Affine actions

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H euclidean space; ~~scalars~~ $\mathbb{R}, \mathbb{C}$ scalars

$\mathrm{Aff}(H) = H \rtimes \mathcal{O}(H)$ $\quad \mathcal{O}(H)$ - orthogonal gp.

$b \in H \quad T_b(x) = x + b \qquad O \in \mathcal{O}(H) \quad O(x) = Ox$

we multiply: $(b+O)(b'+O') = (b + O(b')) + OO'$

G discrete group affine action: $\alpha : G \to \mathrm{Aff}(H)$

have decomposition $\alpha_g = b(g) + \pi_g \quad \pi_g \in \mathcal{O}(H) \quad b(g) \in H$

for α to be a homomorphism, it is equivalent that

$\pi : G \to \mathcal{O}(H)$ representation.

$b(gh) = b(g) + \pi_g(b(h))$ "cocycle condition"

the action is metrically prope : $\Longleftrightarrow$ • $\{g : g \cdot B \cap B \neq \emptyset\}$ is finite $\forall B \subset H$ bounded

$\Longleftrightarrow$ • $\|b(g)\| \to \infty$ as $g \to \infty$

$\Longleftrightarrow$ • each orbit $\to \infty$ as $g \to \infty$

G is <u>a-T-menable</u> iff it admits a proper action on H

Serre (FH): every action on H has a globally fixed pt
- opposite a a-T-menable

all property (T) gps have prop (FH)

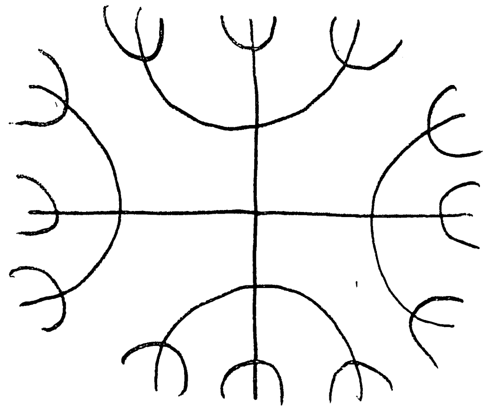
$SL_3(\mathbb{Z})$ is not a-T-menable

Ex $\mathbb{Z}^n$ acts on $\mathbb{R}^2$

(2)

Ex $\mathbb{Z} * \mathbb{Z}_2$ acts on $\mathbb{R}$, $\mathbb{Z}$ translates, $\mathbb{Z}_2$ flips

Ex $G = \mathbb{F}_2$; or generally G acting on a simplicial tree



"fancy free group tree"

Δ_0 vertices

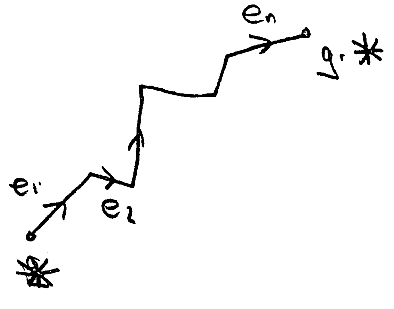
Δ_1 edges - oriented

$$\mathcal{H} = \ell_2^0(\Delta_1) = \{\phi \in \ell_2(\Delta_1) : \phi(\bar{e}) = -\phi(e)\}$$

$\bar{e}$ = reverse of e

$$\pi : G \to \mathcal{O}(\ell_2^0(\Delta_1)) \qquad (\pi_g \phi)(e) = \phi(g^{-1}e)$$

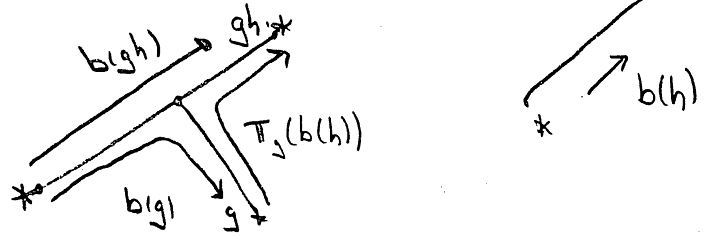
$$b : G \to \ell_2^0(\Delta_1)$$



$$b(g) = \left(\sum_{i=1}^{n} e_i\right) - \left(\sum_{i=1}^{n} \bar{e}_i\right)$$

have to check cocycle rule:

$$b(gh) = \pi_g(b(h)) + b(g)$$



Proper ?

$\| b(g) \|^2 = 2\, d(*, g*)$

Wall spaces : S set $\quad H = \{A, B\}$ wall $\quad A \cup B = S \quad A \cap B = \emptyset \quad A \neq \emptyset \neq$

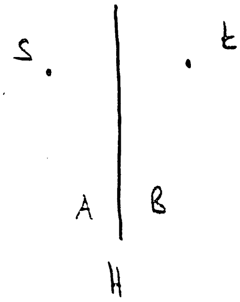
(S, W) wall space – W is a collection of walls

s.t. $\forall s, t \in S : \# \{ H \in W \mid \text{seperates } s,t \} < \infty$

$H = \{A, B\}$ seperates

$s \in A \;\; t \in B$ or

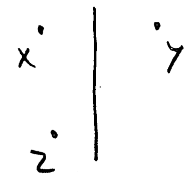
$t \in A \;\; s \in B$



A group acts on (S, W) means it acts on S, preserves W

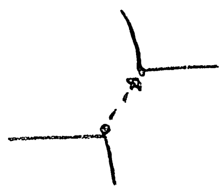
In particular, G permutes walls and half space

$d(x,y) = \# \{ H : \text{seperate } x,y \}$ defines a metric on S

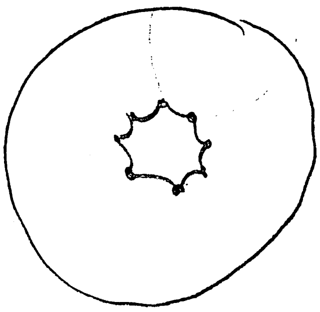


$\rightsquigarrow$ Δ-inequ.

Ex. A tree is a wall space



Ex $G = \pi_1($  $)$ acts properly on a wall space



extend segments

to lines – take orbit under G

this is a wall space

and G acts properly

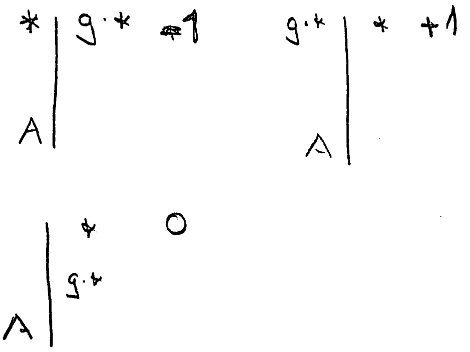
Prop. $G \curvearrowright (S, W)$ wall space, properly

Then G is a-T-menable

pf. $H = \ell_2^\circ(\text{half spaces}) = \{ \phi \in \ell_2(\text{half spaces}) : \phi(\bar{A}) = -\phi(A) \}$

G acts "regularly" w/ a cocycle

$$b(g)(A) = \begin{cases} +1 \\ -1 \\ 0 \end{cases}$$



properness? $\| b(g) \|^2 = d(*, g*)$

Thm. (Robertson–Steger, Cherix–Martin–Valette)

G is a-T-menable $\Leftrightarrow$ G acts properly on a measured wall space.

Ex. $G = BS(2,3)$ a-T-menable, but does not act on a wall space

Thm. (Characterizing a-T-menability)

TFAE

- G is a-T-menable
- G admits a prope[r] function of conditionally negative definite type

  $\psi : G \to [0, \infty)$. $\psi(e) = 0$ $\quad \psi(g^{-1}) = \psi(g)$

  $$\sum a_n = 0 \quad \sum a_i a_j \psi(g_i^{-1} g_j) \leq 0$$
- $C_0(G)$ admits an approximate unit of pos. def functions (Haagerup property

$f_n : G \to \mathbb{C}$ $f_n \in C_0(G)$ $f_n(e)=1$ $f_n(g) \to 1$ ptwise

$\sum a_i \bar{a}_j f_n(g_i^{-1} g_j) \geq 0$

<u>Sketch</u> negative type $\Leftrightarrow$ square of Hilbert norm

① $\Rightarrow$ ② $\|b(g)\|^2$ neg. type ② $\Rightarrow$ ① GNS-type construction

Schoenberg Thm: $e^{-\lambda\psi}$ is pos. def $\forall \lambda > 0$ $\Leftrightarrow$ ψ neg. type

② $\Rightarrow$ ③ ~~Sho~~ Schoenberg ③ $\Rightarrow$ ② odd ~~up~~ $1-f_n$ properly.

<u>Ex.</u> amenable gps are a-T-menable

direct construction of action (--, Valette)

G U_n Følner sets $\frac{|gU_n \Delta U_n|}{|U_n|} \to 0$ $n \to \infty$ $\forall g \in G$

χ_n = normalized characteristic function of $U_n \in \ell^2(G)$

$b(g) = \lambda_g(\chi) - \chi$ is a cocycle for λ on $\ell^2(G)$

$H = \ell^2 G \oplus \ell^2 G \oplus \dots$ $b(g) = \lambda_g(\chi_1) - \chi \oplus \lambda_g(\chi_2) - \chi_2 \oplus \dots$

$\pi = \lambda \oplus \lambda \oplus \dots$ take a subsequence to guarantee $b(g) \in H$ $\forall g$

properness $\{g \in G \mid \|b(g)\|^2 < 4k\} \subset \{g \in G \mid \exists i_1, \dots, i_k \text{ s.t. } \|\lambda_g(\chi_i) - \chi_i\| < 2\}$

$\subset \bigcup_{i=1}^{k} U_i U_i^{-1}$ finite

<u>K-theory</u> - Spectral Picture "warm-up for E-theory"

graded C^*-alg α grading, the eigenspaces split A into $A_0 \oplus A_1$

<u>Ex</u> · $S = C_0(\mathbb{R})$ graded by $\alpha(f)(x) = f(-x)$

· $K = \mathbb{K}(H_0 \oplus H_1)$ $\alpha(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\varepsilon := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ H_i countably inf. dimensial

$K_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $K_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ "inner grading"

<u>Ex</u> · trivial grading

· $A \oplus A$ std. odd grading $\alpha(f,g) = (g,f)$

$K(A) = [S, A \otimes K]$ htpy classes of graded $*$-homs

A ungraded, $A \otimes K$ graded as K

<u>Ex</u> D operator cpt resolvent selfadjoint odd on $H_0 \oplus H_1$

$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ we then have functional calculus

$\phi_D(f) = f(D)$ $\qquad \phi_D : \cancel{K}\ S \to \mathbb{K}$

$[\phi_D] \in K(\mathbb{C})$

Properties:
- abelian gp
- homotopy functor
- stable
- half exact (Rosenberg 80's)

proof w/ Puppe sequences

$0 \to I \to A \xrightarrow{p} A/I \to 0$ have $I \to C_p$

"$C_p \simeq I$ asymptotically"

this induces iso on K-theory

Abelian group? orthogonal sum $\phi, \psi : S \to A \otimes \mathbb{K}$

$\rightsquigarrow \phi \oplus \psi : S \to A \otimes (\mathbb{K} \oplus \mathbb{K})$ use $\mathbb{K} \oplus \mathbb{K} \subset M_2(\mathbb{K}) \cong \mathbb{K}$ by graded unitary

inverse? $\phi_0 : S \to \mathbb{K}$ $\quad -[\phi_D] = [\phi^{op}]$ $\quad \phi^{op} : S \to \mathbb{K}$ $\quad \phi^{op}(f) = f(-D)$

also 'reverse' grading on $H = H_0 \oplus H_1$

homotopy: $\Phi_s(f) = f\begin{pmatrix} D & s \\ s & -D \end{pmatrix}$, check $\begin{pmatrix} D & s \\ s & -D \end{pmatrix}$ is odd on $H \oplus H^{op}$

so Φ_s is formed

$$\begin{pmatrix} D & s \\ s & -D \end{pmatrix}^2 = \begin{pmatrix} D^2+s^2 & 0 \\ 0 & D^2+s^2 \end{pmatrix} \geq s^2 \cdot 1$$ so, $\Phi_s(f) = f\begin{pmatrix} D & s \\ s & -D \end{pmatrix} \to 0 \quad s \to \infty$

so the homotopy runs from $s=0$ to $s=+\infty$

Thm: $K(A) \cong K_{alg}(A)$, via $\qquad$ (A unital)

$$[p]-[q] \mapsto \left(f \mapsto \begin{pmatrix} f(0)p & 0 \\ 0 & f(0)q \end{pmatrix}\right)$$

$K(\mathbb{C}) = K(pt) \cong \mathbb{Z}$:

One can check D as above, but on M closed, then $[\phi_D] = \mathrm{Index}(D) \in K(pt)$

spec(D)

$$\phi_t(f) = \begin{cases} f(t^{-1}D) & 0 < t \leq 0 \\ \begin{pmatrix} f(0)p_+ & 0 \\ 0 & f(0)p_- \end{pmatrix} & t = 0 \end{cases}$$ $\quad p_\pm =$ proj onto ker $D_\pm$

Recap ① a-T-menable gps - examples, wall spaces 25.06.2009
- characterizations

② spectral picture - fctl calculus morphism.

~~Bott class~~:

Clifford algebras

E euclidean space $\rightsquigarrow Cl(E)$ Clifford algebra
real. fin. dim.

① complex algebra; graded; unital

② gen. by $v \in E$ $v^2 = \|v\|^2 \cdot Id$

② Hilbertspace

Linear span of $e_{i_1} \cdot \ldots \cdot e_{i_p}$ $i_1 < \ldots < i_p$
gives ONB.

③ C^*-alg

subalg of $\mathbb{B}$(Hilbertspace)

can define

$e(v) = e \cdot v$ self-adj

$\hat{e}(v) = \alpha(v) e$

④ functoriality : $T \in O(E)$

$\rightsquigarrow \bar{T} : Cl(E) \to Cl(E)$

Notation

$\mathcal{C}(E) = C_\mathbb{Z}(E) = C_0(E, Cl(E))$

$\mathcal{H}(E) = L^2(E, Cl(E))$

$\mathcal{S}(E) =$ schwartz class fcns $: E \to Cl(E)$

each graded by grading of $Cl(E)$

Bott class β $\quad C: E \to C\ell(E) \quad C(v) = v$

Rem. we also consider mult. operator on $\mathcal{H}(E)$

$$M_c(f)(v) = C(v)\, f(v) = v \cdot f(v)$$

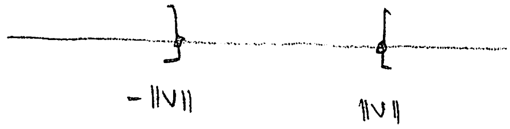
Usually, write C for M_c

We have a $*$-homo $\beta : S \to \mathcal{C}(E)$ $\quad \beta(f) = f(C)$

in other words $\beta(f)(v) = f(C(v)) = f(v)$ (functional calculus

check $\beta(f) \in C_0(E, C\ell(E))$

rem. $v^2 = \|v\|^2$ so spec(v)



$\rightsquigarrow \beta(f) \in C_0(\dots)$

~~rem~~

rem. a related morphism $S \to B(\mathcal{H}(E))$ $\quad f \mapsto f(C) = M_{f(v)}$

Suppose G acts affinely on E: $\alpha_g = \pi_g + b(g)$

Then we get action on $\mathcal{C}(E)$

$$g.\phi(v) = \overline{\pi}_g(\phi(\alpha_g^{-1}(v)))$$

The map β is not G-equivariant. (It is for $G \subset O(E)$)

We would replace β by $\beta_t(f) = f(t^{-1}C)$

a family of $*$-homs $S \to \mathcal{C}(E)$

Lemma. $\alpha \in \mathrm{Aff}(E)$ $\quad \alpha = \pi + b$ $\quad \pi \in O$ $\quad b \in E$

Then $C - \alpha(C) = b$

Lemma $G \in \mathrm{Aff}(E)$. The the family β_t is ~~cep~~ asympt. G-equivariant in the sense that

$$\beta_t(f) - g\cdot\beta_t(f) \to 0 \quad \text{as } t \to \infty$$
$$\forall f \in S \ \forall g \in G$$

pf. reduce to $f = r_\pm$ $\quad r_\pm(x) = (x \pm i)^{-1}$

calculate $\quad (t^{-1}C \pm i)^{-1} - g\,(t^{-1}C \pm i)$

$$= (t^{-1}C \pm i)^{-1}\left(t^{-1}(g(C) - C)\right)(t^{-1}g(C) \pm i)^{-1}$$

$$= t^{-1}\,(t^{-1}C \pm i)^{-1}\, b(g)\,(t^{-1}g(C) \pm i)$$

and $\quad \|.\| \le t^{-1}\,\|\underbrace{b(g)}\| \longrightarrow 0$

constant fnct for $E \to Cl(E)$

Def./Thm Bott class. the (family of) $*$-homs.

$$\beta_t : S \to \mathcal{C}(E)$$

Asymptotic morphism + E-theory

A, B C^*-alg (graded + G-action, –)

$\phi_t : A \to B$ functions is an asymptotic family when

① continuity $\forall a \in A \quad t \mapsto \phi_t(a) : [1,\infty) \to B$ is cont. bd

② algebraic $\forall a, b \in A\ \lambda \in \mathbb{C} \quad \phi_t(ab) - \phi_t(a)\,\phi_t(b) \xrightarrow{t\to\infty} 0$ etc.

③ graded : $\forall a \in A \quad \phi_t(\alpha(a)) - \alpha(\phi_t(a)) \to 0 \quad t \to \infty$

④ G-equivariance: $\forall a \in A \;\; \forall g \in G$

$$\phi_t(g \cdot a) - g \cdot \phi_t(a) \to 0 \quad t \to \infty$$

rem. a cont. of $*$ is an asymp. morph.

β_t Bott morphism is an asymp

Denote: $\mathfrak{T}(B) = C_b([1,\infty), B)$ graded G-C^*-alg

$\mathfrak{T}_0(B) = C_0([1,\infty), B)$ (only if G is discrete)

$\mathfrak{A} = \mathfrak{T}(B) / \mathfrak{T}_0(B)$

rem. $\phi_t : A \to B \Rightarrow \phi : A \to \mathfrak{A}(B)$

$\phi_t \; \psi_t$ are equivalent $\Leftrightarrow$ $\phi_t(a) - \psi_t(a) \to 0$ as $t \to \infty$ $\forall a \in A$

rem. equivalence $\equiv$ $*$-homs $A \to \mathfrak{A}(B)$

Lemma $\mathfrak{A}$ is a functor, defined via

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{T}_0(A) \arrow[r] \arrow[d] & \mathfrak{T}(A) \arrow[r] \arrow[d] & \mathfrak{A}(A) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathfrak{T}_0(B) \arrow[r] & \mathfrak{T}(B) \arrow[r] & \mathfrak{A}(A) \arrow[r] & {}
\end{tikzcd}$$

naive compositions: can compose $*$hom + asymp. morph

$A' \xrightarrow{\psi} A \xrightarrow{\phi_t} \mathfrak{A}(B) \qquad \phi_t \circ \psi$

$A \xrightarrow{\phi_t} \mathfrak{A}(B) \xrightarrow{\mathfrak{A}(\psi)} \mathfrak{A}(B') \qquad \psi \circ \phi_t$

Guentner (5)
25.06.2003

Asymptotic category : $\mathcal{A}$

objects : G - C^*-alg

morphisms : " htpy classes of asymptotic morph "

(Memoirs of AMS – general statement)

A homotopy $A \xrightarrow{\phi_t} B[0,1] \overset{ev_0}{\underset{ev_1}{\rightrightarrows}} B \quad \rightsquigarrow A \to \mathcal{A}(B[0,1])$

Thm There is a functor $C^*\text{-alg}_h \to \mathcal{A}$
which is the identity on objects + maps $*$ homo $\phi: A \to B$
to "constant" asymp. morphism $A \xrightarrow{\phi} B \to \mathcal{A}(B)$

Dirac class α :

E euclidean space ; $\mathcal{Cl}(E)$ $\mathcal{C}(E)$, $\mathcal{H}(E)$, $\mathcal{S}(E)$

$D : \mathcal{H}(E) \to \mathcal{H}(E)$

essentially selfadjoint operator on $\mathcal{S}(E)$
constant coefficient differential op.

$$(Df)(v) = \sum_i^n \hat{e}_i \frac{\partial f}{\partial x_i}(v) \qquad x_i = \text{dual coord. to } e_i$$
$$v = x_1(v)e_1 + \dots + x_n(v)e_n$$

D does not have cpct. resolvent. So cannot just use fctl. calc. morphism.

Thm / Def. The Dirac class α is the class of the asymptotic morphism

$$\alpha_t : S \hat{\otimes} \mathcal{C}\ell(E) \to \cancel{\mathbb{K}(E)}\ K(\mathcal{H}(E))$$

which is asymptotic on basic tensors to

$$\alpha_t(f \otimes h) = f(t^{-1}D)\, M_{h_t} = f(t^{-1}D) h_t$$

$$h_t(v) = h(t^{-1}v)$$

Lemma $\phi_t : A \to C$ $\psi_t : B \to C$ whose images graded commute $[\phi_t(a), \psi_t(b)] \to 0$ $t \to \infty$ $\forall a \in A$ $b \in B$

Then can get an asymptotic morphism

$$\eta_t : A \hat{\otimes} B \to C$$

s.t. $\underline{\eta_t(a \otimes b) - \phi_t(a)\psi_t(b)} \to 0$ $t \to \infty$ $\forall a \in A$ $b \in B$ $*$

pf. $\phi : A \to \mathfrak{A}(C)$ $B \xrightarrow{\psi} \mathfrak{A}(C)$ graded commute

~~and hence η~~

So get $\phi \otimes \psi : A \hat{\otimes} B \to \mathfrak{A}(C)$ and hence η which on basic tensors is asymptotically equal to $(*)$

<u>Lemma</u> $[f(t^{-1}D), M_{h_t}] \to 0 \quad t \to \infty$

$\forall f \in S \ \forall h \in C_0(E)$

pf. Reduce to $f = r_\pm$ + $h \in C_c^\infty$

Then calculate $[(t^{-1}D \pm i)^{-1}, h_t]$

$$= \underbrace{(t^{-1}D \pm i)^{-1}}_{\|.\| \leq 1} \underbrace{[h_t, t^{-1}D]}_{t^{-1}[h_t, D]} \underbrace{(t^{-1}D \pm i)^{-1}}_{\|.\| \leq 1}$$

$\cong t^{-1}$ (clifford mult. by gradient of h_t)

$\to 0 \quad t \to \infty$

Bott-Dirac class

$B = C + D \qquad B : \mathcal{H}(E) \to \mathcal{H}(E)$

compact resolvent, essentially self-adj.

By first lecture have the fctl. calculus morphism: γ

$\gamma_t(f) = f(t^{-1}B)$, $\gamma_t : S \to \mathbb{K}(\mathcal{H}(E))$

cnt. family of $*$-homos

<u>Fundamental Theorem</u> the composition

$$\underbrace{S \to S \hat{\otimes} S \xrightarrow{i \otimes \beta} S \hat{\otimes} \mathcal{C}(E) \xrightarrow{\alpha_t} \mathbb{K}(\mathcal{H}(E))}_{\gamma}$$

is asymptotically commuting

rem. $S \xrightarrow{\Delta} S \otimes S$ is a $*$-homo which maps

$$\Delta(U) = U \otimes U \qquad U(x) = e^{-x^2}$$

$$\Delta(V) = U \otimes V + V \otimes U \qquad V(x) = x e^{-x^2}$$

Explain: $E_G(A,B) = \mathcal{Q}(SAK_G, BK_G)$

How to compose? $E_G(B,C) = \mathcal{Q}(SBK_G, CK_G)$

$$SAK_G - BK_G \overset{\text{tensor}}{\sim} \text{w/ } S$$

$$SAK_G - S^2AK_G - SBK_G - CK_G$$

composition in $\mathcal{Q}$

In this language, the fundamental thm calculates

$$\alpha \in E(\mathcal{C}(E), \mathbb{C})$$

$$\beta \in E(\mathbb{C}, \mathcal{C}(E))$$

$$\alpha \circ \beta = \gamma \in E(\mathbb{C}, \mathbb{C}) \cong K(\mathbb{C}) \cong \mathbb{Z}$$

and $\gamma = 1$ spectral picture

Eric Guentner (1)

28.06.09

Recap $\quad E$ finite dim $\qquad \mathcal{B} = \mathcal{C} + \mathcal{D}$

Bott $\quad \beta : S \to \mathcal{C}(E)$ $\quad G$-equivariant for $G \subset O(E)$

$\beta_t : S \to \mathcal{C}(E)$ asymp. G-equiv. $\quad G \subset \mathrm{Aff}(E)$

$\beta_t(f) = \beta(f_t)$

fctl. calc. for C

Dirac $\quad \alpha : S\mathcal{C}(E) \to \mathbb{K}(H(E))$ asymptotic morphism

asymp. G-equiv. $\quad G \subset O(E)$

Bott-Dirac $\quad \gamma : S \to \mathbb{K}(H(E))$ fctl. calc. for $\mathcal{B}$

Fundamental Thm. The composite

$$\underbrace{S \to S \otimes S \xrightarrow{1 \otimes \beta} S\mathcal{C}(E) \xrightarrow{\alpha_t} \mathbb{K}(H(E))}_{\gamma}$$

commutes asymptotically; also for $G \subset O(E)$

Define $\quad E_G(A,B) = \mathcal{Q}_G(SA\mathbb{K}_G, B\mathbb{K}_G) \qquad \mathbb{K}_G = \mathbb{K}(H_G) \quad H_G = \ell^2 G \otimes (\hat{H}_c)$

rep. : $\lambda \otimes 1$

rem. there is a functor $\mathcal{Q}_G \to E_G$; identity on objects

$\phi \in \mathcal{Q}_G(A,B) \qquad \phi \otimes 1 \in \mathcal{Q}_G(A\mathbb{K}_G, B\mathbb{K}_G)$

$$\rightsquigarrow \left(SA\mathbb{K}_G \xrightarrow{\eta \otimes 1 \otimes 1} A\mathbb{K}_G \xrightarrow{f \otimes 1} B\mathbb{K}_G \right) \in E_G(A,B)$$

$\eta : S \to \mathbb{C} \quad \eta(f) = f(0)$

composition $\qquad \phi \in E_G(A,B) \quad \psi \in E_G(B,C)$

$$SA\mathbb{K}_G \xrightarrow{\Delta \otimes 1} S^2 A\mathbb{K}_G \xrightarrow{1 \otimes \phi} SB\mathbb{K}_G \xrightarrow{\psi} C\mathbb{K}_G$$

$\Delta : S \to S \otimes S$

$\boxed{S \xrightarrow{\Delta} S \otimes S \quad S \xrightarrow{\eta} \mathbb{C} \quad \text{is a monad in } \mathfrak{A}_G^{opp} \quad \text{"Kleisli construction"}}$

So write $\quad \alpha \in E_G(\mathcal{C}_e(E), \mathbb{C}) \quad \beta \in E_G(\mathbb{C}, \mathcal{C}_e(E))$

The Fundamental shows $\quad \alpha\beta = \gamma \in E_G(\mathbb{C}, \mathbb{C})$, $G \subset O(E)$

rem. $E(\mathbb{C}, A) = [\![SK, AK]\!] \cong [S, AK] \cong [S, AK] \cong K(A)$

special feature for $G = 1$ — special feature of S on the ~~lemma~~ Left — spectral picture

so this gives $\gamma = 1 \in E(\mathbb{C}, \mathbb{C})$; also OK w/ $G \subset O(E)$

To prove B.C., we actually need these computations in $E_G(\mathbb{C}, \mathbb{C})$, $G \subset Aff(E)$, properly. To get this, use slick argument of Higson-Kasparov to "switch-off" the translation part of the action

$$\alpha_g^s = \pi_g + s\, b(g) \qquad s \in [0,1] \qquad \begin{array}{l} s = 0 \text{ orthogonal action} \\ s = 1 \text{ give action} \end{array}$$

Then can reinterpret: $\beta : S[0,1] \to \mathcal{C}_e(E)[0,1]$ $\quad \alpha^s$ action over $s \in [0,1]$

and also $\quad \overline{\alpha}_t^! \quad \mathcal{C}_e(E)[0,1] \to \mathbb{K}(\mathcal{H}(E))[0,1] \quad +$ absorb

Here we have to adjust α_t map to get equivariance

Then consider diagram in $\mathfrak{A}_G$

$$\begin{array}{ccccc} \mathbb{C}[0,1] & \xrightarrow{\overline{\beta}_t} & \mathcal{C}_e(E)[0,1] & \xrightarrow{\overline{\alpha}_t^!} & \mathbb{C}[0,1] \\ \uparrow & & & & \downarrow{\scriptstyle ev_s} \\ \mathbb{C} & \longrightarrow & \underbrace{\mathcal{C}_e(E)_s}_{\alpha^s \text{ action}} & \longrightarrow & \mathbb{C} \end{array}$$

Eric Guentner
26.06.09

Rem. Functors in E-theory

$F : C^*\text{-alg} \to C^*\text{-alg}$

when can you promote to $F : E \to E$?

(it is enough to obtain ~~F~~ $F : \mathfrak{A} \to \mathfrak{A}$)

what are morphisms?

$\cancel{A \to}$ $A \to \mathfrak{A}(B)$ morphism $A \to \mathfrak{A}(B[0,1])$ htpy

could apply F: $F(A) \to F(\mathfrak{A}(B))$

but we need $F(A) \to \mathfrak{A}(F(B))$ (mor in $\mathfrak{A}(F(A), F(B))$)

can get nat. transf. $F\mathfrak{A} \to \mathfrak{A}F$ when you have

$$0 \to \mathfrak{T}_0(B) \to \mathfrak{T}(B) \to \mathfrak{A}(B) \to 0$$

and

$$\begin{tikzcd}
0 \arrow[r] & F(\mathfrak{T}_0(B)) \arrow[r] \arrow[d] & F(\mathfrak{T}(B)) \arrow[r] \arrow[d] & F(\mathfrak{A}(B)) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathfrak{T}_0 F(B) \arrow[r] & \mathfrak{T}F(B) \arrow[r] & \mathfrak{A}(F(B)) \arrow[r] & 0
\end{tikzcd}$$

not nec. exact

do not nec. exist

could complete when nothing goes wrong

Infinite dimensional case

No ~~G~~ action — saves time

E euclidean space

construct: $\mathfrak{A}(E)$ C^*-alg (proper free $G \subset \mathrm{Aff}(E)$ properly)

$\beta \in E_G(\mathbb{C}, \mathfrak{A}(E))$ $\alpha \in E_G(\mathfrak{A}(E), \mathbb{C})$

with $x = \alpha\beta = 1 \in E_G(\mathbb{C}, \mathbb{C})$

rem. Clifford alg. $\mathcal{C}_e(E \oplus E') = \mathcal{C}_e(E) \hat{\otimes} \mathcal{C}_e(E')$

$Cl(E \oplus E') = Cl(E) \hat{\otimes} Cl(E')$ $\dim E, E' < \infty$

Bott class (modified)

$\beta_E : \cancel{\text{modified}} : S \to S\,\mathcal{C}_e(E)$ * homo

$\beta_E(f) = f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_E)$

This is the composite $S \xrightarrow{\Delta} S^2 \xrightarrow{1 \otimes \beta^{old}} S\,\mathcal{C}_e(E)$

Now construct $F_a \subset E$ fin dim'l

$E_a \subset E_b$ then $E_{ba} = E_b \ominus E_a$

$\beta_{ba} : S \hat{\otimes} \mathcal{C}_e(E_a) \longrightarrow S\,\mathcal{C}_e(E_{ba}) \hat{\otimes} \mathcal{C}_e(E_a) = S\,\mathcal{C}_e(E_b)$

Lemma $E_a \subset E_b \subset E_c \Rightarrow \beta_{cb}\beta_{ba}$

We form the limit

$$\mathcal{A}(E) = \varinjlim_{E_a \subset E} S\,\mathcal{C}_e(E_a)$$

Bott class β " inclusion of $\{0\} \subset E$ "

$\beta : S \longrightarrow \mathcal{A}(E)$ $\Rightarrow$ say

$\beta_a : S \to S\,\mathcal{C}_e(E_a)$ any $E_a \subset E$

rem. β_t for asymptotic equivariance when $G \subset \mathrm{Aff}(E)$

Eric Guentner (5)
26.06.09

Dirac class ?

Idea for α:

1. $\mathcal{H}(E)$? $\quad \mathbb{K}(\mathcal{H}(E))$; $\mathcal{S}(E)$
2. $E_a \subset E$ define $\alpha_t^a : S\mathcal{C}(E_a) \to \mathbb{K}(\mathcal{H}(E))$
3. check compatibility w/ directed system.
   which means commutativity in

$$E_a \subset E_b \qquad \begin{array}{ccc} S\mathcal{C}(E_a) & \xrightarrow{\alpha_t^a} & \mathbb{K}(\mathcal{H}(E)) \\ \downarrow & & \| \\ S\mathcal{C}(E_b) & \xrightarrow{\alpha_t^b} & \mathbb{K}(\mathcal{H}(E)) \end{array}$$

This is where the Fundamental Thm. is used

we get then $\alpha_t\ \mathcal{C}(E) \to \mathbb{K}(\mathcal{H}(E))$

Fix decomposition: $E = E_0 \oplus E_1 \oplus \dots$

$$F_n = E_0 \oplus \dots \oplus E_n$$

Recall we have $B_n = C_n + D_n$ on each E_n

want $\forall g \in G\ \exists n_0$ s.t. $\forall n \geq n_0\ gE_n \subseteq E_{n+1}$

and ~~every~~ want

$$\mathcal{H}(E_n)\ \mathcal{S}(E_n)$$

and every

$$\Psi_n^a(x) = \pi^{-\frac{1}{4}\dim E_n} e^{-\frac{\|x\|^2}{2}} \text{ spans } \ker B_n$$

This is multiplication $\psi^\circ_E \oplus \psi^\circ_F = \psi^\circ_{E\oplus F}$ dim $E, F < \infty$

Define $\mathcal{H}(E) = \varinjlim \mathcal{H}(F_n)$

$$\begin{aligned} \mathcal{H}(F_n) &\longrightarrow \mathcal{H}(F_{n+1}) = \mathcal{H}(F_n) \hat{\otimes} \mathcal{H}(E_{n+1}) \\ f &\xmapsto[\text{isometry}]{} f \otimes \psi^\circ_{n+1} \end{aligned}$$

$$\mathcal{H}(E) = \hat{\bigotimes}_{n=1}^{\infty} \mathcal{H}(E_n)$$

$$= \overline{\mathrm{sp}\{\phi_0 \ldots \phi_n \psi^\circ_{n+1} \ldots\}}$$

We proceed similarly w/ $\mathcal{S}(E)$.

~~ex~~ rem. $\mathcal{H}(E)$ admits an ONB of elmts. in $\mathcal{S}(E)$

$\{\Phi_0 \ldots \phi_n \psi^\circ_{n+1} \ldots\}$ ϕ_k eigenfunction of B_k

Let $B_{nt} = \underbrace{t_0 D_0 + t_1 D_1 + \ldots + t_{n-1} D_{n-1}}_{D_{t,n-1}} + \widehat{t_n B_n + \ldots}^{D_{tn}}$

$$t_k = 1 + t^{-1} k$$

rem. ① $D_k(\phi_0 \ldots) = (\phi_0 \ldots D_k \phi_k \ldots)$

② $B_k(\phi_0 \ldots) = (\phi_0 \ldots B_k \phi_k \ldots)$

③ $B_k(\phi_0 \ldots \phi_n \psi^\circ_{n+1} \ldots) = 0$ if $k \geq n+1$

So, the inf. sum is finite on $\mathcal{S}(E)$

Lemma B_{nt} is essentially self. adj. + ~~has compact resolvent~~

Eric Guentner (7)
26.06.07

Def $\alpha_t^n : S\mathcal{C}_e(F_n) \to \mathbb{K}(\mathcal{H}(E))$

$$\alpha_t^n(f) = f(t^{-1}B_{nt})h_t$$

Remains to check the asymptotic commutativity

$$\begin{tikzcd} S\mathcal{C}_e(F_n) \arrow[r, "\alpha_t^n"] \arrow[d, "\beta_{E_{n+1}}"'] & \mathbb{K}(\mathcal{H}(F)) \arrow[d, equal] \\ S\mathcal{C}_e(F_{n+1}) \arrow[r, "\alpha_t^{n+1}"] & \mathbb{K}(\mathcal{H}(F)) \end{tikzcd}$$

schematic:

D on $E_0 \oplus \ldots \oplus E_n$
B on $E_{n+1} \oplus \ldots$

$$\begin{tikzcd} E_0 \oplus \ldots \oplus E_n \arrow[r] \arrow[d, "C \text{ on } E_{n+1}"'] & \mathbb{K} \arrow[d, equal] \\ E_0 \oplus \ldots \oplus E_n \oplus E_{n+1} \arrow[r] & \mathbb{K} \end{tikzcd}$$

D on $E_0 \oplus \ldots \oplus E_{n+1}$

B on $E_{n+2} \oplus \ldots$