INDEX THEORY

GEORGE SKANDALIS

Lecture 1: Examples + constructions of Ind_t
Lecture 4: computations + generalizations

1. Unbounded KK-theory (Baaj-Julg)

Recall the definition of cycles of $KK(A, B)$ are triples of form $(\mathcal{E}, \pi, F)$ where
(1) $\mathcal{E} = \mathcal{E}^{(0)} \oplus \mathcal{E}^{(1)}$ is graded Hilbert B-module.
(2) A unital $\pi : A \to \mathscr{L}(\mathcal{E})$ is a graded $*$-representation of A.
(3) $F \in \mathscr{L}(\mathcal{E})$ is of odd grading satisfying $F = F^*$, $F^2 - 1 \in \mathcal{K}(\mathcal{E})$, $[F, \pi(a)] \in \mathcal{K}(\mathcal{E})$.

To define a unbounded cycle $A, \pi, \mathcal{E}$ have the same assumption, and F is replaced by a unbounded D with the following assumption:
(1) $D = D^*$. Recall definition of D^*: if $\mathrm{Dom}(D)$ dense in $\mathcal{E}$ and $\forall\, x, y \in \mathcal{E}, \exists z \in \mathcal{E}$ such that $< Dx, y >=< x, z >$, then $D^*y = z$.
(2) D has compact resolvent,i.e. $(D+\mathrm{i}I)^{-1} \in \mathcal{K}(\mathcal{E})$. Note that $D+\mathrm{i}I$ is invertibe because $\mathrm{Sp}(D) \subset \mathbb{R}$.
(3) There exists a dense algebra $\mathcal{A} \in A$ such that $[\pi(a), D]$ is bounded, $\forall\, a \in \mathcal{A}$.
(4) D is regular, i.e. $(\mathrm{Graph} D)^{\perp} \oplus \mathcal{U}\mathrm{Graph}(D) = \mathcal{E} \oplus \mathcal{E}$, where $\mathrm{Graph}(D) = \{(x, Dx), x \in \mathrm{Dom}\, D\}$ and $U : (x, y) \to (-y, x)$.

Remark 1.1. Every KK-element can be made unbounded. KK-product is sometimes easier for unbounded KK-elements, in particular outer KK-product over $\mathbb{C}$.

Construction form unbounded cycles to a bounded ones:

$$(\mathcal{E}, \pi, D) \to (\mathcal{E}, \pi, D(1 + D^2)^{-\frac{1}{2}})$$

Things to be checked:
(1)$F = D(1 + D^2)^{-\frac{1}{2}}$ extends to a bounded operator on $\mathcal{E}$.
(2)$1 - F^2 = (1 + D^2)^{-1}$ compact.
(3)$[F, \pi(a)] \subset \mathcal{K}(\mathcal{E})$ (It is enough to check on the dense algebra $\mathcal{A}$).
Note that

$$\frac{1}{\sqrt{x}} = \frac{2}{\pi} \int_0^{\infty} \frac{1}{x + t^2} \mathrm{d}t,$$

then

$$(1 + D^2)^{-\frac{1}{2}} = \frac{2}{\pi} \int_0^{\infty} (I(1 + t^2) + D^2)^{-1} \mathrm{d}t$$

is a uniformly convergent integral.
Given $[D, \pi(a)]$ bounded we have $[D(1 + t^2 + D^2)^{-1}, \pi(a)]$ compact, then the norm convergence of the integral

$$[F, \pi(a)] = \frac{2}{\pi} \int_0^{\infty} [D(1 + t^2 + D^2)^{-1}, \pi(a)] \mathrm{d}t$$

gives rise to a compact limit.

2. Examples of elliptic differential operator

2.1. **de Rham operator.** : M compact manifold, $n = \dim M$

Let d be exterior differential on M, $\Omega^0 = C^\infty(M, \mathbb{C})$, $\Omega^1 \cong T^*M \otimes \mathbb{C}$, $\Omega^k = \bigwedge^k T^*M \otimes \mathbb{C}$, $\Omega = \oplus_{k=0}^n \Omega^k$.

Then $\mathrm{d} : C^\infty(\Omega^k) \to C^\infty(\Omega^{k+1})$, $\forall\, \omega_1 \in C^\infty(\Omega^j)$, $\omega_2 \in C^\infty(\Omega^k)$.

$\omega_1 \wedge \omega_2 = (-1)^{jk} \omega_2 \wedge \omega_1$

$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^j \omega_1 \wedge \mathrm{d}\omega_2$.

$\mathrm{d}^2 = 0$.

Choose a Riemannian metric on M, then we have a Hermitian structure on $C^\infty(\Omega)$, denote H as the completion of $C^\infty(\Omega)$ under the inner product:

$H = H^{(0)} \oplus H^{(1)}, H^{(0)} = \bigoplus_{k \text{ even}} L^2(\Omega^k), H^{(1)} = \bigoplus_{k \text{ odd}} L^2(\Omega^k)$.

Let π denote the representation of $C(M)$ on H by point-wise multiplication.

Claim: $(H, \pi, \mathrm{d} + \mathrm{d}^*)$ is an unbounded cycle. In fact:

(1)$\forall f \in C^\infty(M, \mathbb{C})$, $[\mathrm{d}, \pi(f)]\omega = \mathrm{d}f \wedge \omega$ this is bounded (with norm $\|\mathrm{d}f\|_\infty$, where $\| \cdot \|$ is maximal norm)

In addition, $[\mathrm{d}^*, \pi(f)] = -[\mathrm{d}, \pi(f^*)]^*$, so $[\mathrm{d} + \mathrm{d}^*, \pi(f)]$ is bounded on H.

(2)$\mathrm{d} + \mathrm{d}^*$ has compact resolvent.

Remark 2.1. $\mathrm{d} + \mathrm{d}^*$ does not depend on metric and manifold.

Let $M = \mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, then basis for Ω are of the form: $x \to e^{-i<k,x>} dx_I = \omega_{k,I} \in \Omega$ for $k \in \mathbb{Z}^n$, $I \subset \{1, \ldots, n\}$.

One can check that $(\mathrm{d} + \mathrm{d}^*)^2 \omega_{k,I} = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\omega_{k,I} = |k|^2 \omega_{k,I}$, where $|k| = \sum_{i=1}^\infty k_i^2$ and observe that $(\mathrm{d} + \mathrm{d}^*)^2$ has compact inverse.

The analytical index of $\mathrm{d} + \mathrm{d}^*$ is defined as

$$\mathrm{ind}_a(\mathrm{d} + \mathrm{d}^*) = \dim(\ker(\mathrm{d} + \mathrm{d}^*)|_{H^{(0)}}) - \dim(\ker(\mathrm{d} + \mathrm{d}^*)|_{H^{(1)}}).$$

So

$$\mathrm{ind}_a(\mathrm{d} + \mathrm{d}^*) = \sum (-1)^k \beta_k(M), \text{ where } \beta_k(M) =: \dim H^k(M, \mathbb{C}),$$

the Euler number of M.

2.2. **Signature Operator.** M oriented $4k$-dimensional, $\Omega, \mathrm{d} + \mathrm{d}^*$, $\ker(\mathrm{d} + \mathrm{d}^*) = H^*(M) = H^0 \oplus \cdots \oplus H^{4k}$.

We define a quadratic form on Ω:

$$Q(\omega_1, \omega_2) = (-1)^{\frac{l(l-1)}{2}} \int_M \overline{\omega_1} \wedge \omega_2, \forall w_1, w_2 \in \Omega, \deg\omega_1 = l.$$

Note that if $\deg\omega_1 + \deg\omega_2 \neq 4k$, $Q(\omega_1, \omega_2) = 0$.

We define the signature of M as the signature of the quadratic form, note that if $\dim(M) \neq 4k$, the signature is 0.

There exists a grading operator τ on Ω with $\tau^2 = \mathrm{id}, \tau^* = \tau$ satisfying

$$Q(\omega_1, \omega_2) =< \tau\omega_1, \omega_2 > .$$

(τ is the Hodge $*$ operator when $p = 2k$.)

Observe that $\int \mathrm{d}\omega = 0 \Rightarrow \int \mathrm{d}(\overline{\omega_1} \wedge \omega_2) = 0 \Rightarrow \mathrm{d}\tau = -\tau\mathrm{d}^* \Rightarrow \tau(\mathrm{d} + \mathrm{d}^*)\tau = -(\mathrm{d} + \mathrm{d}^*) \Rightarrow \mathrm{d} + \mathrm{d}^*$ is odd in the grading given by τ.

Then by definition

$$\mathrm{ind}(\mathrm{d} + \mathrm{d}^*, \tau) = \dim(\text{eigenspace of } 1) \cap H^* - \dim(\text{eigenspace of (-1)}) \cap H^*)$$

$= \dim((\text{eigenspace of } 1) \cap H^{2k}) - \dim(\text{eigenspace of (-1)}) \cap H^{2k}$=signature of M

Remark 2.2. The de Rham operator and signature operator are the same but acting on spaces with different grading.

2.3. **Dirac Operator.**

Definition 2.3. A Clifford bundle is a graded Hermitian vector bundle E over M together with a smooth vector bundle map $c : T^*M \otimes E \to E$ or $c : T^*M \to \mathscr{L}(E)$, such that $\forall \xi \in T_x^*M$, $c(\xi) = c(\xi)^*$, $c(\xi)^2 = \|\xi\|^2 \operatorname{id}_E$, $c(\xi) \in L(E)^1$.

Remark 2.4. Use the universal property of Clifford algebra, $c : T^*M \to L(E)$ can be extended to an algebra homomorphism $c : \mathrm{Cliff}(T^*M) \to L(E)$, where $\mathrm{Cliff}(T^*M)$ is a bundle over M with each fiber as Clifford algebra generated by T_x^*M.

Definition 2.5. Let E be a vector bundle over M and $C^\infty(E)$ is the set of smooth section of M in E. A connection is a linear map $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ satisfying $\nabla(f\xi) = \mathrm{d}f \otimes \xi + f(\nabla\xi)$, where $f \in C^\infty(M)$, $\xi \in C^\infty(E)$

Remark 2.6. There always exists a connection on E which respects scalar product and grading.

Definition 2.7. Dirac operator $\partial\!\!\!/ : C^\infty(E) \to C^\infty(E)$ is the composition $C^\infty(E) \to C^\infty(T^*M \otimes E) \to C^\infty(E)$ where the first arrow is the connection and the second is the Clifford multiplication of $c(\xi), \xi \in T^*M$.

Remark 2.8. One need to check $/\partial$ has compact resolvent and commute with $\pi(a), \forall a \in C(M)$ up to compact operator ($\pi : C(M) \to \mathcal{L}(L^2(E))$ by multiplication).

Remark 2.9. If $E = \Omega$ and define $c(\xi) = e(\xi) + e(\xi)^*$ where $e(\xi)\omega = \xi \wedge \omega$ then one can check the last two subsections are examples of Dirac Operators.

Question: Let T be a real Euclidean vector bundle over an even dimensional space M, we can form a bundle $\mathrm{Cliff}_{\mathbb{C}} T$ over M with fiber $\mathrm{Cliff}_{\mathbb{C}} T_x \cong M_{2^m}(\mathbb{C})$. Does there exist a graded vector bundle E such that $\mathrm{Cliff}_{\mathbb{C}} T \cong \mathcal{L}(E)$? ($E$ is irreducible representation of $\mathrm{Cliff}_{\mathbb{C}} T$?)

Answer: It is not always true. There is an obstruction (Dixmier-Douady obstruction). Giving such a bundle is what we call a *Spin*^{c} structure on T.

Remark 2.10. Each Riemannian vector bundle E gives rise to a principal $O(n)$ bundle over M. We say E is oriented if we can lift the structure group $O(n)$ to $SO(n)$.

An oriented vector bundle T is spin^c if the structure group $SO(n)$ lifts to to $\mathrm{spin}^c(n) = U(1) \times_{\mathbb{Z}/2} \mathrm{spin}(n)$, where $\mathrm{spin}(n)$ is a double cover of $SO(n)$ (If $n \geq 3$, $\mathrm{spin}(n)$ is the universal cover of $SO(n)$).

3. Topological Index

The Atiyah-Singer Index theorem computes the index of such operators. It can be stated as:

$$\mathrm{ind}_a(P) = \mathrm{ind}_t(\sigma_P)$$

where P is an *elliptic (pseudo)differential operator* and σ_P is (the K-theory class of) its *principal symbol.*

We now give a few explanations on these and define the map $\text{ind}_t : K^0(T^*M) \to \mathbb{Z}$.

Given a elliptic operator $P : C^\infty(M, E) \to C^\infty(M, E)$ on a compact manifold M, its symbol $\sigma(P)$ is a matrix valued map defined on T^*M ($\sigma(P) : T^*M \to \text{End}(V)$). The definition of ellipticity implies $\sigma(P)$ invertible off the zero section.

Using relative K-theory we know the symbol $\sigma(P)$ gives rise to an element in $K^0(T^*M)$. Apply ind_t to this element we get an integer, we will call it topological index of P.

The construction of $\text{ind}_t : K(T^*M) \to \mathbb{Z}$ needs the following ingredients: Thom isomorphism, tubular neighborhood and Bott.

3.1. **Bott element.** Bott element is β is the generator of $K_0\left(C_0\left(\mathbb{R}^{2n}\right)\right)$, the map given by

$$\begin{aligned} K^0(M) &\cong K_0(C(M)) \\ &\to K_0\left(C_0\left(\mathbb{R}^{2n}\right)\right) \otimes K_0\left(C\left(M\right)\right) \cong K_0\left(C_0(M \times \mathbb{R}^{2n})\right) = K^0\left(M \times \mathbb{R}^{2n}\right) : \end{aligned}$$

$K^0(M) \to K^0(M \times \mathbb{R}^{2n}) : p \mapsto \beta \otimes p$ is called Bott map. This map is an isomorphism.

3.2. **Thom isomorpism.**

Theorem 3.1. *If T is a* spin^c *bundle over M then there is an isomorphism $K^0(M) \to K^0(T)$. In particular, any complex vector bundle carries a* spin^c*- structure, so we have Thom isomorphism for complex vector bundles.*

Remark 3.2. The inverse of Thom isomorphism is constructed as follows:

T, spin^c-bundle over M, $\exists T'$ such that $T \oplus T' = M \times \mathbb{R}^{2n}$. There is a natural spin^c structure on T'. Then $T \oplus T'$ is a spin^c bundle over T so

$$K(T) \to K(T \oplus T') \cong K(M \times \mathbb{R}^{2m}) \to K(M),$$

the first arrow is Thom isomorphism and the last one is the inverse Bott map.

3.3. **Tubular neighborhood theorem.** When we embed M in $\mathbb{R}^n$, $TM \to T\mathbb{R}^n \cong \mathbb{R}^{2n}$, the normal bundle N of M also embed in $\mathbb{R}^{2n}(N \oplus TM = M \times \mathbb{R}^n)$.

By the tubular neighborhood theorem, there is an open neighborhood U of M in $\mathbb{R}^n$ such that N is diffeomorphic with $U \subset \mathbb{R}^n$.

3.4. **Construction of** ind_t**.** Embed M in $\mathbb{R}^n$, n is even, $TM \to T\mathbb{R}^n = \mathbb{R}^{2n}$, normal bundle $N \cong U \to \mathbb{R}^n$ (U is a open neighborhood of M in $\mathbb{R}^n$) and $TM \oplus N \cong M \times \mathbb{R}^n$

Since $U \cong N$ is a vector bundle over M, so TU is a vector bundle of TM. In fact,

$$TU \cong \pi^*(N \oplus N) \cong \pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$$

where $\pi : TM \to M$.

By Thom isomorphism, we have

$$K^0(TU) \cong K^0(TM).$$

Since U is open in $\mathbb{R}^n$, so TU is open in $T\mathbb{R}^n \cong \mathbb{R}^{2n}$. Hence we get a map $K(TN) \to \mathbb{R}^{2n}$ induced by inclusion map. Therefore, we have

$$K^0(T^*M) \cong K^0(TM) \cong K^0(TU) \to K^0(\mathbb{R}^{2n}) \cong \mathbb{Z}.$$

The image of $[\sigma(P)] \in K^0(T^*M)$ under these process is defined as the topological index of P.

4. Computation of index of some elliptic operator

(a)Take $M = S^1$. We will construct two elliptic operators whose symbol is respectively the generator of $K^1(S^1) = \mathbb{Z}$ and of $K^0(\mathbb{R} \times S^1) = K^1(S^1) = \mathbb{Z}$.

For the first one, (writing $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and x the real variable) just take

$$-i\frac{\partial}{\partial x} : L^2(S^1) \to L^2(S^1),$$

whose principal symbol is $(x, \xi) \mapsto \xi$. Here $L^2(S^1)$ is generated by $\{z^n\}_{n\in\mathbb{Z}}, z = e^{ix}, x \in S^1$ and denote $e_i = z^n$ and clearly we have $-i\frac{\partial}{\partial x}z^n = nz^n$.

Then making it bounded we obtain H (the Hilbert transform) given by

$$H(e_n) = \begin{cases} e_n & n > 0 \\ -e_n & n < 0 \\ 0 & n = 0 \end{cases} \tag{4.1}$$

whose principal symbol is $(x, \xi) \mapsto \mathrm{sign}(\xi) = \frac{\xi}{|\xi|}$. Put also $P = \frac{H+1}{2}$ whose principal symbol is $\sigma(x, \xi) = \begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$

To define the second element, write $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and use the exact sequence:

$$0 \to C_0(\mathbb{R} \times S^1) \to C(\overline{\mathbb{R}} \times S^1) \to C(S^1) \oplus C(S^1) \to 0$$

The connecting map $K^1(S^1) \oplus K^1(S^1) \to K^0(S^1 \times \mathbb{R}) = K^1(S^1)$ is the map $(a, b) \mapsto a - b$. It follows that the principal symbol of $F = zP + (1 - P)$ is the generator of $K^0(\mathbb{R} \times S^1)$. Therefore, using $K^0(T^*S^1) \cong K^0(S^1 \times \mathbb{R})$:

$$\mathrm{ind}_a : K^0(T^*S^1) \to \mathbb{Z} : \sigma(F) \mapsto \mathrm{ind}(F).$$

Since $F(e_n) = \begin{cases} e_n & n < 0 \\ e_{n+1} & n \geq 0 \end{cases}$, we have $\mathrm{ind}(F) = -1$.

(b) Let Λ be a lattice in $\mathbb{C}$ ($\Lambda = \mathbb{Z}a + \mathbb{Z}b, a, b$ independent over $\mathbb{R}$) then $M = \mathbb{C}/\Lambda \cong \mathbb{T}^2$ compact. E is a graded complex bundle over M, we want to compute the index of the Dirac operator $D : L^2(E^{(0)}) \to L^2(E^{(1)})$, more precisely, the index of the Dolbeault operator $\bar{\partial}_E$ with the coefficient in E. In fact, $\{\bar{\partial}_E\}$, where E is complex vector bundle over X, generates $K_0(M)$, K-homology of M.

An easy example: When E is 1 dim trivial line bundle:

$$\bar{\partial} : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C}) : f \mapsto \frac{\partial}{\partial \bar{z}} f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}),$$

the symbol of which is the clifford multiplication by $i\xi - \eta$. Since

$$f \in \ker \bar{\partial} \Leftrightarrow \frac{\partial}{\partial \bar{z}} f = 0 \Leftrightarrow f \text{ holomorphic on } M$$

and f is bounded on compact M, f can be lifted to a bounded holomorphic function on $\mathbb{C}$ with the same value on the orbit, which implied f to be constant. We have $\dim\ker\bar{\partial} = \dim\ker\bar{\partial}^* = 1$, so

$$\operatorname{ind}(\bar{\partial}) = 0.$$

Remark 4.1. For any complex bundle E over M with dim large enough, E has a complex sub line bundle L ($E = \mathbb{C}^k \oplus L$). It is enough to compute $\operatorname{ind}(\bar{\partial}_L)$. The kernel of $\bar{\partial}_L$ are all holomorphic sections of the line bundle L. (We will assume L a holomorphic line bundle, i.e. the transition function $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ are holomorphic and satisfy the cocycle equality. If ω_i is a nowhere vanishing holomorphic section over U_i, then on $U_i \cap U_j$, we have $\omega_i = g_{ij}\omega_j$.)

Now we define $\bar{\partial}_L$ on the section in L locally by

$$\bar{\partial}_L(fw_i) = (\bar{\partial}f)w_i,$$

it is well defined because g_{ij} is holomorphic.

Now choose a Hermitian metric on L: $\|w_i\| =: \alpha_1$ with $\alpha_i : U_i \to \mathbb{R}^*_+$ smooth. The inner product is locally

$$< fw_i, hw_i >= \int_M \bar{f}h\alpha_i^2 \mathrm{d}x\mathrm{d}y, \forall f \in C_c^\infty(U_i),$$

so $\bar{\partial}_L^*$ locally is

$$< \bar{\partial}_L^* hw_i, fw_i >=< hw_i, \bar{\partial}_L fw_i >= \int (\bar{\partial}f)\bar{h}\alpha_i^2 \mathrm{d}x\mathrm{d}y = -\int f\bar{\partial}(h\alpha_i^2)\mathrm{d}x\mathrm{d}y.$$

This means we have an anti-linear bundle map $\varphi_L : L \to L^* = \operatorname{Hom}(L, \mathbb{C})$ where the holomorphi c section on L^* is given by w_i^*, $\varphi_L(hw_i) = \bar{h}\alpha_i^2 w_i^*$ is a well-defined isometric map. One need to check that $\bar{\partial}_L^* = -\varphi^{-1} \circ \bar{\partial}_{L^*} \circ \varphi_L$, therefore,

$$\operatorname{ind}(\bar{\partial}_L) = \dim(\ker\bar{\partial}_L) - \dim(\ker\bar{\partial}_{L^*}).$$

Definition 4.2. A divisor on M is a function $D : M \to \mathbb{Z}$ with finite discrete support, denoted by $D = \sum_{p\in M} D(p)p$. The degree of D is $\deg(D) = \sum_{p\in M} D(p)$. Divisor of a meromorphic function f on M is $D(f) = \sum_{p\in M} s_{f(p)}p$, where $s_{f(p)} = 1$ if p is a simple zero of f. In general, $s_{f(p)}$ is equal to the multiplicity of the zero p or minus the multiplicity of the pole p.

Given a divisor $D : a_1, \cdots a_k \in M, n_1, \cdots, n_k \in \mathbb{Z}$, we can construct a holomorphic line bundle L as follows:

Let $U_0 = M \setminus a_1, \cdots, a_k$, $U_i =$ (disjointed) disc around $a_i, (i > 0)$ be the open cover of M with the transition function $g_{0i}(z) = (z - a_i)^{-n_i}$. ($L^*$ is constructed through $g_{0i} = (z - a_i)^{n_i}$).

For any global holomorphic section in L, locally we have $fw_0 = f_iw_i, w_0 = g_{0i}w_i$, and $f = (z - a_i)^{n_i} f_i$ near a_i, then a holomorphic section correspond to a unique meromorphic function f on M such that the multiplicity of the pole at a_i is no more than $-n_i$ if n_i and the multiplicity of zero at a_i is no less than n_i if $n_i > 0$, this is equivalent to say that $D(f) \geq D$, so

$$\dim(\ker\bar{\partial}_L) = \dim(\text{holo. sections in L}) = \dim\{f\text{mero. on M}|D(f) \geq D\}.$$

Similarly,

$$\dim(\ker\bar{\partial}_{L^*}) = \dim(\text{holo. sections in } L^*) = \dim\{f \text{ mero. on M}|D(f) \geq -D\}.$$

So we have

$$\begin{aligned}\operatorname{ind}(\bar{\partial}_L) &= \dim\{f \text{ mero.} | D(f) - D \geq 0\} - \dim\{f \text{ mero.} | D(f) + D \geq 0\} \\ &= g - 1 - D = -\deg(D),\end{aligned}$$

where L is the line bundle correspond to divisor D. The second equality is due to Riemann Roch theorem.

Remark 4.3. It is fact that for any holomorphic line bundle L it correspond to a divisor $D = \sum n_i[a_i]$, then $\operatorname{ind} \bar{\partial}_L = -\sum n_i$.

5. Generalization of proof of the index theorem

Recall in the proof of Atiyah-Singer index theorem we use the tangent groupoid $G \rightrightarrows M \times [0,1]$ where $G = (0,1] \times M \times M \cup \{0\} \times T^*M$. G as a continuous field over $[0,1]$, $M \times M$ continuous deform into T^*M. The normal bundle with respect the inclusion $M \to M \times M$ is T^*M

Analogously if there is an embedding $i : M \to V$ we form a normal bundle $N_i(x) = T_{i(x)}/d_i(T_x M), x \in M$, and define a manifold

$$D(i) = N_i \times \{0\} \cup V \times \mathbb{R}^*$$

with the smooth topology defined by $((x_n, \lambda_n) \in V \times \mathbb{R}^*) \to ((x, \xi, 0) \in N_i \times \{0\}) \Leftrightarrow \lambda_n \to 0, x_n \to i(x), p(x_n - i(x))/\lambda_i \to \xi$, where p is the projection to the quotient in the definition of $N_i(x)$. (deformation to the normal cone)

(a)Atiyah-Singer index theorem for families:

Let $p : M \to Y$ be a map with fiber $M_y = p^{-1}(y)$ and $(M_y)_{y \in Y}$ a family of manifold. Define the groupoid $G = M \times_Y M = \{(x, y) : p(x) = p(y), x, y \in M\}$ and the inclusion $M \to M \times_Y M : x \mapsto (x, x)$.

By the normal cone method we can construct $ind : K(T^*M) \to K(Y)$ and get the fiberwised index theorem.

(b)Non-commutative fiberation:

Let M be a compact manifold and $\tilde{M}$ is the universal covering space of M, and $\Gamma = \pi_1(M)$, then $M = \tilde{M}/\Gamma$.

Construct groupoid $G = \tilde{M} \times \tilde{M}/\Gamma \rightrightarrows \tilde{M}/\Gamma$ by $s((\tilde{x}, \tilde{y})/\Gamma) = \tilde{y}/\Gamma$ and $r((\tilde{x}, \tilde{y})/\Gamma) = \tilde{x}/\Gamma$. $(\tilde{x}, \tilde{y})$ and $(\tilde{y'}, \tilde{z'})$ is compossible if $\exists g \in \Gamma$ such that $\tilde{y} = \tilde{y'}g$ and the composition is $(\tilde{x}, \tilde{z}g)$. Clearly it is well defined.

The groupoid G is transitive, i.e. $\forall x, y \in M, \exists r \in G$ such that $s(r) = x, r(r) = y$. Also $G_x^x = \{r \in G | s(r) = r(r) = x\} \cong \Gamma$.

Take the inclusion $i : M \to G$ and the normal bundle to this is the cotangent bundle to M, then $D(i) = T^*M \cup (0,1] \times G$ and we get a index map

$$ind : K^0(T^*M) \cong K^0(D(i)) \to K_0(G) \cong K_0(C_r^*\Gamma).$$

Remark 5.1. Baum-Connes Conjecture: If Γ is torsion free, then

1.Every element in $K_0(C_r^*\Gamma)$ can be constructed using the index of elliptic operator in the above way.

2.If two elliptic operator have the same index, there is a good topological reason. (This will imply Novikov conjecture).