ with $\|S\| \leq 1$. The intertwining conditions become $S^*S - 1, SS^* - 1 \in \mathcal{K}_B(\mathbb{H}_B)$, $S\phi_1(a) - \phi_0(a)S \in \mathcal{K}_B(\mathbb{H}_B)$ for all $a \in A$. Homotopy becomes homotopy of triples (ϕ_0,ϕ_1,S) (with strong continuity).[1] In this picture modules are denoted by

$$(E_0 \oplus E_1, \phi_0 \oplus \phi_1, S) \quad \text{where} \quad S \in \mathcal{L}_B(E_0, E_1).$$

In particular, if $A = \mathbb{C}$, then

$$KK(\mathbb{C},B) \cong \{[T] : T \in \mathcal{L}_B(\mathbb{H}_B),\ T^*T - 1, TT^* - 1 \in \mathcal{K}_B(\mathbb{H}_B)\}.$$

THEOREM 84. *$KK(\mathbb{C},B) \cong K_0(B)$ for B trivially graded and σ-unital.*

Proof. Three methods of proof:

(1) Assuming $KK(\mathbb{C},B)$ can be described as the set of all triples $(\hat{\mathbb{H}}_B,\phi,T)$ where ϕ is unital, $T = T^*$, $\|T\| \leq 1$ and $T^2 - 1 \in \mathcal{K}(\hat{\mathbb{H}}_B)$ modulo the equivalence relations generated by
   (a) operator homotopy and
   (b) addition of degenerate cycles with unital $\mathbb{C}$-action,

   i.e. we assume that $KK(\mathbb{C},B) = \widehat{KK}(\mathbb{C},B)$. Then for all such triples T has the form $T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$. The condition on T is equivalent to $\pi(S)$ being unitary in $Q = \mathcal{L}_B(\mathbb{H}_B)/\mathcal{K}_B(\mathbb{H}_B) = \mathcal{L}_B/\mathcal{K}_B$, where $\pi : \mathcal{L}_B(\mathbb{H}_B) \to Q$ is the canonical projection. So every cycle $\mathcal{E}$ for $KK(\mathbb{C},B)$ gives an element in $K_1(Q)$. The exact sequence $0 \to \mathcal{K}_B \to \mathcal{L}_B \to Q \to 0$ gives a long exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(\mathcal{K}_B) & \longrightarrow & K_0(\mathcal{L}_B)(=0) & \longrightarrow & K_0(Q) \\ \uparrow{\scriptstyle index} & & & & \downarrow \\ K_1(Q) & \longleftarrow & K_1(\mathcal{L}_B)(=0) & \longleftarrow & K_1(\mathcal{K}_B) \end{array}$$

[1] This is not very precise and actually hardly correct. One should instead consider strictly continuous functions if we regard $\mathcal{L}(\mathbb{H}_B)$ as the multiplier algebra $M(\mathcal{K} \otimes B)$; moreover, Michael Joachim has pointed out to me that it is necessary to require the additional condition that for all $a \in A$ the function $t \mapsto S\phi_{1,t}(a) - \phi_{0,t}(a)S$ is not only strictly/strongly continuous but norm-continuous; here $t \mapsto \phi_{i,t}$ denotes the homotopies of representations of A on $\mathcal{L}(\mathbb{H}_B)$.

So $K_1(Q) \cong K_0(\mathcal{K}_B) = K_0(\mathcal{K} \otimes B) \cong K_0(B)$. So we obtain a map from $KK(\mathbb{C}, B)$ to $K_0(B)$ after observing that the K_1 elements are invariant under the elementary moves (operator homotopy and degenerate element addition). By a general lifting argument you can lift homotopies from Q to $\mathcal{L}_B$, so Φ is injective. It is clearly surjecitve and a homomorphism.

(2) Let B be unital. Let $(\hat{\mathbb{H}}_B, \phi, T)$ be a cycle as above, so $T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$. We try to define an index of $S : \mathbb{H}_B \to \mathbb{H}_B$ as an element of $K_0(B)$.

Problem: The image of S does not have to be closed and $\ker S, \operatorname{coker} S$ do not have to be finitely generated and projective.

Solution: One can show that there is an $S' \in \mathcal{L}_B(\mathbb{H}_B)$ such that

$$S - S' \in \mathcal{K}_B(\mathbb{H}_B)$$

and $\ker S', \operatorname{coker} S'^*$ are finitely generated and projective.

Definition: *index*$(S) = [\ker S'] - [\operatorname{coker} S'^*] \in K_0(B)$.

Exercise:

(a) Is this well-defined and a homomorphism?
(b) Is this invariant under homotopy?
(c) Is it bijective on the level of $KK(\mathbb{C}, B)$?

(3) (after Vincent Lafforgue) We define a map from $K_0(B) \to KK(\mathbb{C}, B)$ for B unital. Start with a finitely generated projective B-module E. Find a B-valued inner product on E (one can show that there is an essentially unique one). Define $\Phi([E]) = (E \rightleftharpoons_0^0 0) \in \mathbb{E}(\mathbb{C}, B)$. Moreover, define $\Phi([-E]) = (0 \rightleftharpoons_0^0 E)$. Then $\Phi([E] \oplus [-E]) = (E \rightleftharpoons_0^0 E) \sim (E \rightleftharpoons_{\mathrm{id}}^{\mathrm{id}} E) \sim 0$ because $\mathrm{id}_E \in \mathcal{K}_B(E)$ (which one has to show). So Φ is well-defined as a map from $K_0(B)$ to $KK(\mathbb{C}, B)$. We indicate how to show that it is surjective.

Let $\mathcal{E} = (E_0 \rightleftharpoons_g^f E_1) \in \mathbb{E}(\mathbb{C}, B)$. Find an $n \in \mathbb{N}$, $R \in \mathcal{K}_B(B^n, E_1), S \in \mathcal{K}_B(E_1, B^n)$ such that

$$\|1 - fg - RS\| < \frac{1}{2}$$

which means that every compact operator almost factors through some B^n. Then $fg + RS$ is invertible in $\mathcal{L}_B(E_1)$. Define $w = (fg + RS)^{-1}$. Note that $w \in 1 + \mathcal{K}_B(E_1)$. Now

$$(E_0 \underset{g}{\overset{f}{\rightleftharpoons}} E_1) \oplus (B^n \underset{0}{\overset{0}{\rightleftharpoons}} 0) = (E_0 \oplus B^n \underset{(g,0)}{\overset{(f,0)}{\rightleftharpoons}} E_1)$$

$$\sim (E_0 \oplus B^n \underset{\breve{g}=(g,S)w}{\overset{\breve{f}=(f,R)}{\rightleftharpoons}} E_1) = (*).$$

Observe that

$$\breve{f}\breve{g} = fgw + RSw = (fg + RS)w = \mathrm{id}_E\,.$$

Hence $\breve{p} = \breve{g}\breve{f} \in \mathcal{L}_B(E_0 \oplus B^n)$ is an idempotant. Let us assume that $\breve{p} = \breve{p}^*$, Then $E_0 \oplus B^n \cong \operatorname{Im} \breve{p} \oplus \operatorname{Im}(1 - \breve{p})$. This implies

$$(*) = (\operatorname{Im} \breve{p} \underset{\breve{g}}{\overset{\breve{f}}{\rightleftharpoons}} E_1) \oplus (\operatorname{Im}(1 - \breve{p}) \underset{0}{\overset{0}{\rightleftharpoons}} 0),$$

where $(\operatorname{Im} \breve{p} \rightleftharpoons_{\breve{g}}^{\breve{f}} E_1) \sim 0$ in $KK(\mathbb{C}, B)$. Observe $\breve{f}\breve{p} = \breve{f}$ and $\breve{p}\breve{g} = \breve{g}$. Note

$$1 - \breve{p} \in \mathcal{K}_B(E_0 \oplus B^n).$$

Then $\mathrm{Im}(1-\check{p})$ has a compact identity. This implies $\mathrm{Im}(1-\check{p})$ is finitely generated and projective. Hence

$$[\mathcal{E}] = [\mathrm{Im}(1-\check{p})] - [B^n] \in \Phi(K_0(B)).$$

Injectivity is similar.

□

3. The Kasparov product

THEOREM 85. *Let A, B, C, D be graded σ-unital C^*-algebras. Let A be separable. Then there exists a map*

$$\hat{\otimes}_B : KK(A,B) \times KK(B,C) \to KK(A,C),$$

called the Kasparov product, that has the following properties:

(1) *biadditivity:*

$$(x_1 \oplus x_2)\hat{\otimes}_B y = x_1\hat{\otimes}_B y \oplus x_2\hat{\otimes}_B y$$

and

$$x\hat{\otimes}_B(y_1 \oplus y_2) = x\hat{\otimes}_B y_1 \oplus x\hat{\otimes}_B y_2.$$

(2) *associativity, if B is separable as well, then*

$$x\hat{\otimes}_B(y\hat{\otimes}_C Z) = (x\hat{\otimes}_B y)\hat{\otimes}_C Z,$$

for all $x \in KK(A,B), y \in KK(B,C)$ and $z \in KK(C,D)$.

(3) *unit elements: if we define $1_A = [\mathrm{id}_A] \in KK(A,A)$ and $1_B = [\mathrm{id}_B] \in KK(B,B)$, then for all $x \in KK(A,B)$:*

$$1_A\hat{\otimes}_A x = x = x\hat{\otimes}_B 1_B.$$

(4) *functoriality: if $\phi : A \to B$ and $\psi : B \to C$ are graded $*$-homomorphism, then*

$$x\hat{\otimes}_B[\psi] = \psi_*(x) \quad \text{and} \quad [\phi]\hat{\otimes}_B y = \phi^*(y)$$

for all $x \in KK(A,B)$ and $y \in KK(B,C)$.

(5) *it generalizes the product of Morita cycles defines before.*

REMARK 86.

(1) The separable graded C^*-algebras form an additive category when equipped with the KK-groups as morphism sets and the flipped Kasparov product as compositions. The $\psi \to [\psi]$ is a functor from the category of separable graded C^*-algebras with graded $*$-homomorphism in this category.
(2) isomorphisms in this category are also called KK-equivalences. Consequently we know that Morita equivalences give KK-equivalences. In particular, KK-theory is also Morita invariant in the first component.

Idea of proof. Let $(E_1, \phi_1, T_1) \in \mathbb{E}(A,B)$ and $(E_2, \phi_2, T_2) \in \mathbb{E}(B,C)$. As module for the product we can take $E_{12} = E_1\hat{\otimes}E_2$ and as module action we can take $\phi_{12} = \phi_1\hat{\otimes}1$. The problem is to find the operator.
A very naive approach is to define $T_{12} = T_1\hat{\otimes}1 + 1\hat{\otimes}T_2$. $T_1\hat{\otimes}1$ is okay, but $1\hat{\otimes}T_2$ does not make any sense as long as T_2 is not B-linear on the left. If we neglect this problem, then we calculate

$$T_{12}^2 = T_1^2\hat{\otimes}1 + 1\hat{\otimes}T_2^2,$$

so we end up with something which is rather 2 than 1 up to compact operators. So the idea is to find suitable "coefficient" operators $M, N \in \mathcal{L}_C(E_{12})$ such that $M^2 + N^2 = 1$ and $M, N \geq 0$. Define

$$T_{12} = MT_1\hat{\otimes}1 + N1\hat{\otimes}T_2.$$

Then

$$T_{12}^2 \approx M^2T_1^2\hat{\otimes}1 + N^2 1\hat{\otimes}T_2^2 + \text{rest} \approx 1 + \text{rest}.$$

The critical point is that we need a lemma which ensures the existence of such coefficients such that the calculations are justified and rest=0 up to compact operators. This is the subject of "Kasparov's Technical Lemma".
To give a sense to an expression like $1\hat{\otimes}T_2$ is subject of the theory of connections. Such connections will only be unique up to "compact perturbation" and also the technical lemma involves some choices, so there is need for a contition when two operators are homotopic so that they give the same element in KK. These are the three tools which we introduce before we come to the proof of the existence of the product. □

PROPOSITION 87 (A sufficient condition for operator homotopy)**.** *Let A, B be graded C^*-algebras, $\mathcal{E} = (E, \phi, T), \mathcal{E}' = (E, \phi, T') \in \mathbb{E}(A, B)$. If*

$$\forall\ a \in A: \quad \phi(a)[T, T']\phi(a^*) \geq 0 \quad \textit{mod}\ \mathcal{K}_B(E),$$

where mod means that $\phi(a)[T, T']\phi(a^*) + k \geq 0$ for some $k \in \mathcal{K}_B(E)$, then $\mathcal{E}$ and $\mathcal{E}'$ are operator homotopic.

DEFINITION 88. If (B, β) is a graded C^*-algebra and $A \subset B$ is a sub-C^*-algebra then A is called graded if $\beta(A) \subset A$. [All subalgebras of graded algebras will be assumed graded.]

DEFINITION 89. Let B be a C^*-algebra and $A \subset B$ a subalgebra. Let $\mathcal{F} \subset B$ be a subset. We say that $\mathcal{F}$ derives A if $\forall\ a \in A, f \in \mathcal{F},\ [f, a] \in A$, where it is a graded commutator.

THEOREM 90. *Let B be a graded σ-unital C^*-algebra. Let A_1, A_2 be σ-unital sub-C^*-algebras of $M(B)$ and let $\mathcal{F}$ be a separable, closed linear subspace of $M(B)$ such that $\beta_B(\mathcal{F}) = \mathcal{F}$. Assume that*

1. $A_1 \cdot A_2 \subset B \quad [A_1 \perp A_2$ *mod* $B]$;
2. $[\mathcal{F}, A_1] \subset A_1 \quad [\mathcal{F}$ *derives* $A_1]$.

Then there exist elements $M, N \in M(B)$ of degree 0 such that $M + N = 1,\ M, N \geq 0,\ MA_1 \subset B,\ NA_2 \subset B,\ [N, \mathcal{F}] \subset B$.

REMARK 91.

1. The larger A_1, A_2 and $\mathcal{F}_1$, the stronger the lemma;
2. we can always assume WLOG: $B \subset A_1, A_2$.

   Proof. We can replace A_i with $A_i + B = A_i'$. A_i' is a graded sub-C^*-algebra that is σ-unital. If b is strictly positive in B and a_i is strictly positive in A_i then $b + a_i$ is strictly positive in A_i' because $b + a_i \geq 0$ and $(a_i + b)(A_i + B) \supset a_iA + bB$ (dense in A_i'.) □

3. we will use the lemma in the case $B = \mathcal{K}(E), M(B) = \mathcal{L}(E)$ for a countably generated Hilbert module E.

Exercise 92. Let X be a locally compact, σ-compact Hausdorff space and $\delta X = \beta X \backslash X$ its "corona space". Then δX is stonean, i.e. the closure of open sets are open or $\forall U, V \subset \delta X$ open, $U \cap V = \emptyset$ then $\exists\ f : \delta X \to [0,1]$ continuous such that $f|_U = 0,\ f|_V = 1$.

Next we will define connections. In this part let B, C be graded C^*-algebras, E_1 a Hilbert B-module, E_2 a Hilbert C-module, $\phi : B \to \mathcal{L}_C(E_2)$ a graded $*$-homomorphism, $E_{12} = E_1 \hat{\otimes}_B E_2$.

REMARK 93. Let $T_2 \in \mathcal{L}_C(E_2)$ and assume that

$$(*) \quad \forall\ b \in B : [\phi(b), T_2] = 0.$$

Define $1\hat{\otimes}T_2 \in \mathcal{L}_C(E_{12})$ on elementary tensors by

$$(1\hat{\otimes}T_2)(e_1\hat{\otimes}e_2) = (-1)^{\delta T_2 \delta e_1} e_1 \hat{\otimes} T_2(e_2).$$

in the sense that you first split T_2 into odd and even parts......
If T_2 is just B-linear up to compact operators, i.e. if

$$(**) \quad \forall\ b \in B\ [\phi(b), T_2] \in \mathcal{K}_C(E_2),$$

then this construction no longer works. We can however construct a substitute for $1\hat{\otimes}T_2$ "up to compact operators".

DEFINITION 94. For any $x \in E_1$ define

$$T_x : E_2 \to E_{12}, \quad e_2 \to x\hat{\otimes}e_2.$$

LEMMA 95. *If $T_2 \in \mathcal{L}_C(E_2)$ satisfies (*), then*

$$(***)1 \qquad \begin{array}{ccc} E_2 & \xrightarrow{T_2} & E_2 \\ {\scriptstyle T_x}\big\downarrow & & \big\downarrow{\scriptstyle T_x} \\ E_{12} & \longrightarrow & E_{12} \end{array}$$

gradedly commutes for all $x \in E_1$ (i.e. $T_x \circ T_2 = (1\hat{\otimes}T_2) \circ T_x \cdot (-1)^{\delta x \delta T_2}$). Similarly

$$(***)2 \qquad \begin{array}{ccc} E_2 & \xrightarrow{T_2} & E_2 \\ {\scriptstyle T_x^*}\big\uparrow & & \big\uparrow{\scriptstyle T_x^*} \\ E_{12} & \xrightarrow[1\otimes T_2]{} & E_{12} \end{array}$$

gradedly commutes.

LEMMA 96. *For all $x \in E$, we have $T_x \in \mathcal{L}_C(E_2, E_{12})$ with $T_x^* : E_{12} \to E_2$, $e_1 \otimes e_2 \to \phi(\langle x, e_1 \rangle)e_2$.*

DEFINITION 97. Let $T_2 \in \mathcal{L}_C(E_2)$. Then an operator $F_{12} \in \mathcal{L}_C(E_{12})$ is called a T_2-connection for E_1 (on E_{12}) if for all $x \in E_1$ the diagrams (***)1 and (***)2 commute up to compact operators.

PROPOSITION 98. *Let $T_2, T_2' \in \mathcal{L}_C(E_2)$, let T_{12} be a T_2-connection and T_{12}' be a T_2'-connection.*

1. *T_{12}^* is a T_2^*-connection;*
2. *$T_{12}^{(i)}$ is a $T_2^{(i)}$-connection for $i = 0, 1$;*
3. *$T_{12} + T_{12}'$ is a $(T_2 + T_2')$-connection;*

(4) $T_{12} \cdot T'_{12}$ *is a* $(T_2 T'_2)$*-connection;*
(5) *if* T_2 *and* T_{12} *are normal, then* $f(T_{12})$ *is an* $f(T_2)$*-connection for every continuous function* f *such that the spectra of* T_2 *and* T_{12} *are contained in its domain of definition.*
(6) *if* E_3 *is a Hilbert* D*-module,* $\psi : C \to \mathcal{L}_D(E_3)$ *is a graded* $*$*-homomorphism and* $T_3 \in \mathcal{L}_D(E_3)$ *with* $[T_3, \psi(C)] \subset \mathcal{K}_D(E_3)$*, and if* T_{23} *is a* T_3*-connection on* $E_2 \hat{\otimes}_C E_3$ *and if* T *is a* T_{23}*-connection on* $E = E_1 \hat{\otimes}_B (E_2 \hat{\otimes}_C E_3)$*, then* T *is a* T_3*-connection on* $E \cong (E_1 \hat{\otimes}_B E_2) \hat{\otimes}_C E_3$*.*
(7) *if* $E_1 = E'_1 \oplus E''_1$ *and if we identify* $E_1 \hat{\otimes}_B E_2$ *with* $E'_1 \hat{\otimes}_B E_2 \oplus E''_1 \hat{\otimes}_B E_2$*, then* T_2 *has the form* $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ *and* T_{12} *has the form* $\begin{pmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{pmatrix}$ *and* A_{12} *is an* A_2*-connection on* $E'_1 \hat{\otimes}_B E_2$ *and* D_{12} *is a* D_2*-connection on* $E''_1 \hat{\otimes}_B E_2$*. Conversely if* $T_2 = \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix}$ *and* A_{12}/D_{12} *is an* A_2/D_2*-connection, then* $\begin{pmatrix} A_{12} & 0 \\ 0 & D_{12} \end{pmatrix}$ *is a* T_2*-connection.*

PROPOSITION 99. *Let* $T_2 \in \mathcal{L}_C(E_2)$ *and let* T_{12} *be a* T_2*-connection.*
(1) $\forall\ k \in \mathcal{K}_B(E_1) :\ [T_{12}, k \otimes 1] \in \mathcal{K}_C(E_{12})$.
(2) T_{12} *is a zero-connection on* E_{12} *if and only if*

$$\forall\ k \in \mathcal{K}_B(E_1) :\ T_{12}(k \hat{\otimes} 1),\ (k \hat{\otimes} 1) T_{12} \in \mathcal{K}_C(E_{12}).$$

Proof. (1) Let $k \in \mathcal{K}_B(E_1)$. WLOG $k = \theta_{y,x}$ for $x, y \in E_1$. WLOG x, y, T_2, T_{12} are homogeneous with $\delta T_2 = \delta T_{12}$. Then

$$\theta_{y,x} \hat{\otimes} 1 = T_y T_x^*$$

by definition of T_x, T_y. Hence

$$(\theta_{y,x} \hat{\otimes} 1) \circ T_{12} = T_y \circ T_x^* \circ T_{12} = T_y \circ (-1)^{\delta x \delta T_2} T_2 \circ T_x^*$$

$$= (-1)^{\delta x \delta T_2} (-1)^{\delta y \delta T_2} T_{12} \circ T_y \circ T_x^* = (-1)^{\delta \theta_{y,x} \delta T_2} T_{12} \circ (\theta_{y,x} \hat{\otimes} 1) \mod \mathcal{K}_C(E_{12})$$

i.e. $[k, T_{12}] \in \mathcal{K}_C(E_{12})$.

(2) T_{12} is a 0-connection if and only if $\forall\ z \in E_1 : T_z^* T_{12}, T_{12} T_z$ are compact. Let $k \in \mathcal{K}_B(E_1)$. As above, WLOG $k = \theta_{y,x}$ for $x, y \in E_1$, we hence have $T_{12}(k \hat{\otimes} 1) = T_{12}(T_y T_x^*) = (T_{12} T_y) T_x^*$ is compact if and only if T_{12} is a 0-connection. This shows $\Rightarrow$.

Conversely, if $T_{12}(k \hat{\otimes} 1)$ is compact for all k, then $T_{12}(\theta_{z,z} \hat{\otimes} 1) T_{12}^* = T_{12} T_z T_z^* T_{12}^*$ is compact for all $z \in E_1$. So $(T_{12} T_z)(T_{12} T_z)^* \in \mathcal{K}_C(E_{12})$, hence by a lemma from the first section: $T_{12} T_z \in \mathcal{K}_C(E_1, E_{12})$. Similarly for $T_z^* T_{12}$. So T_{12} is a 0-connection.

□

LEMMA 100. *Let* $T_2, T'_2 \in \mathcal{L}_C(E_2)$ *such that* $\forall\ b \in B : \phi(b)(T_2 - T'_2),\ (T_2 - T'_2)\phi(b) \in \mathcal{K}_C(E_2)$*. Then* T_{12} *is a* T_2*-connection if and only if* T_{12} *is a* T'_2*-connection.*

Proof. Let T_{12} be a T_2-connection. Let $x \in E_1$. Find $\tilde{x} \in E_1, b \in B$ such that $x = \tilde{x} b$. Then $T_x = T_{\tilde{x}} \circ \phi(b)$.

$$T_{12} \circ T_x = (-1)^{\delta x \delta T_{12}} T_x \circ T_2 = (-1)^{\delta x \delta T_{12}} T_{\tilde{x}} \circ \phi(b) \circ T_2$$

$$(-1)^{\delta x \delta T_{12}} T_{\tilde{x}} \circ \phi(b) \circ T'_2 = (-1)^{\delta x \delta T_{12}} T_x \circ T'_2 \mod \mathcal{K}_C(E_2, E_{12})$$

and similarly for $T_x^* \circ T_{12}$. □

THEOREM 101 (Existence of connections). *Let E be a countably generated Hilbert B-module, E_2 a Hilbert C-module, $\phi : B \to \mathcal{L}_C(E_2)$ a graded $*$-homomorphism. If $T_2 \in \mathcal{L}_C(E_2)$ satisfies $\forall\ b \in B : [T_2, \phi(b)] \in \mathcal{K}_C(E_2)$, then there exists an T_2-connection on $E_1 \hat{\otimes}_B E_2$.*

Proof.

(1) Assume $\forall\ b \in B,\ [T_2, \phi(b)] = 0$. Then $1\hat{\otimes}_B T_2$ is a T_2-connection. In particular, 0 is a 0-connection, and if $B = \mathbb{C}$ and ϕ is unital, then the above result always applies.

(2) Assume $\phi : B \to \mathcal{L}_C(E_2)$ non-degenerate and $E_1 = B$. Then $\Phi : B\hat{\otimes}_B E_2 \to E_2$ via $b \otimes e_2 \to be_2$ is an isomorphism. This implies $T_{12} = \Phi^* T_2 \Phi \in \mathcal{L}_C(B\hat{\otimes}_B E_2)$ is a T_2-connection because $\phi(b) = \Phi \circ T_b$ for all $b \in B$ and hence
$$T_{12} T_b = \Phi^* T_2 \Phi T_b = \Phi^* T_2 \phi(b)$$
$$= (-1)^{\delta b \delta T_2} \Phi^* \phi(b) T_2 = (-1)^{\delta b \delta T_2} T_b T_2 \bmod \mathcal{K}_C(E_2, E_{12})$$
and similarly for T_{12}^*.

(3) Assume that B is unital, ϕ is unital and $E_1 = \hat{\mathbb{H}}_B$. Note that
$$\hat{\mathbb{H}}_B \hat{\otimes}_B E_2 \cong (\hat{\mathbb{H}} \hat{\otimes}_{\mathbb{C}} B) \otimes_B E_2 \cong \hat{\mathbb{H}} \hat{\otimes}_{\mathbb{C}} (B \hat{\otimes}_B E_2).$$
From (2), we know that there is a T_2-connection T_{23} on $B\hat{\otimes}_B E_2$. From (1) we know that there is a T_{23}-connection T on $\hat{\mathbb{H}}_B \hat{\otimes}_B E_2$.It follows that T is a T_2-connection on $\hat{\mathbb{H}}_B \hat{\otimes}_B E_2$.

(4) B is unital, ϕ is unital and E_1 is arbitrary. We have $E_1 \hat{\otimes} \hat{\mathbb{H}}_B \cong \hat{\mathbb{H}}_B$. By case (3) there is a T_2-connection on $\hat{\mathbb{H}}_B \hat{\otimes}_B E_2$. Hence there is also a T_2-connection on $E_1 \hat{\otimes}_B E_2$.

(5) general case: Let B^+ be the unital algbra $B \oplus \mathbb{C}$ and $\phi^+ : B^+ \to \mathcal{L}_C(E_2)$ be the unital extension of ϕ. Then E_1 is also a graded B^+-Hilbert module. The notion of a T_2-connection does not depend on this change of coefficients and $E_1 \hat{\otimes}_{B^+} E_2 = E_1 \hat{\otimes}_B E_2$. Also $[T_2, \phi^+(b + \lambda 1)] \in \mathcal{K}_C(E_2)$ for all $b + \lambda 1 \in B^+$. So there is a T_2-connection on $E_1 \hat{\otimes}_B E_2$ by case (4).

□

Exercise 102. Show: For every $(E, \phi, T) \in \mathbb{E}(A, B)$ there is some $(E', \phi', T') \in \mathbb{E}(A, B)$ homotopic to (E, ϕ, T) with ϕ' non-degenerate (actually, you can take $E' = A \cdot E$).

DEFINITION 103 (Kasparov product). $\mathcal{E}_{12} = (E_{12}, \phi_{12}, T_{12})$ is called a Kasparov product for (E_1, ϕ_1, T_1) and (E_2, ϕ_2, T_2) if

(1) $(E_{12}, \phi_{12}, T_{12}) \in \mathbb{E}(A, C)$;
(2) T_{12} is a T_2-connection on E_{12};
(3) $\forall\ a \in A : \phi_{12}(a)[T_1 \hat{\otimes} 1, T_{12}] \phi_{12}(a)^* \geq 0 \bmod \mathcal{K}_C(T_{12})$.

The set of all operators T_{12} on E_{12} such that $\mathcal{E}_{12}$ is a Kasparov product is denoted by $T_1 \# T_2$.

THEOREM 104. *Assume that A is separable. Then there exists a Kasparov product $\mathcal{E}_{12}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. It is unique up to operator homotopy and T_{12} can be chosen self-adjoint if T_1 and T_2 are self-ajoint. [It remains to show that the product is well-defined on the level of KK-theory.]*

Example 105.

(1) Assume $T_2 = 0$, i.e. $(E_2, \phi_2, 0) \in \mathbb{M}(B, C)$. Then $T_{12} = T_1\hat{\otimes}1$ is a Kasparov product of T_1 and 0.
 (a) $(E_{12}, \phi_{12}, T_1\hat{\otimes}1) \in \mathbb{E}(A, C)$ as stated above.
 (b) $T_1\hat{\otimes}1$ is a 0-connection because $(k\hat{\otimes}1)(T_1\hat{\otimes}1) = (kT_1)\hat{\otimes}1 \in \mathcal{K}_C(E_{12})$ because $\phi_2(B) \subset \mathcal{K}_C(E_2)$. (Also $T_1 k\hat{\otimes}1 \in \mathcal{K}_C(E_{12})$) for all $k \in \mathcal{K}_B(E_1)$.
 (c) let $a \in A$. Then $\phi_{12}(a)[T_1\hat{\otimes}1, T_1\hat{\otimes}1]\phi_{12}(a)^* = \phi_{12}(a)2T_1^2\hat{\otimes}1\phi_{12}(a)^* = 2\phi_{12}(a)\phi_{12}(a)^* \geq 0$ mod compact.

 So the multiplication between $\mathbb{E}(A, B)$ and $\mathbb{M}(B, C)$ defined earlier agrees with the Kasparov product.

(2) In particular, the push-forward along a $*$-homomorphism is a Kasparov product.
(3) Also the pull-back is a special case of the Kasparov product. Assume that we have shown that the product is well-defined on the level of homotopy classes.

 Let $\phi : A \to B$ be a $*$-homomorphism. Then one can assume WLOG that $\phi_2 : B \to \mathcal{L}_C(E_2)$ is non-degenerate. Then $B\hat{\otimes}_B E_2 \cong E_2$ and we can regard T_2 as a T_2-connection. The action of A on E_2 under this identification is $\phi_2 \circ \varphi$. It is easy to see that we obtain an element in $0\#T_2$ which is isomorphic to $\varphi^*(\mathcal{E}_2)$.
(4) In particualr, $1_A\hat{\otimes}_A x = x = x\hat{\otimes}_B 1_B$ for all $x \in KK(A, B)$.

Proof of the main theorem. □

Also the product lifts to a biadditive, associative map on the level of KK.

LEMMA 106. *Let A, B, C be as above. $\mathcal{E}_1 = (E_1, \phi_1, T_1) \in \mathbb{E}(A, B)$ with $T_1^* = T_1$ and $\|T_1\| \leq 1$ and $\mathcal{E}_2 = (E_2, \phi_2, T_2) \in \mathbb{E}(B, C)$. Let G be any T_2-connection of degree 1 on $E_{12} = E_1\hat{\otimes}_B E_2$. Define*

$$T_{12} = T_1\hat{\otimes}1 + [(1 - T_1^2)^{\frac{1}{2}}\hat{\otimes}1]G.$$

Then $\phi_{12}(a)(T_{12}^2 - 1)$ and $\phi_{12}(a)(T_{12} - T_{12}^*)$ are in $\mathcal{K}_C(E_{12})$ and $\phi_{12}(a)[T_{12}, T_1\hat{\otimes}1]\phi_{12}(a)^* \geq 0$ mod $\mathcal{K}_C(E_{12})$ for all $a \in A$. Suppose $[T_{12}, \phi_{12}(a)] \in \mathcal{K}(E_{12})$ for all $a \in A$, then $\mathcal{E}_{12} = (E_{12}, \phi_{12}, T_{12}) \in \mathbb{E}(A, C)$ and $\mathcal{E}_{12}$ is operator homotopic to an element of $\mathcal{E}_1\#\mathcal{E}_2$.

Proof. Let $L = (1 - T_1^2)^{\frac{1}{2}}\hat{\otimes}1$. $\phi_{12}(a)(T_{12}^2 - 1) = \phi_{12}(a)[T_1^2\hat{\otimes}1 + (T_1\hat{\otimes}1)LG + LG(T_1\hat{\otimes}1) + LGLG - 1]$. Now $\phi_{12}(a)(T_1\hat{\otimes}1)LG = \phi_{12}(a)L(T_1\hat{\otimes}1)G$ and $\phi_{12}(a)L \in \mathcal{K}_B(E_1)\hat{\otimes}1$, so $\phi_{12}L(T_1\hat{\otimes}1) \in \mathcal{K}_B(E_1)\hat{\otimes}1$, so $[\phi_{12}(a)L(T_1\hat{\otimes}1), G] \in \mathcal{K}_C(E_{12})$ and hence

$$\phi_{12}(a)L(T_1\hat{\otimes}1)G \overset{modK}{=} -(-1)^{\delta a}G\phi_{12}(a)L(T_1\hat{\otimes}1) \overset{modK}{=} -\phi_{12}(a)LG(T_1\hat{\otimes}1).$$

Similarly $\phi_{12}(a)LGLG = (-1)^{\delta a}G\phi_{12}(a)L^2G = (-1)^{\delta a + \delta a}\phi_{12}(a)L^2G^2$. So $\phi_{12}(a)(T_{12}^2 - 1) = \phi_{12}(a)((T_1^2 - 1)\hat{\otimes}1 = ((1 - T_1^2)\hat{\otimes}1)G^2) = [\phi_1(a)(T_1^2 - 1)]\hat{\otimes}1(1 - G^2) \in \mathcal{K}_C(E_{12})$. Similarly for $\phi_{12}(a)(T_{12} - T_{12}^*) \in \mathcal{K}_C(E_{12})$ and $\phi_{12}(a)[T_{12}, T_1\hat{\otimes}1]\phi_{12}(a)^* \geq 0$ mod $\mathcal{K}_C(E_{12})$.
Now find M and N as in the existence proof of the product such that

$$\tilde{T}_{12} = M^{\frac{1}{2}}(F_1\hat{\otimes}1) + N^{\frac{1}{2}}G$$

defines a Kasparov product $\tilde{\mathcal{E}}_{12} = (E_{12}, \phi_{12}, \tilde{T}_{12}) \in \mathbb{E}(A, C)$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. $\mathcal{E}_{12}$ is operator homotopy to $\tilde{\mathcal{E}}_{12}$ via:

$$T_t = [tM + (1-t)]^{\frac{1}{2}}(T_1 \hat{\otimes} 1) + [tN + (1-t)((1-T_1^2)^{\frac{1}{2}} \hat{\otimes} 1)]^{\frac{1}{2}} G.$$

□

The general form of the product. Let A_1, A_2, B_1, B_2 and D be graded σ-unital C^*-algebras and $x \in KK(A_1, B_1 \hat{\otimes} D)$, $y \in KK(D \hat{\otimes} A_2, B_2)$. If A_1 and A_2 are separable, then we define

$$x \otimes_D y = (x \hat{\otimes} 1_{A_1}) \hat{\otimes}_{B_1 \hat{\otimes} D \hat{\otimes} A_2} (1_{B_1} \hat{\otimes} y) \in KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

If $\mathbb{C} = D$, then we obtain a product

$$\otimes_{\mathbb{C}} : KK(A_1, B_1) \otimes KK(A_2, B_2) \to KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

It is commutative in the following sense. Let

$$\Sigma_{A_1, A_2} : A_1 \hat{\otimes} A_2 \to A_2 \hat{\otimes} A_1, \; a_1 \hat{\otimes} a_2 \to (-1)^{\delta a_1 \delta a_2} a_2 \hat{\otimes} a_1$$

and define Σ_{B_1, B_2} analogously. Then

$$x \otimes_{\mathbb{C}} y = \Sigma_{B_1, B_2}^{-1} \circ y \otimes_{\mathbb{C}} x \circ \Sigma_{A_1, A_2}.$$