1. Hilbert modules and adjointable operators

Let $B$ be a $C^*$-algebra.

**Definition 1.** A (right) pre-Hilbert module $E$ over $B$ is a complex vector space $E$ which is at the same time a (right) $B$-module compatible with the vector space structure of $E$ and is equipped with a map

$$\langle \cdot, \cdot \rangle : E \times E \to B,$$

such that

1. $\langle \cdot, \cdot \rangle$ is sesquilinear (linear in the right component);
2. $\forall b \in B$ and $\forall e, f \in E$, $\langle e, fb \rangle = \langle e, f \rangle b$;
3. $\forall e, f \in E$, $\langle e, f \rangle^* = \langle f, e \rangle \in B$;
4. $\forall e \in E$, $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0$ if and only if $e = 0$.

Define $\|e\| = \sqrt{\langle e, e \rangle}$ for all $e \in E$. If $E$ is complete with respect to this norm, then we call $E$ a Hilbert $B$-module. $E$ is called full if $\langle E, E \rangle = B$.

**Exercise 2.** Show that $\|\cdot\|$ defines a norm on $E$.

**Example 3.**

1. If $B = \mathbb{C}$, then a Hilbert module over $B$ is the same as a Hilbert space;
2. $B$ itself is a $B$-module with the module action

$$e \cdot b = eb \quad \forall e, b \in B$$

and the inner product

$$\langle e, f \rangle = e^* f \in B \quad \forall e, f \in B;$$
3. More generally, any closed right ideal $I \leq B$ is a right Hilbert $B$-module;
4. Let $(E_i)_{i \in I}$ be a family of pre-Hilbert $B$-modules. Then the direct sum $\bigoplus_{i \in I} E_i$ is a pre-Hilbert $B$-module with the inner product

$$\langle (e_i), (f_i) \rangle = \sum_{i \in I} \langle e_i, f_i \rangle_{E_i}.$$

Because the completion of a pre-Hilbert $B$-module is a Hilbert $B$-module, we can form the completion of $\bigoplus_{i \in I} E_i$, and also call it $\bigoplus_{i \in I} E_i$;
5. In the above example, let $I = \mathbb{N}$ and $E_i = B$. Define $\mathbb{H}_B = \bigoplus_{i \in \mathbb{N}} B$ to be the Hilbert $B$-module.
Example 4. Define

$$\ell^2(N, B) = \left\{ (b_i)_{i \in N} | b_i \in B \ \forall \ i \in N \text{ and } \sum_{i \in N} \|b_i\|^2 < \infty \right\}.$$ 

Show that $\ell^2(N, B) \subset H_B$ and find an example such that $\ell^2(N, B) \neq H_B$.

**Lemma 5.** If $E$ is a pre-Hilbert $B$-module, then for all $e, f \in E$

$$\|e\| \|f\| \geq \|\langle e, f \rangle\|.$$ 

**Proof.** If $f \neq 0$, define $b = -\frac{\langle f, e \rangle}{\|f\|^2}$. Then the inequality follows from $\langle e + fb, e + fb \rangle \geq 0$. 

**Remark 6.** Let $H$ be a Hilbert space and $T \in L(H)$. Then $T^*$ is the unique operator such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for all $x, y \in H$. Such $T^*$ always exists and this star operator turns $L(H)$ into a $C^*$-algebra.

**Definition 7.** Let $E_B$ and $F_B$ be Hilbert $B$-modules. Let $T$ be a map from $E$ to $F$. Then $T^* : F \to E$ is called the adjoint of $T$ if for all $e \in E, f \in F$

$$\langle Te, f \rangle = \langle e, T^*f \rangle.$$ 

If such $T^*$ exists, we call $T$ adjointable. The set of all such operator is denoted by $L(E, F)$.

**Exercise 8.** Find an example such that a continuous linear map $T : E \to F$ is not adjointable.

**Proposition 9.** Let $E, F$ be Hilbert $B$-modules, and let $T$ be an adjointable map from $E$ to $F$. Then

1. $T^*$ is unique, and $T^*$ is also adjointable and $(T^*)^* = T$,
2. $T$ is linear, $B$-linear and continuous,
3. $\|T\|^2 = \|T^*\|^2 = \|TT^*\| = \|T^*T\|$.

**Proposition 10.** Let $E, F$ be Hilbert $B$-modules, then $L(E) = L(E, E)$ is a $C^*$-algebra and $L(E, F)$ is a Banach space.

**Definition 11.** Let $E, F$ be Hilbert $B$-modules. For all $e \in E, f \in F$, define

$$\theta_{f, e} : E \to F$$

by

$$\theta_{f, e}(e') = f \langle e, e' \rangle_E.$$ 

**Proposition 12.** In the above situation, we have

1. $\theta_{f, e} \in L(E, F)$ and $\theta^{*}_{f, e} = \theta_{e, f}$,
2. for all $T \in L(F)$ and $S \in L(E)$, we have

$$T \circ \theta_{f, e} = \theta_{Tf, e}, \ \ \ \theta_{f, e} \circ S = \theta_{f, S^*e}.$$ 

**Definition 13.** Define $K(E, F) = K_B(E, F)$ to be the closed linear span of $\{\theta_{f, e} | e \in E, f \in F\}$. Elements in $K(E, F)$ is called compact operators.
PROPOSITION 14.
\[ \mathcal{L}(F) \mathcal{K}(F, E) = \mathcal{K}(F, E); \]
\[ \mathcal{K}(E, F) \mathcal{L}(F) = \mathcal{K}(E, F); \]
\[ \mathcal{K}(E, F) = \mathcal{K}(F, E). \]

In particular, \( \mathcal{K}(E) = \mathcal{K}(E, E) \) is a closed, \( \ast \)-closed two-sided ideal of \( \mathcal{L}(E) \).

LEMMA 15. Let \( E, F \) be Hilbert \( B \)-modules. Then
\[ \mathcal{K}(E, F) = \{ T \in \mathcal{L}(E, F) | TT^* \in \mathcal{K}(F) \}. \]

Proof. “\( \subseteq \)” is obvious.

“\( \supseteq \)” : Let \( \{ U_\lambda \}_\lambda \) be a bounded approximate unit for \( \mathcal{K}(F) \). Then using \( U_\lambda = U_\lambda^* \),
\[ \| U_\lambda T - T \|^2 = \| U_\lambda TT^* U_\lambda - U_\lambda TT^* - TT^* U_\lambda + TT^* \|. \]
Since \( TT^* \in \mathcal{K}(F) \) implies \( U_\lambda T \to T \in \mathcal{L}(E, F) \) and \( U_\lambda T \in \mathcal{K}(E, F) \), we have \( T \in \mathcal{K}(E, F) \).

Example 16.
(1) Let \( B = \mathbb{C} \), and let \( H \) be a Hilbert space. Then \( \mathcal{K}(H) \) is the usual algebra of compact operators,
(2) If \( B \) is arbitrary, and if you regard \( B \) as a Hilbert \( B \)-module, then \( \mathcal{K}(B) = B \).

Proof. Define \( \Phi : B \to \mathcal{L}(B) \) by \( b(b') = bb' \) for all \( b' \in B \). Then \( \Phi \) is a \( \ast \)-homomorphism and \( \Phi(b^*c) = \theta_{b,c} \) for all \( b, c \in B \). So \( \Phi(B \cdot B) \subseteq \mathcal{K}(B) \). But \( B \cdot B = B \).

(3) If \( E = E_1 \oplus E_2 \) and \( F = F_1 \oplus F_2 \), then
\[ \mathcal{K}(E, F) = \bigoplus_{i=1,2} \bigoplus_{j=1,2} \mathcal{K}(E_i, F_j), \]
and every \( T \in \mathcal{K}(E, F) \) can be expressed as a matrix
\[ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \]
(4) As a consequence of above, we have \( \mathcal{K}(B^m, B^n) = M_{m \times n}(B) \).

DEFINITION 17. If \( B \) is a C*-algebra, then we define
\[ M(B) = \mathcal{L}(B). \]
\( M(B) \) is called the multiplier algebra of \( B \). For example \( M(C_0(X)) = C_b(X) \) if \( X \) is a locally compact space.

PROPOSITION 18. If \( E \) is a Hilbert \( B \)-module, then
\[ M(\mathcal{K}(E)) = \mathcal{L}(E). \]

Sketch of proof. If \( T \in \mathcal{L}(E) \), then \( S \to TS \) defines an element \( T \cdot \in M(\mathcal{K}(E)) = \mathcal{L}(\mathcal{K}(E)) \). This defines a \( \ast \)-homomorphism \( \Psi : \mathcal{L}(E) \to M(\mathcal{K}(E)) \). For \( T \in \ker(\Psi) \): Let \( e \in E \).
\[ 0 = \langle \Psi(T)(\theta_{e,T}e)(Te), \Psi(T)(\theta_{e,T}e)(Te) \rangle = \langle (T\theta_{e,T}e)(Te), (T\theta_{e,T}e)(Te) \rangle = \langle Te, Te \rangle^3 \]
So \( Te = 0 \) for all \( e \in E \). Hence \( T = 0 \) and \( \Psi \) is injective.
If $m \in M(K(E))$ and $e \in E$, we define
\[ T(e) = \lim_{\epsilon \to 0} m(\theta_{e,e})(e)(\langle e, e \rangle + \epsilon)^{-1}. \]
Then this is a well-defined element of $L(E)$ and $\Psi(T) = m$. So $\Psi$ is surjective. \( \square \)

**DEFINITION 19.** Let $B, B'$ be $C^*$-algebras, and let $\psi : B \to B'$ be a $\ast$-homomorphism. Let $E_B$ be a Hilbert $B$-module and $E_B'$ is a Hilbert $B'$-module. A homomorphism with coefficient map $\psi$ from $E_B$ to $E_B'$ is a map $\Phi : E_B \to E_B'$ such that

1. $\Phi$ is $\mathbb{C}$-linear,
2. $\phi(eb) = \Phi(e)\psi(b)$ for all $e \in E_B$ and $b \in B$,
3. $\langle \Phi(e), \Phi(f) \rangle = \phi(\langle e, f \rangle) \in B'$ for all $e, f \in E_B$.

We denote such a map also by $\Phi_\psi$, by emphsizing $\psi$.

**REMARK 20.** From the definition, it follows that $\|\Phi(e)\| \leq \|e\|$ for all $e \in E_B$ and equality holds when $\psi$ is injective.

**REMARK 21.** There is an obvious composition of homomorphisms with coefficient maps: for $\Phi_\psi : E_B \to E_B'$ and $\Psi_\chi : E_B' \to E_{B''}$, we have a homomorphism $(\Psi \circ \Phi)_\chi \psi : E_B \to E_{B''}$.

Also $(\text{Id}_E)_{\text{Id}_B} : E_B \to E_B$ is a homomorphism.

**DEFINITION 22.** Two Hilbert $B$-modules $E_B$ and $E_B'$ are called isomorphic if there is a homomorphism $\Phi_{\text{Id}_B} : E_B \to E_B'$ which is bijective. Then $\Phi_{\text{Id}_B}^{-1} : E_B' \to E_B$. Write $E_B \cong E_B'$. Note that in this case, $\Phi_{\text{Id}_B} \in L(E_B, E_B')$ and $\Phi_{\text{Id}_B}^{-1} = \Phi_{\text{Id}_B}^{-1}$.

**DEFINITION 23.** A $C^*$-algebra $B$ is called $\sigma$-unital if there exists a countable bounded approximate unit.

**DEFINITION 24.** A positive element $h \in B$ is called strictly positive if $\phi(h) > 0$ for all states $\phi$ of $B$.

**LEMMA 25.** $B$ is $\sigma$-unital if and only if $B$ contains a strictly positive element.

**LEMMA 26.** A positive element $h \in B$ is strictly positive if and only if $hB = B$.

**LEMMA 27.** Let $E$ be a Hilbert $B$-module, and let $T \in L(E)$ be positive. Then $T$ is strictly positive if and only if $\overline{T(E)} = E$.

**DEFINITION 28.** A Hilbert $B$-module $E$ is called countably generated if there is a set $\{x_n : x_n \in E, \forall n \in \mathbb{N}\}$ such that the span of the set $\{x_n b : x_n \in E, b \in B, \forall n \in \mathbb{N}\}$ is dense in $E$.

We will show that $E$ is countably generated if and only if $K(E)$ is $\sigma$-unital. This is a consequence of the following important theorem.

**THEOREM 29** (Kasparov’s Stabilization Theorem). If $E$ is a countably generated Hilbert $B$-module, then $E \oplus H_B \cong H_B$.

*Proof.* Without loss of generality, we assume that $B$ is unital. We want to define a unitary $V : H_B \to E \oplus H_B$.

Instead of defining $V$ directly, we define $T \in L(H_B, E \oplus H_B)$ such that $T$ and $|T| = (T^* T)^{\frac{1}{2}}$ have dense range. Then the isometry $V$ defined by $V(|T|(x)) = T(x)$
can be extended to an isometry from $\mathbb{H}_B$ to $E \oplus \mathbb{H}_B$ with $\text{Range}(V) \supset \text{Range}(T)$ (which is dense, so $V$ is a unitary).

Let $\xi_n$ be the $n$-th standard basis vector in $\mathbb{H}_B$, and let $(\eta_n)$ be a generating sequence of $E$ such that for all $n \in \mathbb{N}$, $\{l \in \mathbb{N} | \eta_n = \eta_l\}$ is an infinite set. WLOG, we assume that $\|\eta_n\| \leq 1$ for all $n \in \mathbb{N}$. Define

$$T = \sum_k 2^{-k} \theta_{(\eta_k, 2^{-l} \xi_k), \xi_k}.$$ 

(1) $T$ has a dense range: Let $k \in \mathbb{N}$. Then for any $l \in \mathbb{N}$ with $\eta_k = \eta_l$, we have that $T(\xi_l) = 2^{-l}(\eta_k, 2^{-l} \xi_l)$, so

$$T(2^l \xi_l) = (\eta_k, 2^{-l}) \to (\eta_k, 0)$$

as $l \to \infty$. Hence $(\eta_k, 0) \in \text{Range}(T(\mathbb{H}_B))$, and also $2^l((\eta_k, 2^{-l} \xi_l) - (\eta_k, 0)) = (0, \xi_l) \in \text{Range}(T(\mathbb{H}_B))$;

(2) $T^* T$ has dense range:

$$T^* T = \sum_{k,l} 2^{-k-l} \theta_{\xi_k(\eta_k, \eta_l) + (2^{-k} \xi_k, 2^{-l} \xi_l), \xi_l}$$

$$= \sum_k 4^{-2k} \theta_{\xi_k, \xi_k} + \left( \sum_k 2^{-k} \theta_{(\eta_k, 0), \xi_k} \right)^2 \left( \sum_k 2^{-k} \theta_{(\eta_k, 0), \xi_k} \right)$$

$$\geq \sum_k 4^{-2k} \theta_{\xi_k, \xi_k} \quad \text{(def } S).$$

$S$ is positive and has dense range, so it is strictly positive in $\mathcal{K}(\mathbb{H}_B)$. Hence $T^* T$ is strictly positive in $\mathcal{K}(H)$ and has dense range;

(3) $|T|$ has dense range because $\text{Range}(|T|) \supset \text{Range}(T^* T)$.

\[\square\]

**COROLLARY 30.** $E_B$ is countably generated if and only if $\mathcal{K}(E)$ is $\sigma$-unital.

**Proof.**

(1) If $B$ is unital and $E = \mathbb{H}_B$. Let $\xi_i$ be the standard $i$-th basis vector in $\mathbb{H}_B$. Then

$$h = \sum_i 2^{-i} \theta_{\xi_i, \xi_i}$$

is strictly positive in $\mathcal{K}(E)$ since it has dense range;

(2) If $B$ is unital and $E = P \mathbb{H}_B$ for some $P \in \mathcal{L}(\mathbb{H}_B)$ with $P^* = P = P^2$. (This is almost generic my the above theorem.) Then

$$PhP = \sum_i 2^{-i} \theta_{p\xi_i, p\xi_i}$$

is strictly positive in $\mathcal{K}(E)$;

(3) $B$ is countably generated if and only if $B^+$ is countably generated. So $\mathcal{K}_{B+}(E)$ is $\sigma$-unital if and only if $\mathcal{K}_B(E)$ is $\sigma$-unital since $\mathcal{K}_{B+}(E) = \mathcal{K}_B(E)$.

\[\square\]

**DEFINITION 31.** Let $B, C$ be $C^*$-algebras, and let $E_B$ and $F_C$ be Hilbert $B, C$-modules respectively and let $\phi : B \to \mathcal{L}(F_C)$ be a $*$-homomorphism. On $E \otimes_{alg} F \times E \otimes_{alg} F$, define

$$\langle e \otimes f, e' \otimes f' \rangle = \langle f, \phi(e, e')f' \rangle \in C.$$
This defines a $C$-valued bilinear map. Define $N = \{ t \in E \otimes_{\text{alg}} F | \langle t, t \rangle = 0 \}$. Then $\langle \cdot, \cdot \rangle$ defines an inner product on $E \otimes_{\text{alg}} F/N$ which turns it to be a pre-Hilbert $C$-module.

The completion is called the inner tensor product of $E$ and $F$ and is denoted by $E \otimes_B F$ or $E \otimes_{\phi} F$.

**Lemma 32.** Let $E_{1B}, E_{2B}$ and $F_C$ be Hilbert $B, C$ module respectively, and let $\phi : B \to \mathcal{L}(F)$ be a $*$-homomorphism. Let $T \in \mathcal{L}(E_1, E_2)$. Then $e_1 \otimes f \to T(e_1) \otimes f$ defines a map $T \otimes 1 \in \mathcal{L}(E_1 \otimes_B F, E_2 \otimes_B F)$ such that $(T \otimes 1)^* = T^* \otimes 1$ and $\| T \otimes 1 \| \leq \| T \|$. If $\phi(B) \subset \mathcal{K}(F)$, then $T \in \mathcal{K}(E_1, E_2)$ implies $T \otimes 1 \in \mathcal{K}(E_1 \otimes F, E_2 \otimes F)$.

**Proof.** We only prove the last assertion here. The map $T \to T \otimes 1$ is linear and contractive from $\mathcal{L}(E_1, E_2)$ to $\mathcal{L}(E_1 \otimes F, E_2 \otimes F)$. So it suffices to consider $T$ of the form $\theta_{e_2, e_1}$ with $e_1 \in E_1$ and $e_2 \in E_2$. Because $E_2 = E_2 \cdot B$, it suffices to consider $\theta_{e_2, e_1}$ with $b \in B$. Now for all $e_1' \otimes f \in E_1 \otimes F$,

\[
(\theta_{e_2, e_1} \otimes 1)(e_1' \otimes f) = \theta_{e_2, e_1}(e_1') \otimes f = e_2 b(e_1, e_1') \otimes f = e_2 \otimes \phi(b) \phi((e_1, e_1')) f = (M_{e_2} \circ \phi(b) \circ N_{e_1})(e_1' \otimes f),
\]

where $M_{e_2} : F \to E_2 \otimes_B F$ by $f' \to e_2 \otimes f'$ and $N_{e_1} : E_1 \otimes_B F \to F$ by $e_1' \otimes f' \to \phi((e_1, e_1')) f'$. Because $M_{e_2} \in \mathcal{L}(F, E_2 \otimes_B F), N_{e_1} \in \mathcal{L}(E_1 \otimes_B F, F)$ and $\phi(b) \in \mathcal{K}(F)$, we have $\theta_{e_2, e_1} \otimes 1 \in \mathcal{K}(E_1 \otimes F, E_2 \otimes F)$.

**Lemma 33.** Let $B$ and $C$ be $C^*$-algebras, and let $\phi : B \to C$ be a $*$-homomorphism. Define $\tilde{\phi} : B \to \mathcal{L}(C) = M(C)$ by $b \to (c \to \phi(b)c)$. Then $\tilde{\phi}(B) \subset \mathcal{K}(C)$.

**Definition 34.** Let $E_B$ be a Hilbert $B$-module, and let $\phi : B \to C$ be a $*$-homomorphism. Define the push-forward $\phi_*(E)$ as $E \otimes_B C = E \otimes_{\phi} C$.

**Lemma 35.**

(1) $(\text{id}_B)_*(E) = E \otimes_B B \cong E$ canonically;

(2) $\psi_* (\phi_*(E)) \cong (\psi \circ \phi)_*(E)$ naturally, where $\psi : C \to D$ is a $*$-homomorphism.

**Lemma 36.** $T \in \mathcal{K}(E_1, E_2)$ implies $\phi_*(T) \in \mathcal{K}(\phi_*(E_1), \phi_*(E_2))$. Moreover,

$\phi_*(\theta_{c_2, c_1}) = \theta_{\phi(b_2), \phi(b_1)}$ for all $b_1, b_2 \in B, c_1 \in E_1$ and $c_2 \in E_2$.

**Remark 37.**

(1) The push-forward has the following universal property. If $\phi : B \to C$ and if $E_B$ is a Hilbert $B$-module, then there is a natural homomorphism $\Phi_\phi : E_B \cong E_B \otimes_B C = \phi_*(E)$ defined by $\Phi_\phi(e \otimes b) = e \otimes \phi(b)$. If $\Psi_\phi : E_B \to F_C$ is any homomorphism with coefficient map $\phi$, there is a unique homomorphism $\Phi_{\text{id}_C} : \phi_*(E)_C \to F_C$ defined by $\Psi_\phi(c) = \Psi_\phi(c)_C$ such that the following diagram commutes

\[
\begin{array}{ccc}
E & \xrightarrow{\Psi_\phi} & F \\
\downarrow \Phi_\phi & & \downarrow \Psi_{\text{id}_C} \\
\phi_*(E) & \xrightarrow{\cdot} & 
\end{array}
\]
DEFINITION 38. Let $B, B'$ be $C^*$-algebras, and let $E_B, E'_B$ be Hilberts $B, B'$ modules respectively. Then define a bilinear map

$$\langle \cdot, \cdot \rangle : E \otimes_{alg} E' \times E \otimes_{alg} E' \to B \otimes B'$$

by

$$\langle e_1 \otimes e'_1, e_2 \otimes e'_2 \rangle = \langle e_1, e_2 \rangle \otimes \langle e'_1, e'_2 \rangle.$$ 

This defines an inner product on $E \otimes_{C} E'$. Its completion, denoted by $E \otimes E'$, is a Hilbert $B \otimes B'$-module, called the external tensor product of $E$ and $E'$.

DEFINITION 39. A graded $C^*$-algebra is a $C^*$-algebra $B$ equipped with an order two $*$-homomorphism $\beta_B$, called the grading automorphism of $B$, i.e. $\beta_B^2 = \beta_B$. A $*$-homomorphism $\phi$ from a graded algebra $(B, \beta_B)$ to a graded algebra $(C, \beta_C)$ is graded if $\beta_C \circ \phi = \phi \circ \beta_B$.

REMARK 40. Note we have

$$B_0 \cdot B_1 \subset B_1 \quad B_1 \cdot B_0 \subset B_1$$

$$B_0 \cdot B_0 \subset B_0 \quad B_1 \cdot B_1 \subset B_0.$$ 

Moreover, $\phi : B \to C$ is graded if and only if $\phi(B_i) \subset C_i$ for $i = 0, 1$.

DEFINITION 41 (Definition and lemma). If $B$ is graded, then the graded commutator of $B$ is defined on homogeneous elements $a, b, c$ by

$$[a, b] = ab - (-1)^{\deg(a) \deg(b)} ba.$$ 

It satisfies the following properties.

1. $[a, b] = -(-1)^{\deg(a) \deg(b)} [b, a]$;
2. $[a, bc] = [a, b]c + (-1)^{\deg(a) \deg(b)} b[a, c]$;
3. $(-1)^{\deg(a) \deg(c)} [a, b]c + (-1)^{\deg(a) \deg(b)} [b, c]a + (-1)^{\deg(b) \deg(c)} [c, a]b = 0$.

DEFINITION 42. Let $A$ and $B$ be graded $C^*$-algebras. Define their graded tensor product as follows. On $A \otimes_{alg} B$, define

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\deg(a_1) \deg(b_1)} (a_1 a_2 \hat{\otimes} b_1 b_2)$$

and

$$(a_1 \hat{\otimes} b_1)^* = (-1)^{\deg(a_1) \deg(b_1)} (a_1^* \hat{\otimes} b_1^*)$$

for all homogeneous element $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Define a grading automorphism by $\beta_{A \hat{\otimes} B} = \beta_A \otimes \beta_B$.

Just as in the ungraded case, there are several feasible norms on $A \otimes_{alg} B$ and among them there is a maximal one. Completed for this norm the algebra $A \otimes_{alg} B$ becomes the maximal graded tensor product $A \otimes_{max} B$. There is also a spacial graded tensor product $A \hat{\otimes} B$. In general these completions can be different from there ungraded counterparts, but in the cases we are interested in, they agree. Hence we will not make a fuss about these norms.
PROPOSITION 43. The spatial graded tensor product $A \hat{\otimes} B$ is associative $(A \hat{\otimes} (B \hat{\otimes} C) = (A \hat{\otimes} B) \hat{\otimes} C)$ and commutative $(A \hat{\otimes} B \cong B \hat{\otimes} A)$ via $a \hat{\otimes} b \rightarrow (-1)^{\deg(a) \deg(b)} b \hat{\otimes} a$.

Example 44.

1. If $A$ is an ungraded $C^*$-algebra, then $id_A$ is a grading automorphism on $A$ which we call the trivial grading. With this grading, $A$ is called trivially graded.
2. If $A$ is a $C^*$-algebra and $u \in M(A)$ satisfies $u = u^* = u^{-1}$, then one can define a grading on $A$ by $a \rightarrow uau$. Such a grading is called an inner grading. We will see later that inner gradings are the less interesting gradings.
3. On $\mathbb{C}(1) = \mathbb{C} \oplus \mathbb{C}$, define the following grading automorphism:

   $$(a, b) \rightarrow (b, a).$$

   Then $(\mathbb{C}(1))_0 = \{(a, a) | a \in \mathbb{C}\}$ and $(\mathbb{C}(1))_1 = \{(a, -a) | a \in \mathbb{C}\}$. This grading is called the standard odd grading.

4. More generally, define the odd grading also on $A(1) = A \oplus A$ for any $C^*$-algebra $A$. Note that $A(1) \cong A \hat{\otimes} \mathbb{C}(1)$.

5. Alternatively, define $\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C}$ as follows.

   The multiplication is given by
   $$(1,0)(1,0) = (0,1)(0,1) = (1,0);$$
   $$(1,0)(0,1) = (0,1)(1,0) = (0,1).$$

   The involution is given by $(a, b)^* = (\bar{a}, \bar{b})$.

   The norm is given by $\| (a, b) \| = \max \{ |a + b|, |a - b| \}$.

   The grading is given by $(a, b) \rightarrow (a, -b)$.

   Then $\mathbb{C}_1$ is a graded $C^*$-algebra. Also $\mathbb{C}_1 \cong \mathbb{C}(1)$ as a graded $C^*$-algebra. Let $\mathbb{C}_1$ act on $\mathbb{C} \oplus \mathbb{C}$ by
   $$(a, b) \rightarrow \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$ 

   This is a faithful representation.

DEFINITION 45. Let $n \in \mathbb{N}$. Let $\mathbb{C}_n$ be the universal unital $\mathbb{C}$-algebra defined in the following way, called the $n$-th complex Clifford algebra:

1. there is an $\mathbb{R}$-linear map $i : \mathbb{R}^n \rightarrow \mathbb{C}_n$ such that
   $$i(v) \cdot i(v) = \langle v, v \rangle \cdot 1_{\mathbb{C}_n} \in \mathbb{C}_n$$
   for all $v \in \mathbb{R}^n$;

2. if $\phi : \mathbb{R}^n \rightarrow A$ is any $\mathbb{R}$-linear map from $\mathbb{R}^n$ to a unital $\mathbb{C}$-algebra satisfying the above condition, then there is a unique unital $\mathbb{C}$-linear homomorphism $\hat{\phi} : \mathbb{C}_n \rightarrow A$ such that $\phi = \hat{\phi} \circ i$.

Consider the complexified exterior algebra $\Lambda^*_\mathbb{C}\mathbb{R}^n$. It has a canonical Hilbert space structure. Let $\mathbb{C}_n$ act on $\Lambda^*_\mathbb{C}\mathbb{R}^n$ as follows: if $v \in \mathbb{R}^n$ then define $\mu(v) = ext(v) + ext(v)^* \in \mathcal{L}(\Lambda^*_\mathbb{C}\mathbb{R}^n)$. From the universal property of the Clifford algebra we obtain a homomorphism from $\mathbb{C}_n$ to $\mathcal{L}(\Lambda^*_\mathbb{C}\mathbb{R}^n)$.

On $\mathbb{C}_n$ we have an involution induced by the map

$$(v_1 \cdot v_2 \cdots v_k)^* = v_k \cdot v_{k-1} \cdots v_1$$

for all $v_1, \cdots, v_k \in \mathbb{R}^n$. With this involution, $\mathbb{C}_n$ is a $*$-algebra and $\mu : \mathbb{C}_n \rightarrow \mathcal{L}(\Lambda^*_\mathbb{C}\mathbb{R}^n)$ a $*$-homomorphism. It defines a $C^*$-algebra structure on $\mathbb{C}_n$. 

Example 46.
(1) $\mathbb{C}_1$ is the two-dimensional algebra defined above;
(2) $\mathbb{C}_2$ is the four-dimensional algebra with the basis $1,e_1,e_2,e_1e_2$ such that $e_1^2 = e_2^2 = 1$ and $e_1e_2 = -e_2e_1$.

**Definition 47.** The unitary map $v \rightarrow -v$ in $\mathbb{R}^n$ lifts to an isomorphism $\beta_n : \mathbb{C}_n \rightarrow \mathbb{C}_n$ such that $(\beta_n)^2 = 1$. It is a grading on $\mathbb{C}_n$.

**Exercise 48.** Show that $\mathbb{C}_2$ is isomorphic to $\mathbb{M}_{2 \times 2}(\mathbb{C})$ with the inner grading given by
\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

**Proposition 49.** We have $\mathbb{C}_{m+n} \cong \mathbb{C}_m \otimes \mathbb{C}_n$ for all $m,n \in \mathbb{N}$.

**Proof.** Define $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$. Let $i_m : V \rightarrow C_m$, $i_n : W \rightarrow C_n$ and $i_{m+n} : V \oplus W \rightarrow C_{m+n}$ be the canonical injections. Let $\pi_V : V \oplus W \rightarrow V$ and $\pi_W : V \oplus W \rightarrow W$ be the canonical projections. Then
\[
i = (i_V \hat{\otimes} 1) \circ \pi_V \oplus (1 \hat{\otimes} i_W) \circ \pi_W : V \oplus W \rightarrow C_m \otimes \mathbb{C}_n
\]
satisfies $i(x)i(x) = \langle x,x \rangle 1_{C_m} \otimes \mathbb{C}_n$, so there is a unital $\mathbb{C}$-linear homomorphism $\hat{i} : C_{m+n} \rightarrow C_m \otimes \mathbb{C}_n$ such that $i = \hat{i} \circ i_{m+n}$. Similarly, one can construct homomorphisms $\mathbb{C}_m \rightarrow C_{m+n}$ and $\mathbb{C}_n \rightarrow C_{m+n}$ which gradedly commute, so there is a homomorphism $\mathbb{C}_m \otimes \mathbb{C}_n \rightarrow C_{m+n}$. It is an inverse of $\hat{i}$. \qed

**Proposition 50.** If $n \in \mathbb{N}$ is even, then $\mathbb{C}_n \cong \mathbb{M}_{2^n \times 2^n}(\mathbb{C})$ with an inner grading. If $n = 2m+1$ is odd, then $\mathbb{C}_n \cong \mathbb{M}_{2^m \times 2^m}(\mathbb{C}) \oplus \mathbb{M}_{2^m \times 2^m}(\mathbb{C})$ with standard odd grading.

**Definition 51.** Let $(B, \beta_B)$ be a graded $C^*$-algebra and $E_B$ be a Hilbert $B$-module. A grading automorphism $\sigma_E : E \rightarrow E$ is a homomorphism with coefficient map $\beta_B$ such that $\sigma_E^2 = \text{id}_E$, i.e.
\[
\langle \sigma_E(e),\sigma_E(f) \rangle = \beta_E(\langle e,f \rangle)
\]
and $\sigma_E(cb) = \sigma_E(c)\beta_B(b)$ for all $c,f \in E$ and $b \in B$.

**Remark 52.** With $E_0 = \{ e \in E|\sigma_E(e) = e \}$ and $E_1 = \{ e \in E|\sigma_E(e) = -e \}$, we have
\[
(E_i,E_j) \subset B_{i+j}
\]
and $E_iB_j \subset E_{i+j}$.

If $B$ is trivially graded, then it still makes sense to consider graded Hilbert $B$-modules; they are just orthogonal direct sums of two Hilbert $B$-modules.

**Definition 53** (Definition and Lemma). If $E$ and $F$ are graded Hilbert modules over the graded $C^*$-algebra $B$, then define
\[
\sigma_{L(E,F)}(T) = \sigma_F \circ T \circ \sigma_E
\]
for all $T \in L(E,F)$.

This map satisfies:
(1) $\sigma_{L(E,F)}^2(T) = T$ for all $T \in L(E,F)$;
(2) $\sigma_{L(E,F)}(T^*) = [\sigma_{L(E,F)}(T)]^*$ for all $T \in L(E,F)$;
(3) $\sigma_{L(E,G)}(T \circ S) = \sigma_{L(F,G)}(T) \circ \sigma_{L(E,F)}(S)$ for all $T \in L(F,G)$ and $S \in L(E,F)$ where $G_B$ is another Hilbert $B$-module.
(4) \( \sigma_{\mathcal{L}(E,F)}(K(E,F)) \subset K(E,F) \) with \( \sigma_{\mathcal{L}(E,F)}(\theta_{f,e}) = \theta_{\sigma_{\mathcal{L}(f,e)}(e)} \) for all \( e \in E \) and \( f \in F \).

**Corollary 54.** If \( E \) is a graded Hilbert \( B \)-module, then \( \mathcal{L}(E) \) and \( K(E) \) are graded \( C^* \)-algebras.

**Definition 55.** The elements of \( \mathcal{L}(E,F)_0 \) are called even, written \( \mathcal{L}(E,F)^{even} \), the elements of \( \mathcal{L}(E,F)_1 \) are called off, written \( \mathcal{L}(E,F)^{odd} \).

**Remark 56.** An even element of \( \mathcal{L}(E,F) \) maps \( E_0 \) to \( F_0 \) and \( E_1 \) to \( F_1 \), and an odd element maps \( E_0 \) to \( F_1 \) and \( E_1 \) to \( F_0 \).

**Remark 57.** The following concepts and results can easily be adapted from the trivially graded case to the general graded case.

1. graded homomorphism with graded coefficient maps;
2. Kasparov stabilization theorem: \( \mathbb{H}_B \) has to be replaced by \( \mathbb{H}_B = \mathbb{H}_B \oplus \mathbb{H}_B \) with grading \( S = (\beta_B, \beta_B, \cdots) \) on the first summand and \(-S\) on the second summand;
3. the interior tensor product of graded Hilbert modules;
4. the exterior tensor product of graded Hilbert modules. The inner product is defined by
   \[
   \langle e_1 \hat{\otimes} f_1, e_2 \hat{\otimes} f_2 \rangle = (-1)^{\text{deg}(f_1)(\text{deg}(e_1)+\text{deg}(e_2))} \langle e_1, e_2 \rangle \hat{\otimes} \langle f_1, f_2 \rangle.
   \]
5. the push-forward along graded \( * \)-homomorphisms.

2. **The definition of \( KK \)-theory**

All \( C^* \)-algebras \( A, B, C, \cdots \) in this section will be \( \sigma \)-unital. Let \( A, B \) be graded \( C^* \)-algebras.

**Definition 58.** A Kasparov \( A-B \)-module or a Kasparov \( A-B \)-cycle is a triple \( E = (E, \phi, T) \) where \( E \) is a countably generated graded Hilbert \( B \)-module, \( \phi : A \to \mathcal{L}(E) \) is a graded \( * \)-homomorphism and \( T \in \mathcal{L}(E) \) is an odd operator such that

1. \( \forall a \in A : [\phi(a), T] \in K(E); \)
2. \( \forall a \in A : \phi(a)(T^2 - \text{id}_E) \in K(E); \)
3. \( \forall a \in A : \phi(a)(T - T^*) \in K(E). \)

Note that the commutator in 1) is graded. The class of all Kasparov \( A-B \)-modules will be denoted by \( E(A,B) \). Sometimes we denote elements of \( E(A,B) \) also as pairs \((E,T)\) without making reference to the action \( \phi \).

**Remark 59.** We are not going to discuss many examples at this point. They will occur later in the talks dedicated to applications of \( KK \)-theory.

**Definition 60** (Definition and Lemma).

1. If \( E_1 = (E_1, \phi_1, T_1) \) and \( E_2 = (E_2, \phi_2, T_2) \) are elements of \( E(A,B) \), then
   \( E_1 \oplus E_2 := (E_1 \oplus E_2, \phi_1 \oplus \phi_2, T_1 \oplus T_2) \in E(A,B); \)
2. If \( C \) is another graded \( C^* \)-algebra and \( \psi : B \to C \) is an even \( * \)-homomorphism and \( E = (E, \phi, T) \in E(A,B) \) then
   \( \psi_*(E) := (\psi_*(E), \phi \hat{\otimes} 1, \psi_*(T) = T \hat{\otimes} 1) \in E(A,C). \)
3. If \( C \) is another graded \( C^* \)-algebra, \( \phi : A \to B \) is an even \( * \)-homomorphism and \( E = (E, \phi, T) \in E(B,C), \) then
   \( \phi^*(E) := (E, \phi \circ \varphi, T) \in E(A,C); \)
(4) If $E = (E, \phi, T) \in \mathbb{E}(A, B)$ then
\[-E := (-E, \phi_-, -T) \in \mathbb{E}(A, B),\]
where $-E$ is the same Hilbert $B$-module as $E$ but with the grading $\sigma_{-E} := -\sigma_E$, and $\phi_- := \phi \circ \beta_A$ where $\beta_A$ is the grading on $A$.

**Proof.** We only show parts of (2). Let $a \in A$. Then
\[
(\hat{\phi} \hat{1})(a)((T \hat{1})^2 - \text{id}_{E \hat{\otimes} C}) = (\hat{\phi}(a) \hat{1})((T^2 \text{id}_C - \text{id}_E \hat{\otimes} \text{id}_C)
\]
\[
= (\hat{\phi}(a)(T^2 - \text{id}_E)) \otimes \text{id}_C
\]
\[
= \psi_*(\hat{\phi}(a)(T^2 - \text{id}_E)) \in K(\psi_*E).
\]
Here we use that $\phi(a)(T^2 - \text{id}_E) \in K(E)$. The other conditions follow similarly. □

**DEFINITION 61.** Let $\varphi : A \to A'$ and $\psi : B \to B'$ be *-homomorphisms and let $E = (E, \phi, T) \in \mathbb{E}(A, B)$ and $E' \in \mathbb{E}(A', B')$. A homomorphism from $E$ to $E'$ with coefficient maps $\varphi$ and $\psi$ is a homomorphism $\Phi_\psi$ from $E_B$ to $E_B'$ such that
\[
\begin{align*}
(1) & \quad \forall a \in A \forall e \in E, \Phi_\psi(\phi(a)e) = \phi'(\varphi(a))\Phi_\psi(e) \text{ i.e. } \Phi \text{ has coefficient map } \varphi \text{ on the left}; \\
(2) & \quad \phi \circ T = T' \circ \Phi;
\end{align*}
\]
The most important case is the case that $\Phi$ is bijective and $\varphi = \text{id}_A$, $\psi = \text{id}_B$. Then $E$ and $E'$ are called isomorphic.

**LEMMA 62.** We have up to isomorphism (for all $E, E_1, E_2, E_3 \in \mathbb{E}(A, B)$):
\[
\begin{align*}
(1) & \quad (E_1 \oplus E_2) \oplus E_3 \cong E_1 \oplus (E_2 \oplus E_3); \\
(2) & \quad E_1 \oplus E_2 \cong E_2 \oplus E_1; \\
(3) & \quad E \oplus (0, 0, 0) \cong E; \\
(4) & \quad \text{If } \psi : B \to C \text{ and } \psi' : C \to C' \text{ then } \\
& \quad \psi'_*(\psi_*(E)) \cong (\psi' \circ \psi)_*(E); \\
(5) & \quad (\text{id}_B)_*(E) \cong E; \\
(6) & \quad \text{If } \phi : A' \to A \text{ and } \phi' : A'' \to A \text{ then } \\
& \quad \phi'^*(\phi^*(E)) = (\phi \circ \phi')^*(E), \quad \text{id}_A^*(E) = E; \\
(7) & \quad \psi_*(E_1 \oplus E_2) \cong \psi_*(E_1) \oplus \psi_*(E_2), \quad \psi_*(-E) = -\psi_*(E); \\
(8) & \quad \phi^*(E_1 \oplus E_2) \cong \phi^*(E_1) \oplus \phi^*(E_2), \quad \phi^*(-E) = -\phi^*(E); \\
(9) & \quad \phi^*(\psi_*(E)) = \psi_*(\phi^*(E)).
\end{align*}
\]

**DEFINITION 63.** Let $C$ be a graded $C^*$-algebra and $E = (E, \phi, T) \in \mathbb{E}(A, B)$. We now give the definition of a cycle $\tau_E(E) = E \otimes \text{id}_C \in \mathbb{E}(A \hat{\otimes} C, B \hat{\otimes} C)$: the module is $E_B \hat{\otimes} C_C$, the action of $A \hat{\otimes} C$ is $\phi \otimes \text{id}_C$ and the operator is $T \otimes \text{id}_C$.

**Example 64.** If $C = C([0, 1]) = \{ f : [0, 1] \to C, \ f \text{ continuous} \}$, then $A \hat{\otimes} C \cong A[0, 1] = \{ f : [0, 1] \to A, \ f \text{ continuous} \}$ and $B \hat{\otimes} C \cong B[0, 1]$. Similarly $E_B \hat{\otimes} C_C \cong E[0, 1]$ if $E = (E, \phi, T) \in \mathbb{E}(A, B)$. Now $\tau_E([0, 1]) \cong (E[0, 1], \phi[0, 1], T[0, 1]) \in \mathbb{E}(A[0, 1], B[0, 1])$ under this identifications.

**DEFINITION 65** (Notions of homotopy). Let $E_0$ and $E_1$ be in $\mathbb{E}(A, B)$:
\[
(1) \quad \text{An operator homotopy from } E_0 \text{ to } E_1 \text{ is a norm-continuous path } (T_t)_{t \in [0, 1]} \text{ in } \mathcal{L}(E) \text{ for some graded Hilbert } B \text{-module } E \text{ equipped with a graded left action } \phi : A \to \mathcal{L}(E) \text{ such that } \]
\[
\begin{align*}
(\text{a}) & \quad \forall t \in [0, 1] : \quad (E, \phi, T_t) \in \mathbb{E}(A, B); \\
\end{align*}
\]
12 AN INTRODUCTION TO KK-THEORY

LEMMA 66. Homotopy is an equivalence relation on $\mathbb{E}(A,B)$.

Proof.

1) Reflexivity: let $\mathcal{E} = (E, \phi, T) \in \mathbb{E}(A,B)$. Then $i_A^*(\tau_{C[0,1]}(\mathcal{E})) \cong (E[0,1], \phi[0,1] \circ i_A, T[0,1])$ is a homotopy from $\mathcal{E}$ to $\mathcal{E}$, where $i_A : A \to A[0,1]$ is the inclusion as constant functions.

2) Symmetry: let $\mathcal{E} \in \mathbb{E}(A,B[0,1])$ and $\psi : B[0,1] \to B[0,1]$, $\beta \to (t \to \beta(1-t))$. Then $ev_B^R(\psi_*(\mathcal{E})) = (ev_B^R \circ \psi)(\mathcal{E}) = (ev_{B[0,1]}^R(\mathcal{E}))$, where $ev_B^R \circ \psi = ev_{B[0,1]}^R$.

3) Transitivity: this is a non-trivial exercise.

$\square$

DEFINITION 67. Define $KK(A,B) := \mathbb{E}(A,B)/ \sim$. If $\mathcal{E} \in \mathbb{E}(A,B)$ then we denote the corresponding element of $KK(A,B)$ by $[\mathcal{E}]$.

LEMMA 68. $KK(A,B)$ is an abelian group when equipped with the well-defined operation

$$[\mathcal{E}_1] \oplus [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2].$$

In particular, $KK(A,B)$ is a set. We have

$$[\mathcal{E}] \oplus [-\mathcal{E}] = [0,0,0],$$

where $[0,0,0]$ is the zero element of $KK(A,B)$.

Before we come to the proof of this important lemma, we define:

DEFINITION 69. The class $\mathbb{D}(A,B) \subset \mathbb{E}(A,B)$ of degenerate Kasparov $A-B$-modules is the class of all elements $(E, \phi, T)$ such that $[\phi(a), T], \phi(a)(T^2 - 1), \phi(a)(T - T^*) = 0$ for all $a \in A$.

LEMMA 70. If $\mathcal{E} = (E, \phi, T) \in \mathbb{D}(A,B)$, then $\mathcal{E} \sim 0$.

Proof. We construct a homotopy using a mapping cylinder, in this case for the rather trivial homomorphism $0 \xrightarrow{\sigma} E$. Consider the following diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & E[0,1]_{B[0,1]} \\
\downarrow & & \downarrow ev_0^E \\
0_B & \xrightarrow{\sigma} & E_B
\end{array}
$$

The pull-back $Z$ in this diagram can be identified with the Hilbert $B[0,1]$-module $E(0,1) = \{ \epsilon : [0,1] \to E, \epsilon \text{ continuous and } \epsilon(0) = 0 \}$. On $E(0,1)$ define an $A$-action by $(a \cdot \epsilon)(t) = a(\epsilon(t))$ for all $a \in A$, $\epsilon \in E(0,1)$ and $t \in [0,1]$. Define $T \in \mathcal{L}(E(0,1))$, $\epsilon \to T \circ \epsilon$. Then $\mathcal{E} = (E(0,1), T) \in \mathbb{E}(A,B[0,1])$ and $ev_0^B(\mathcal{E}) \cong 0$ and $ev_{1,*}(\mathcal{E}) \cong \mathcal{E}$. $\square$

Proof of the important lemma. It is obvious that $KK(A,B)$ is a set because the class of isomorphism classes of countable generated Kasparov $A-B$-modules is small. Moreover, the direct sum is well-defined and $[0]$ is the zero element. The
addition is commutative. What is left to show is that $E \oplus -E \sim 0$ for $E = (E, \phi, T) \in \mathcal{E}(A, B)$. Define $G_t \in \mathcal{L}(E \oplus -E)$ to be the element given by the matrix:

$$G_t = \begin{pmatrix}
\cos t \cdot T & \sin t \cdot \text{id}_E \\
\sin t \cdot \text{id}_E & -\cos t \cdot T
\end{pmatrix}.$$

Then $G_0 = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} = (T \oplus (-T))$, so $(E \oplus -E, \phi \oplus \phi_-, G_0) = (E \oplus -E, \phi \oplus \phi_-, T \oplus -T)$. Also $G_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $(E \oplus -E, \phi \oplus \phi_-, G_1) \in \mathcal{D}(A, B)$. That $G_t$ is odd and $(E \oplus -E, \phi \oplus \phi_-, G_t) \in \mathcal{E}(A, B)$ for all $t \in \mathbb{R}$ can be checked by direct calculations.

**Lemma 71.** $KK(A, B)$ is a bifunctor from the category of graded ($\sigma$-unital) $C^*$-algebras and graded $\ast$-homomorphism to the category of abelian groups.

**Proof.** Let $\psi : B \to C$ be a graded $\ast$-homomorphism. Then $E \to \psi_*(E)$ lifts to a map $\psi_* : KK(A, B) \to KK(A, C)$. Here using the diagram

$$\begin{array}{ccc}
B[0, 1] & \xrightarrow{\psi[0, 1]} & C[0, 1] \\
\downarrow ev^B & & \downarrow ev^C \\
B & \longrightarrow & C
\end{array}$$

It is a group homomorphism and the construction is functorial.

**Definition 72.** Define $M(A, B) \subset \mathcal{E}(A, B)$ be the class of what I call Morita cycles from $A$ to $B$ by $(E, \phi, T) \in M(A, B)$ if and only if $\phi(A) \subset K(E)$. If $\psi : A \to B$ is a graded $\ast$-homomorphism, then we define $(\psi) = (B, \psi, 0) \in M(A, B) \subset \mathcal{E}(A, B)$. We define $[\psi] = [(\psi)] \in KK(A, B)$.

If $A E_B$ is a graded Morita equivalence, then $A \cong K(E)$, and if $\phi$ is the left action of $A$ on $E$ then $(E, \phi, 0) \in M(A, B) \subset \mathcal{E}(A, B)$, we write $(E)$ for $(E, \phi, 0) \in \mathcal{E}(A, B)$ and $[E]$ for $[(E)] \in KK(A, B)$.

**Definition 73** (Definition and lemma). If $E = (E, \phi, T) \in \mathcal{E}(A, B)$ and $F = (F, \phi', 0) \in M(B, C)$ then define $E \otimes_B F = (E \otimes_B F, \phi \otimes 1, T \otimes 1)$. Then $\mathcal{E} \otimes_B F \in \mathcal{E}(A, C)$. This defines a group homomorphism

$$\hat{\otimes}_B F : KK(A, B) \to KK(A, C)$$

such that

1. $\mathcal{E} \otimes_B (\psi) = \psi_*(E)$ for all $\psi : B \to C$;
2. $(\mathcal{E} \otimes_B F) \otimes_C \mathcal{F}' \cong \mathcal{E} \otimes_B (\mathcal{F} \otimes_C \mathcal{F}')$ for all $\mathcal{F}' \in M(C, D)$;
3. $\mathcal{E} \otimes_B (\psi_*(E)) \otimes_C \mathcal{F}' \cong \mathcal{E} \otimes_B (\psi_*(F'))$.

**Proof.** (1) $\hat{\otimes}_B F$ is well-defined on the level of $KK$. If $\hat{\mathcal{E}} \in \mathcal{E}(A, B[0, 1])$ then, because $\mathcal{F}[0, 1] \in M(B[0, 1], C[0, 1])$,

$$ev^C_{\hat{\mathcal{E}}} (\mathcal{E} \otimes_B [0, 1]) \cong ev^B_{\hat{\mathcal{E}}} (\mathcal{E} \otimes_B \mathcal{F}).$$

(2) $\hat{\otimes}_B F$ is a group homomorphism. If $\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2 \in \mathcal{E}(A, B)$, then

$$(\mathcal{E}_1 \otimes \mathcal{E}_2) \hat{\otimes}_B F \cong \mathcal{E}_1 \hat{\otimes}_B F \otimes \mathcal{E}_2 \hat{\otimes}_B F.$$
COROLLARY 74. If $B$ and $B'$ are (gradedly) Morita equivalent with Morita equivalence $B E E'$, then $\otimes_B E$ is an isomorphism.

$$KK(A, B) \cong KK(A, B')$$

Proof. Let $B, \bar{E}$ denote the flipped equivalence. Then

$$B E \otimes_B \bar{E} \cong B B \quad \text{and} \quad B' \bar{E} \otimes_B E B' \cong B' B'',$$

so

$$(\mathcal{E} \otimes_B \bar{E}) \bar{E} \cong \mathcal{E} \otimes_B (E \otimes_B \bar{E}) \cong \mathcal{E} \otimes_B B = \text{id}_{B,*}(\mathcal{E}) \cong \mathcal{E}$$

and likewise

$$\mathcal{E}' \otimes_B \bar{E} \otimes_B E \cong \mathcal{E}'$$

for all $\mathcal{E} \in \mathcal{E}(A, B)$ and $\mathcal{E}' \in \mathcal{E}(A, B')$. □

LEMMA 75 (Stability of KK-theory). Let $\mathbb{K}$ carry the grading given by $(1, -1)$ under an identification $\mathbb{K} \cong M_2(\mathbb{K})$.

1. $\tau_\mathbb{K}$ is an isomorphism $KK(A, B) \cong KK(A \otimes \mathbb{K}, B \otimes \mathbb{K})$.

2. We have $KK(A, B) \cong KK(A \otimes \mathbb{K}, B) \cong KK(A, B \otimes \mathbb{K})$.

LEMMA 76 (Homotopy invariance). Let $\psi_0, \psi_1 : B \to C$ be graded $*$-homomorphisms and $\psi : B \to C^* [0, 1]$ such that $\psi_t = ev^C_t \circ \psi$ for $t = 0, 1$. Then $[\psi_0] = [\psi_1] \in KK(B, C)$ and $(\psi)$ is a homotopy from $(\psi_0)$ to $(\psi_1)$. It follows that $\psi_0,*(\mathcal{E}) \sim \psi_1,*(\mathcal{E})$ for all $\mathcal{E} \in \mathcal{E}(A, B)$.

COROLLARY 77. If $A \sim 0$ is contractible, then $KK(A, A) \cong KK(A, 0) \cong 0$.

PROPOSITION 78. If $B$ is $\sigma$-unital, then it suffices in the definition of $KK(A, B)$ to consider only those triples $(E, \phi, T)$ where $E = \mathbb{H}_B$.

Proof. $(\mathbb{H}_B, 0, 0) \in \mathbb{D}(A, B)$ and hence $(E, \phi, T) \sim (E \oplus \mathbb{H}_B, \phi \oplus 0, T \oplus 0)$. (and $ev^B_{t,*(\mathbb{H}_B[0, 1])} \cong \mathbb{H}_B$ for all $t \in [0, 1]$.) □

DEFINITION 79. Let $\mathcal{E} = (E, \phi, T) \in \mathcal{E}(A, B)$. Then a “compact perturbation” of $T$ (or of $\mathcal{E}$) is an operator $T'$ (or the cycle $(E, \phi, T')$) such that

$$\forall a \in A : \phi(a)(T - T') \in K_B(E).$$

LEMMA 80. In this case: $\mathcal{E}' = (E, \phi, T') \in \mathcal{E}(A, B)$ and $\mathcal{E} \sim \mathcal{E}'$.

Proof. Consider the straight line segment. □

PROPOSITION 81. If $(E, \phi, T) \in \mathcal{E}(A, B)$, then there is a compact perturbation $(E, \phi, S)$ of $(E, \phi, T)$ such that $S = S$, so in the definition of $KK(A, B)$ it suffices to consider only those triples with self-adjoint operator; and compact perturbations, homotopies and operator homotopies may be taken within this class.

Proof. Replace $T$ with $\frac{T - T^*}{2}$. □

PROPOSITION 82. If $(E, \phi, T) \in \mathcal{E}(A, B)$, then there is a compact perturbation $(E, \phi, S) \in \mathcal{E}(A, B)$ of $(E, \phi, T)$ with $S = S^*$ and $\|S\| \leq 1$. If $A$ is unital we may in addition obtain an $S$ with $S^2 - 1 \in \mathbb{K}(E)$, compact perturbations, homotopies and operator homotopies may be taken within this class.
Proof. WLOG, $T^* = T$, use functional calculus for

$$f(x) = \begin{cases} 
1, & x > 1 \\
x, & -1 \leq x \leq 1 \\
-1, & x < -1.
\end{cases}$$

\[\square\]

**REMARK 83** (The Fredholm picture of $KK(A, B)$.) If $A$ is unital: $P = \phi(1)$. Replace $S$ with $PSP + (1 - P)S(1 - P)$. Let $A$ be unital (the $\sigma$-unital case is more complicated). In the definition of $KK$-theory it suffices to consider only those triples $(E, \phi, T)$ with $\phi$ unital (replace $E$ with $PE$ and $T$ with $PTP$). If there exists a unital graded $*$-homomorphism from $A$ to $L_B(\mathbb{H}_B)$, then WLOG $E = \mathbb{H}_B$. If $A$ and $B$ are trivially graded: Identity $L(\mathbb{H}_B)$ with $M_2(L(\mathbb{H}_B))$ with grading given by $\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$. Let $\phi = \left( \begin{array}{cc} \phi_0 & 0 \\ 0 & \phi_1 \end{array} \right)$ with $\phi_i : A \to L_B(\mathbb{H}_B)$ unital. $T = \left( \begin{array}{cc} 0 & S^* \\ S & 0 \end{array} \right)$ for some $S \in L_B(\mathbb{H}_B)$ with $||S|| \leq 1$. The intertwining conditions become $S^*S - 1, SS^* - 1 \in K_B(\mathbb{H}_B)$, $S\phi_1(a) - \phi_0(a)S \in K_B(\mathbb{H}_B)$ for all $a \in A$. Homotopy becomes homotopy of triples $(\phi_0, \phi_1, S)$ (with strong continuity).\(^1\) In this picture modules are denoted by

$$(E_0 \oplus E_1, \phi_0 \oplus \phi_1, S) \text{ where } S \in L_B(E_0, E_1).$$

In particular, if $A = C$, then

$$KK(C, B) \cong \{ [T] : T \in L_B(\mathbb{H}_B), \ T^*T - 1, TT^* - 1 \in K_B(\mathbb{H}_B) \}.$$ 

**THEOREM 84.** $KK(C, B) \cong K_0(B)$ for $B$ trivially graded and $\sigma$-unital.

Proof. Three methods of proof:

1. Assuming $KK(C, B)$ can be described as the set of all triples $(\mathbb{H}_B, \phi, T)$ where $\phi$ is unital, $T = T^*$, $||T|| \leq 1$ and $T^2 - 1 \in K(\mathbb{H}_B)$ modulo the equivalence relations generated by
   - operator homotopy and
   - addition of degenerate cycles with unital $\mathbb{C}$-action,

   i.e. we assume that $KK(C, B) = K\widehat{K}(C, B)$. Then for all such triples $T$ has the form $T = \left( \begin{array}{cc} 0 & S^* \\ S & 0 \end{array} \right)$. The condition on $T$ is equivalent to $\pi(S)$ being unitary in $Q = L_B(\mathbb{H}_B)/K_B(\mathbb{H}_B) = L_B/K_B$, where $\pi : L_B(\mathbb{H}_B) \to Q$ is the canonical projection. So every cycle $E$ for $KK(C, B)$ gives an element in $K_1(Q)$. The exact sequence $0 \to K_B \to L_B \to Q \to 0$ gives a long exact sequence in $K$-theory:

$$\begin{array}{cccc}
K_0(K_B) & \longrightarrow & K_0(L_B)(= 0) & \longrightarrow \\
\uparrow \text{index} & & & \downarrow \\
K_1(Q) & \leftarrow & K_1(L_B)(= 0) & \leftarrow K_1(K_B)
\end{array}$$

\(^1\)This is not very precise and actually hardly correct. One should instead consider strictly continuous functions if we regard $L(\mathbb{H}_B)$ as the multiplier algebra $M(\mathbb{K} \otimes B)$; moreover, Michael Joachim has pointed out to me that it is necessary to require the additional condition that for all $a \in A$ the function $t \mapsto S\phi_{1, t}(a) - \phi_{0, t}(a)S$ is not only strictly/strongly continuous but norm-continuous; here $t \mapsto \phi_{i, t}$ denotes the homotopies of representations of $A$ on $L(\mathbb{H}_B)$.
So \( K_1(Q) \cong K_0(K_B) = K_0(\mathcal{K} \otimes B) \cong K_0(B) \). So we obtain a map from \( KK(\mathbb{C}, B) \) to \( K_0(B) \) after observing that the \( K_1 \) elements are invariant under the elementary moves (operator homotopy and degenerate element addition). By a general lifting argument you can lift homotopies from \( Q \) to \( \mathcal{L}_B \), so \( \Phi \) is injective. It is clearly surjective and a homomorphism.

(2) Let \( B \) be unital. Let \((\mathbb{H}_B, \phi, T)\) be a cycle as above, so \( T = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \).

We try to define an index of \( S: \mathbb{H}_B \to \mathbb{H}_B \) as an element of \( K_0(B) \).

**Problem**: The image of \( S \) does not have to be closed and \( \ker S, \text{coker} S \) do not have to be finitely generated and projective.

**Solution**: One can show that there is an index of \( S' \in \mathcal{L}_B(\mathbb{H}_B) \) such that

\[
S - S' \in K_B(\mathbb{H}_B)
\]

and \( \ker S', \text{coker} S'^* \) are finitely generated and projective.

**Definition**: index of \( S = [\ker S'] - [\text{coker} S'^*] \in K_0(B) \).

**Exercise**:

(a) Is this well-defined and a homomorphism?
(b) Is this invariant under homotopy?
(c) Is it bijective on the level of \( KK(\mathbb{C}, B) \)?

(3) (after Vincent Lafforgue) We define a map from \( K_0(B) \to KK(\mathbb{C}, B) \) for \( B \) unital. Start with a finitely generated projective \( B \)-module \( E \). Find a \( B \)-valued inner product on \( E \) (one can show that there is an essentially unique one). Define \( \Phi([E]) = (E \mapsto 0, 0) \in \mathbb{E}(\mathbb{C}, B) \). Moreover, define \( \Phi([-E]) = (0 \mapsto 0, E) \). Then \( \Phi([E] \oplus [-E]) = (E \mapsto 0, E) \sim (E \mapsto \text{id}_E) \sim 0 \) because \( \text{id}_E \in K_B(E) \) (which one has to show). So \( \Phi \) is well-defined as a map from \( K_0(B) \) to \( KK(\mathbb{C}, B) \). We indicate how to show that it is surjective.

Let \( E = (E_0 \mapsto \hat{f} E_1) \in \mathbb{E}(\mathbb{C}, B) \). Find an \( n \in \mathbb{N} \), \( R \in K_B(B^n, E_1) \), \( S \in K_B(E_1, B^n) \) such that

\[
\|1 - fg - RS\| < \frac{1}{2}
\]

which means that every compact operator almost factors through some \( B^n \). Then \( fg + RS \) is invertible in \( \mathcal{L}_B(E_1) \). Define \( w = (fg + RS)^{-1} \). Note that \( w \in 1 + K_B(E_1) \). Now

\[
(E_0 \overset{\hat{f}}{\underset{g}{\rightarrow}} E_1) \oplus (B^n \overset{0}{\underset{0}{\rightarrow}} 0) = (E_0 \oplus B^n \overset{(f, 0)}{\underset{(g, 0)}{\rightarrow}} E_1)
\]

\[
\sim (E_0 \oplus B^n \overset{\hat{f} = f, R}{\underset{g = (g, S)}{\rightarrow}} E_1) = (*).
\]

Observe that

\[
\hat{f} \hat{g} = fgw + RSw = (fg + RS)w = \text{id}_E.
\]

Hence \( \tilde{p} = \hat{g} \hat{f} \in \mathcal{L}_B(E_0 \oplus B^n) \) is an idempotent. Let us assume that \( \tilde{p} = \tilde{p}^* \). Then \( E_0 \oplus B^n \cong \text{Im} \tilde{p} \oplus \text{Im}(1 - \tilde{p}) \). This implies

\[
(*) = (\text{Im} \tilde{p} \overset{\hat{f}}{\underset{\hat{g}}{\rightarrow}} E_1) \oplus (\text{Im}(1 - \tilde{p}) \overset{0}{\underset{0}{\rightarrow}} 0),
\]

where \( (\text{Im} \tilde{p} \overset{\hat{f}}{\underset{\hat{g}}{\rightarrow}} E_1) \sim 0 \) in \( KK(\mathbb{C}, B) \). Observe \( \hat{f} \tilde{p} = \hat{f} \) and \( \hat{p} \hat{g} = \hat{g} \). Note

\[
1 - \tilde{p} \in K_B(E_0 \oplus B^n).
\]
Then $\text{Im}(1 - \tilde{p})$ has a compact identity. This implies $\text{Im}(1 - \tilde{p})$ is finitely generated and projective. Hence

$$[E] = [\text{Im}(1 - \tilde{p})] - [B^n] \in \Phi(K_0(B)).$$

Injectivity is similar.

\[ \square \]

3. The Kasparov Product

**Theorem 85.** Let $A, B, C, D$ be graded $\sigma$-unital $C^*$-algebras. Let $A$ be separable. Then there exists a map

$$\hat{\otimes}_B : KK(A, B) \times KK(B, C) \to KK(A, C),$$

called the Kasparov product, that has the following properties:

1. biadditivity:

$$x_1 \oplus x_2 \hat{\otimes}_B y = x_1 \hat{\otimes}_B y \oplus x_2 \hat{\otimes}_B y$$

and

$$x \hat{\otimes}_B (y_1 \oplus y_2) = x \hat{\otimes}_B y_1 \oplus x \hat{\otimes}_B y_2.$$

2. associativity, if $B$ is separable as well, then

$$x \hat{\otimes}_B (y \hat{\otimes}_C Z) = (x \hat{\otimes}_B y) \hat{\otimes}_C Z,$$

for all $x \in KK(A, B), y \in KK(B, C)$ and $z \in KK(C, D)$.

3. unit elements: if we define $1_A = [\text{id}_A] \in KK(A, A)$ and $1_B = [\text{id}_B] \in KK(B, B)$, then for all $x \in KK(A, B)$:

$$1_A \hat{\otimes}_A x = x = x \hat{\otimes}_B 1_B.$$

4. functoriality: if $\phi : A \to B$ and $\psi : B \to C$ are graded $*$-homomorphism, then

$$x \hat{\otimes}_B [\psi] = \psi_* (x) \quad \text{and} \quad [\phi] \hat{\otimes}_B y = \phi^* (y)$$

for all $x \in KK(A, B)$ and $y \in KK(B, C)$.

5. it generalizes the product of Morita cycles defined before.

**Remark 86.**

1. The separable graded $C^*$-algebras form an additive category when equipped with the $KK$-groups as morphism sets and the flipped Kasparov product as compositions. The $\psi \to [\psi]$ is a functor from the category of separable graded $C^*$-algebras with graded $*$-homomorphism in this category.

2. Isomorphisms in this category are also called $KK$-equivalences. Consequently we know that Morita equivalences give $KK$-equivalences. In particular, $KK$-theory is also Morita invariant in the first component.

**Idea of proof.** Let $(E_1, \phi_1, T_1) \in \mathbb{E}(A, B)$ and $(E_2, \phi_2, T_2) \in \mathbb{E}(B, C)$. As module for the product we can take $E_{12} = E_1 \hat{\otimes} E_2$ and as module action we can take $\phi_{12} = \phi_1 \hat{\otimes} 1$. The problem is to find the operator.

A very naïve approach is to define $T_{12} = T_1 \hat{\otimes} 1 + 1 \hat{\otimes} T_2$. $T_1 \hat{\otimes} 1$ is okay, but $1 \hat{\otimes} T_2$ does not make any sense as long as $T_2$ is not $B$-linear on the left. If we neglect this problem, then we calculate

$$T_{12}^2 = T_1^2 \hat{\otimes} 1 + 1 \hat{\otimes} T_2^2,$$
so we end up with something which is rather $2$ than $1$ up to compact operators. So the idea is to find suitable “coefficient” operators $M, N \in \mathcal{L}_C(E_{12})$ such that $M^2 + N^2 = 1$ and $M, N \geq 0$. Define

$$T_{12} = MT_1 \hat{\otimes} 1 + N1 \hat{\otimes} T_2.$$ 

Then

$$T_{12}^2 \approx M^2T_1^2 \hat{\otimes} 1 + N^21 \hat{\otimes} T_2^2 + \text{rest} \approx 1 + \text{rest}.$$ 

The critical point is that we need a lemma which ensures the existence of such coefficients such that the calculations are justified and rest=0 up to compact operators. This is the subject of “Kasparov’s Technical Lemma”.

To give a sense to an expression like $1 \hat{\otimes} T_2$ is subject of the theory of connections. Such connections will only be unique up to “compact perturbation” and also the technical lemma involves some choices, so there is need for a condition when two operators are homotopic so that they give the same element in $KK$. These are the three tools which we introduce before we come to the proof of the existence of the product. 

\begin{proposition} \label{prop:operator_homotopy}
(A sufficient condition for operator homotopy). Let $A, B$ be graded $C^*$-algebras, $E = (E, \phi, T), E' = (E, \phi, T') \in \mathcal{E}(A, B)$. If

$$\forall a \in A : \quad \phi(a)[T, T']\phi(a^*) \geq 0 \mod \mathcal{K}_B(E),$$

where mod means that $\phi(a)[T, T']\phi(a^*) + k \geq 0$ for some $k \in \mathcal{K}_B(E)$, then $E$ and $E'$ are operator homotopic.
\end{proposition}

\begin{definition} \label{def:graded_subalgebra}
If $(B, \beta)$ is a graded $C^*$-algebra and $A \subset B$ is a sub-$C^*$-algebra then $A$ is called graded if $\beta(A) \subset A$. [All subalgebras of graded algebras will be assumed graded.]
\end{definition}

\begin{definition} \label{def:derives}
Let $B$ be a $C^*$-algebra and $A \subset B$ a subalgebra. Let $\mathcal{F} \subset B$ be a subset. We say that $\mathcal{F}$ derives $A$ if $\forall a \in A, f \in \mathcal{F}, \ [f, a] \in A$, where it is a graded commutator.
\end{definition}

\begin{theorem} \label{thm:graded_unital}
Let $B$ be a graded $\sigma$-unital $C^*$-algebra. Let $A_1, A_2$ be $\sigma$-unital sub-$C^*$-algebras of $M(B)$ and let $\mathcal{F}$ be a separable, closed linear subspace of $M(B)$ such that $\beta_B(\mathcal{F}) = \mathcal{F}$. Assume that

\begin{enumerate}
  \item $A_1 \cdot A_2 \subset B$ \quad $[A_1 \perp A_2 \ mod \ B]$;
  \item $[\mathcal{F}, A_1] \subset A_1$ \quad $[\mathcal{F} \ derives \ A_1]$.
\end{enumerate}

Then there exist elements $M, N \in M(B)$ of degree $0$ such that $M + N = 1, M, N \geq 0, MA_1 \subset B, NA_2 \subset B, [N, \mathcal{F}] \subset B$.
\end{theorem}

\begin{remark} \label{rem:graded_unital}
\begin{enumerate}
  \item The larger $A_1, A_2$ and $\mathcal{F}_1, \$ the stronger the lemma;
  \item we can always assume WLOG: $B \subset A_1, A_2$.
\end{enumerate}

\begin{proof}
We can replace $A_i$ with $A_i + B = A_i'$, $A_i'$ is a graded sub-$C^*$-algebra that is $\sigma$-unital. If $b$ is strictly positive in $B$ and $a_i$ is strictly positive in $A_i$ then $b + a_i$ is strictly positive in $A_i'$ because $b + a_i \geq 0$ and $(a_i + b)(A_i + B) \supset a_iA + bB$ (dense in $A_i'$)
\end{proof}

\begin{enumerate}
  \item we will use the lemma in the case $B = \mathcal{K}(E), M(B) = \mathcal{L}(E)$ for a countably generated Hilbert module $E$.
\end{enumerate}

Exercise 92. Let $X$ be a locally compact, $\sigma$-compact Hausdorff space and $\delta X = \beta X \setminus X$ its “corona space”. Then $\delta X$ is stonean, i.e. the closure of open sets are open or $\forall U, V \subset \delta X$ open, $U \cap V = \emptyset$ then $\exists f : \delta X \to [0,1]$ continuous such that $f|U = 0$, $f|V = 1$.

Next we will define connections. In this part let $B, C$ be graded $C^*$-algebras, $E_1$ a Hilbert $B$-module, $E_2$ a Hilbert $C$-module, $\phi : B \to \mathcal{L}(E_2)$ a graded $\ast$-homomorphism, $E_{12} = E_1 \hat{\otimes} B E_2$.

Remark 93. Let $T_2 \in \mathcal{L}(E_2)$ and assume that
\[(\ast) \quad \forall b \in B : [\phi(b), T_2] = 0.\]

Define $1 \hat{\otimes} T_2 \in \mathcal{L}(E_{12})$ on elementary tensors by
\[(1 \hat{\otimes} T_2)(e_1 \hat{\otimes} e_2) = (-1)^{\delta x \delta e_1} e_1 \hat{\otimes} T_2(e_2).\]
in the sense that you first split $T_2$ into odd and even parts.....

If $T_2$ is just $B$-linear up to compact operators, i.e. if
\[(\ast\ast) \quad \forall b \in B : [\phi(b), T_2] \in K_C(E_2),\]
then this construction no longer works. We can however construct a substitute for $1 \hat{\otimes} T_2$ “up to compact operators”.

Definition 94. For any $x \in E_1$ define
\[T_x : E_2 \to E_{12}, \quad e_2 \to x \hat{\otimes} e_2.\]

Lemma 95. If $T_2 \in \mathcal{L}(E_2)$ satisfies $(\ast)$, then
\[
\begin{array}{ccc}
E_2 & \xrightarrow{T_2} & E_2 \\
\downarrow T_x & & \downarrow T_x \\
E_{12} & \longrightarrow & E_{12}
\end{array}
\]
gradedly commutes for all $x \in E_1$ (i.e. $T_x \circ T_2 = (1 \hat{\otimes} T_2) \circ T_x \cdot (-1)^{\delta x \delta T_2}$). Similarly
\[
\begin{array}{ccc}
E_2 & \xrightarrow{T^{(2)}_2} & E_2 \\
\downarrow T^*_x & & \downarrow T^*_x \\
E_{12} & \xrightarrow{\hat{\otimes} T_2} & E_{12}
\end{array}
\]
gradedly commutes.

Lemma 96. For all $x \in E$, we have $T_x \in \mathcal{L}(E_2, E_{12})$ with $T^*_x : E_{12} \to E_2$, $e_1 \otimes e_2 \to \phi((x, e_1)) e_2$.

Definition 97. Let $T_2 \in \mathcal{L}(E_2)$. Then an operator $F_{12} \in \mathcal{L}(E_{12})$ is called a $T_2$-connection for $E_1$ (on $E_{12}$) if for all $x \in E_1$ the diagrams $(\ast\ast\ast)$1 and $(\ast\ast\ast)$2 commute up to compact operators.

Proposition 98. Let $T_2, T'_2 \in \mathcal{L}(E_2)$, let $T_{12}$ be a $T_2$-connection and $T'_{12}$ be a $T'_2$-connection.

1. $T^*_x$ is a $T'_2$-connection;
2. $T^{(i)}_{12}$ is a $T^{(i)}_2$-connection for $i = 0, 1$;
3. $T_{12} + T'_{12}$ is a $(T_2 + T'_2)$-connection.
(4) $T_{12} \cdot T_{12}'$ is a $(T_{2}T_{2}')$-connection;
(5) if $T_{2}$ and $T_{12}$ are normal, then $f(T_{12})$ is an $f(T_{2})$-connection for every
continuous function $f$ such that the spectra of $T_{2}$ and $T_{12}$ are contained in
its domain of definition.
(6) if $E_{3}$ is a Hilbert $D$-module, $\psi : C \to L_{D}(E_{3})$ is a graded $*$-homomorphism
and $T_{3} \in L_{D}(E_{3})$ with $[T_{3}, \psi(C)] \subset K_{D}(E_{3})$, and if $T_{23}$ is a $T_{3}$-connection
on $E_{2} \hat{\otimes} C E_{3}$ and if $T$ is a $T_{23}$-connection on $E = E_{1} \hat{\otimes} B E_{2} \hat{\otimes} C E_{3}$, then $T$
is a $T_{3}$-connection on $E \equiv (E_{1} \hat{\otimes} B E_{2}) \hat{\otimes} C E_{3}$.
(7) if $E_{1} = E_{1}' \oplus E_{1}''$ and if we identify $E_{1} \hat{\otimes} B E_{2}$ with $E_{1}' \hat{\otimes} B E_{2} \oplus E_{1}'' \hat{\otimes} B E_{2}$,
then $T_{2}$ has the form $\left( \begin{array}{cc} A_{2} & B_{2} \\ C_{2} & D_{2} \end{array} \right)$ and $T_{12}$ has the form $\left( \begin{array}{cc} A_{12} & B_{12} \\ C_{12} & D_{12} \end{array} \right)$
and $A_{12}$ is an $A_{2}$-connection on $E_{1}' \hat{\otimes} B E_{2}$ and $D_{12}$ is a $D_{2}$-connection
on $E_{1}'' \hat{\otimes} B E_{2}$. Conversely if $T_{2} = \left( \begin{array}{cc} A_{2} & 0 \\ 0 & D_{2} \end{array} \right)$ and $A_{12}/D_{12}$ is an $A_{2}/D_{2}$-connection,
then $\left( \begin{array}{cc} A_{12} & 0 \\ 0 & D_{12} \end{array} \right)$ is a $T_{2}$-connection.

**Proposition 99.** Let $T_{2} \in L_{C}(E_{2})$ and let $T_{12}$ be a $T_{2}$-connection.

(1) $\forall k \in K_{B}(E_{1}) : [T_{12}, k \otimes 1] \in K_{C}(E_{12}).$

(2) $T_{12}$ is a zero-connection on $E_{12}$ if and only if
$\forall k \in K_{B}(E_{1}) : T_{12}(k \otimes 1), (k \otimes 1)T_{12} \in K_{C}(E_{12}).$

**Proof.**
(1) Let $k \in K_{B}(E_{1})$. WLOG $k = \theta_{y,x}$ for $x, y \in E_{1}$. WLOG $x, y, T_{2}, T_{12}$ are homogeneous with $\delta T_{2} = \delta T_{12}$. Then
$\theta_{y,x} \otimes 1 = T_{y}T_{x}^{*}$
by definition of $T_{x}, T_{y}$. Hence
$(\theta_{y,x} \otimes 1) \circ T_{12} = T_{y} \circ T_{x}^{*} \circ T_{12} = T_{y} \circ (-1)^{\delta y \delta T_{2}} T_{2} \circ T_{x}^{*}$
$= (-1)^{\delta y \delta T_{2}}(-1)^{\delta y \delta T_{2}} T_{12} \circ T_{y} \circ T_{x}^{*} = (-1)^{\delta y \delta \theta_{y,x} \delta T_{2}} T_{12} \circ (\theta_{y,x} \otimes 1) \mod K_{C}(E_{12})$
i.e. $[k, T_{12}] \in K_{C}(E_{12})$.
(2) $T_{12}$ is a 0-connection if and only if $\forall z \in E_{1} : T_{z}^{*} T_{12}, T_{12} T_{z}$ are compact.
Let $k \in K_{B}(E_{1})$. As above, WLOG $k = \theta_{y,x}$ for $x, y \in E_{1}$, we hence have $T_{12}(k \otimes 1) = T_{12}(T_{y} T_{x}^{*}) = (T_{12}T_{y}) T_{x}^{*}$ is compact if and only if $T_{12}$ is a 0-connection. This shows $\Rightarrow$.

Conversely, if $T_{12}(k \otimes 1)$ is compact for all $k$, then $T_{12}(\theta_{z,x} \otimes 1)T_{12} = T_{12} T_{z}^{*} T_{12}$ is compact for all $z \in E_{1}$. So $(T_{12}T_{z})(T_{12}T_{z})^{*} \in K_{C}(E_{12})$, hence by a lemma from the first section: $T_{12} T_{z} \in K_{C}(E_{1}, E_{12})$. Similarly
for $T_{2} T_{z}$. So $T_{12}$ is a 0-connection.

**Lemma 100.** Let $T_{2}, T_{2}' \in L_{C}(E_{2})$ such that $\forall b \in B : \phi(b)(T_{2} - T_{2}')$, $(T_{2} - T_{2}') \phi(b) \in K_{C}(E_{2})$. Then $T_{12}$ is a $T_{2}$-connection if and only if $T_{12}$ is a $T_{2}'$-connection.

**Proof.** Let $T_{12}$ be a $T_{2}$-connection. Let $x \in E_{1}$. Find $\tilde{x} \in E_{1}, b \in B$ such that
$x = \tilde{x} b$. Then $T_{x} = T_{\tilde{x}} \circ \phi(b)$.

$T_{12} \circ T_{x} = (-1)^{\delta x \delta T_{12}} T_{x} \circ T_{2} = (-1)^{\delta x \delta T_{12}} T_{\tilde{x}} \circ \phi(b) \circ T_{2}$
$(-1)^{\delta x \delta T_{12}} T_{\tilde{x}} \circ \phi(b) \circ T_{2}' = (-1)^{\delta x \delta T_{12}} T_{x} \circ T_{2}' \mod K_{C}(E_{2}, E_{12})$
and similarly for $T_2^* \circ T_{12}$.

**THEOREM 101** (Existence of connections). Let $E$ be a countably generated Hilbert $B$-module, $E_2$ a Hilbert $C$-module, $\phi : B \to \mathcal{L}_C(E_2)$ a graded $*$-homomorphism. If $T_2 \in \mathcal{L}_C(E_2)$ satisfies $\forall \ b \in B : [T_2, \phi(b)] \in \mathcal{K}_C(E_2)$, then there exists an $T_2$-connection on $\hat{E}_1 \hat{\otimes}_B E_2$.

**Proof.**

1. Assume $\forall \ b \in B : [T_2, \phi(b)] = 0$. Then $1 \hat{\otimes}_B T_2$ is a $T_2$-connection. In particular, $0$ is a $0$-connection, and if $B = \mathbb{C}$ and $\phi$ is unital, then the above result always applies.

2. Assume $\phi : B \to \mathcal{L}_C(E_2)$ non-degenerate and $E_1 = B$. Then $\Phi : B \hat{\otimes}_B E_2 \to E_2$ via $b \otimes e_2 \to be_2$ is an isomorphism. This implies $T_{12} = \Phi^* T_2 \Phi \in \mathcal{L}_C(B \hat{\otimes}_B E_2)$ is a $T_2$-connection because $\phi(b) = \Phi \circ T_b$ for all $b \in B$ and hence

$$T_{12} T_b = \Phi^* T_2 \Phi T_b = \Phi^* T_2 \phi(b)$$

$$= (-1)^{\delta b \delta T_2} \phi(b) T_2 = (-1)^{\delta b \delta T_2} (b T_2) \mod \mathcal{K}_C(E_2, 1_{12})$$

and similarly for $T_{12}^*$.

3. Assume that $B$ is unital, $\phi$ is unital and $E_1 = \mathbb{H}_B$. Note that

$$\mathbb{H}_B \hat{\otimes}_B E_2 \cong (\mathbb{H} \hat{\otimes}_C B) \hat{\otimes}_B E_2 \cong \mathbb{H} \hat{\otimes}_C (B \hat{\otimes}_B E_2).$$

From (2), we know that there is a $T_2$-connection $T_{23}$ on $B \hat{\otimes}_B E_2$. From (1) we know that there is a $T_{23}$-connection $T$ on $\mathbb{H}_B \hat{\otimes}_B E_2$. It follows that $T$ is a $T_2$-connection on $\mathbb{H}_B \hat{\otimes}_B E_2$.

4. $B$ is unital, $\phi$ is unital and $E_1$ is arbitrary. We have $E_1 \hat{\otimes}_B \mathbb{H}_B \cong \mathbb{H}_B$. By case (3) there is a $T_2$-connection on $\mathbb{H}_B \hat{\otimes}_B E_2$. Hence there is also a $T_2$-connection on $E_1 \hat{\otimes}_B E_2$.

5. General case: Let $B^+$ be the unital algebra $B \oplus \mathbb{C}$ and $\phi^+ : B^+ \to \mathcal{L}_C(E_2)$ be the unital extension of $\phi$. Then $E_1$ is also a graded $B^+$-Hilbert module. The notion of a $T_2$-connection does not depend on this change of coefficients and $E_1 \hat{\otimes}_{B^+} E_2 = E_1 \hat{\otimes}_B E_2$. Also $[T_2, \phi^+(b + \lambda 1)] \in \mathcal{K}_C(E_2)$ for all $b + \lambda 1 \in B^+$. So there is a $T_2$-connection on $E_1 \hat{\otimes}_B E_2$ by case (4).

**Exercise 102.** Show: For every $(E, \phi, T) \in \mathbb{E}(A, B)$ there is some $(E', \phi', T') \in \mathbb{E}(A, B)$ homotopic to $(E, \phi, T)$ with $\phi'$ non-degenerate (actually, you can take $E' = A \cdot E$).

**DEFINITION 103** (Kasparov product). $\mathcal{E}_{12} = (E_{12}, \phi_{12}, T_{12})$ is called a Kasparov product for $(E_1, \phi_1, T_1)$ and $(E_2, \phi_2, T_2)$ if

1. $(E_{12}, \phi_{12}, T_{12}) \in \mathbb{E}(A, C)$;
2. $T_{12}$ is a $T_2$-connection on $E_{12}$;
3. $\forall a \in A : \phi_{12}(a) [T_{12} \hat{\otimes} 1, T_{12}] \phi_{12}(a) \geq 0 \mod \mathcal{K}_C(T_{12})$.

The set of all operators $T_{12}$ on $E_{12}$ such that $\mathcal{E}_{12}$ is a Kasparov product is denoted by $T_{1 \# T_2}$.

**THEOREM 104.** Assume that $A$ is separable. Then there exists a Kasparov product $\mathcal{E}_{12}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. It is unique up to operator homotopy and $T_{12}$ can be chosen self-adjoint if $T_1$ and $T_2$ are self-adjoint. [It remains to show that the product is well-defined on the level of $KK$-theory.]
Example 105.

1. Assume $T_2 = 0$, i.e. $(E_2, \phi_2, 0) \in M(B, C)$. Then $T_{12} = T_1 \otimes 1$ is a Kasparov product of $T_1$ and $0$.
   (a) $(E_{12}, \phi_{12}, T_{12} \otimes 1) \in E(A, C)$ as stated above.
   (b) $T_1 \otimes 1$ is a 0-connection because $(k \otimes 1)(T_1 \otimes 1) = (kT_1) \otimes 1 \in K_C(E_{12})$ because $\phi_2(B) \subset K_C(E_2)$. (Also $T_1k \otimes 1 \in K_C(E_{12})$ for all $k \in K_B(E_1)$.
   (c) Let $a \in A$. Then $\phi_{12}(a)[T_1 \otimes 1, T_1 \otimes 1] \phi_{12}(a^*) = \phi_{12}(a)2T_1^{2} \otimes 1 \phi_{12}(a)^* = 2\phi_{12}(a)\phi_{12}(a)^* \geq 0$ mod compact.

So the multiplication between $E(A, B)$ and $M(B, C)$ defined earlier agrees with the Kasparov product.

2. In particular, the push-forward along a $\ast$-homomorphism is a Kasparov product.

3. Also the pull-back is a special case of the Kasparov product. Assume that we have shown that the product is well-defined on the level of homotopy classes.

Let $\phi : A \to B$ be a $\ast$-homomorphism. Then one can assume WLOG that $\phi_2 : B \to L_C(E_2)$ is non-degenerate. Then $B \otimes_B E_2 \cong E_2$ and we can regard $T_2$ as a $T_2$-connection. The action of $A$ on $E_2$ under this identification is $\phi_2 \circ \phi$. It is easy to see that we obtain an element in $0\#T_2$ which is isomorphic to $\phi^\ast(E_2)$.

4. In particular, $1_A \otimes_A x = x = x \otimes_B 1_B$ for all $x \in KK(A, B)$.

Proof of the main theorem. □

Also the product lifts to a biadditive, associative map on the level of $KK$.

Lemma 106. Let $A, B, C$ be as above. $\mathcal{E}_1 = (E_1, \phi_1, T_1) \in E(A, B)$ with $T_1^\ast = T_1$ and $\|T_1\| \leq 1$ and $\mathcal{E}_2 = (E_2, \phi_2, T_2) \in E(B, C)$. Let $G$ be any $T_2$-connection of degree 1 on $E_{12} = E_1 \otimes_B E_2$. Define

$$T_{12} = T_1 \otimes 1 + [(1 - T_1^2)^{1/2} \otimes 1]G.$$

Then $\phi_{12}(a)(T_{12}^2 - 1)$ and $\phi_{12}(a)(T_{12} - T_{12}^2)$ are in $K_C(E_{12})$ and $\phi_{12}(a)[T_{12}, T_1 \otimes 1] \phi_{12}(a)^* \geq 0 \mod K_C(E_{12})$ for all $a \in A$. Suppose $[T_{12}, \phi_{12}(a)] \in K(E_{12})$ for all $a \in A$, then $\mathcal{E}_{12} = (E_{12}, \phi_{12}, T_{12}) \in E(A, C)$ and $\mathcal{E}_{12}$ is operator homotopic to an element of $\mathcal{E}_{1}\#E_2$.

Proof. Let $L = (1 - T_1^2)^{1/2} \otimes 1$. $\phi_{12}(a)(T_{12}^2 - 1) = \phi_{12}(a)(T_1^2 \otimes 1 + (T_1 \otimes 1)LG + LG(T_1 \otimes 1) + (TG_1 - 1)]$. Now $\phi_{12}(a)(T_1 \otimes 1)LG = \phi_{12}(a)L(T_1 \otimes 1)G$ and $\phi_{12}(a)L \in K_B(E_1) \otimes 1$, so $\phi_{12}L(T_1 \otimes 1) \in K_B(E_1) \otimes 1$, so $[\phi_{12}(a)L(T_1 \otimes 1), G] \in K_C(E_{12})$ and hence

$$\phi_{12}(a)L(T_1 \otimes 1)G \mod K = -(-1)^{\delta a}G\phi_{12}(a)L(T_1 \otimes 1) \mod K = -\phi_{12}(a)LG(T_1 \otimes 1).$$

Similarly $\phi_{12}(a)LGLG = (1 - T_1^2)^{1/2} \phi_{12}(a)L^2G = (1 - T_1^2)^{1/2} \phi_{12}(a)L^2G^2$. So $\phi_{12}(a)(T_1^2 - 1) = \phi_{12}(a)((T_1^2 - 1) \otimes 1) = (1 - (T_1^2 - 1)) \otimes 1(1 - G^2) \in K_C(E_{12})$.

Similarly for $\phi_{12}(a)(T_{12} - T_{12}^2) \in K_C(E_{12})$ and $\phi_{12}(a)[T_{12}, T_1 \otimes 1] \phi_{12}(a)^* \geq 0 \mod K_C(E_{12})$.

Now find $M$ and $N$ as in the existence proof of the product such that

$$\tilde{T}_{12} = M \hat{x}(F_1 \otimes 1) + N \hat{x}G$$
defines a Kasparov product \( \tilde{\mathcal{E}}_{12} = (E_{12}, \phi_{12}, \tilde{T}_{12}) \in \mathcal{E}(A, C) \) of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). \( \mathcal{E}_{12} \) is operator homotopy to \( \tilde{\mathcal{E}}_{12} \) via:

\[
T_t = [tM + (1 - t)]^\frac{1}{2} (T_1 \otimes 1) + [tN + (1 - t)((1 - T_1^2)^\frac{1}{2} \otimes 1)]^\frac{1}{2} G.
\]

The general form of the product. Let \( A_1, A_2, B_1, B_2 \) and \( D \) be graded \( \sigma \)-unital \( C^* \)-algebras and \( x \in KK(A_1, B_1 \hat{\otimes} D) \), \( y \in KK(D \hat{\otimes} A_2, B_2) \). If \( A_1 \) and \( A_2 \) are separable, then we define

\[
x \otimes_D y = (x \hat{\otimes} 1_{A_1}) \hat{\otimes} B_1 \hat{\otimes} (1_{B_1} \hat{\otimes} y) \in KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).
\]

If \( C = D \), then we obtain a product

\[
\otimes_C : KK(A_1, B_1) \otimes KK(A_2, B_2) \to KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).
\]

It is commutative in the following sense. Let

\[
\Sigma_{A_1, A_2} : A_1 \hat{\otimes} A_2 \to A_2 \hat{\otimes} A_1, \quad a_1 \hat{\otimes} a_2 \to (-1)^{\delta a_1 \delta a_2} a_2 \hat{\otimes} a_1
\]

and define \( \Sigma_{B_1, B_2} \) analogously. Then

\[
x \otimes_C y = \Sigma_{B_1, B_2}^{-1} \circ y \otimes_C x \circ \Sigma_{A_1, A_2}.
\]