16/06/09

2. Juli

K-theory of the C^*-algebras of free groups

F_n: free group on n generators $a_1, -, a_n$.

- Universal property: for any G, the data of a group morphism $F_n \longrightarrow G$ is equivalent to the dat of elements $g_1, -, g_n$ in G.
- For any C^*-algebra A, a C^*-morphism $C^*F_n \longrightarrow A$ is equivalent to the datum of a family of unitaries $u_1, -, u_n$ in $\mathcal{U}(A)$. (No such thing for $C^*_r F_n$)
- Left regular representation: $\lambda : C^*F_n \longrightarrow C^*_r F_n \subset \mathcal{B}(\ell^2(F_n))$

Special case: $n=1$ $C^*\mathbb{Z} \xrightarrow[\lambda]{\sim} C^*_r\mathbb{Z} \simeq C(S^1)$

For $n \geq 2$, $\underset{(U_1, -, U_n)}{\overset{\cup}{C^*F_n}} \overset{\lambda}{\twoheadrightarrow} \underset{(\lambda(a_1), -, \lambda(a_n))}{\overset{\cup}{C^*_r F_n}}$ is not an isomorphism since F_n is not amenable.

Th 1 [Cuntz] $K_0(C^*F_n) \simeq \mathbb{Z}$, generated by the class of 1
$K_1(C^*F_n) \simeq \mathbb{Z}^n$, generated by classes of $u_1, -, u_n$

Th 2 [Pimsner-Voiculescu] $K_0(C^*_r F_n) \simeq \mathbb{Z}$, generated by 1
$K_1(C^*_r F_n) \simeq \mathbb{Z}^n$, generated by $\lambda(a_1), -, \lambda(a_n)$

Th 3 [Cuntz] F_n is K-amenable, which implies that

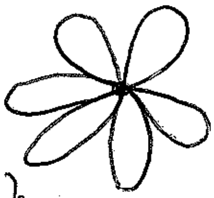
$$K_*(C^*F_n) \xrightarrow{\lambda_*} K_*(C^*_r F_n)$$

is an isomorphism.

Remark:

Th 1 + Th 2 $\Rightarrow$ $K_*(C^*F_n) \underset{\sim}{\xrightarrow{\lambda_*}} K_*(C^*_r F_n)$

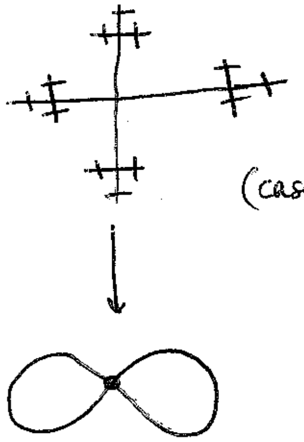
Topology : F_n can be seen as the fundamental group $\pi_1(X_n)$, X_n = n circles with one common point.



Continuous functions on X_n :

$$C(X_n) = \{ (f_1, -, f_n) \mid \forall i \in [\![1,n]\!], f_i \in C(S^1), f_1(1) = \dots = f_n(1) \}$$

Universal cover of X_n $\longrightarrow$ $\tilde{X}_n$ is a tree



Exercise : what is the K-theory of X_n ? (of $C(X_n)$)

$$0 \longrightarrow C_0(\mathbb{R}) \oplus - \oplus C_0(\mathbb{R}) \longrightarrow C(X_n) \longrightarrow \mathbb{C} \longrightarrow 0$$

$\rightsquigarrow$ six-term exact sequence $\Rightarrow$ $K_0 = \mathbb{Z}$
$K_1 = \mathbb{Z}^n$

Construct $\pi : C^*F_n \xrightarrow{(\pi_1, -, \pi_n)} C(X_n)$

For $j \in [\![1,n]\!]$, $\pi_j : F_n \longrightarrow \mathbb{Z}$, $\pi_j(a_i) = 0$ if $i \neq j$
$\pi_j(a_j) = 1$

$$\begin{array}{ll} C^*F_n & \\ \cup & \\ a & \longmapsto (\pi_1(a), -, \pi_n(a)) \in C(X_n) \\ & \quad \cap \\ & \ C^*\mathbb{Z} \\ & \quad \wr \\ & \ C(S^1) \end{array}$$

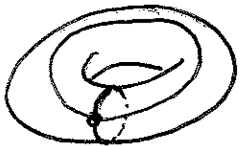
$f_i = \pi_i(a)$ function on S^1: $f_i(1) = 1_{\mathbb{Z}} \cdot \pi_i(a) = 1_{F_n}(a)$.

Rk : π doesn't factorize through $C^*_r F_n$ (unless $n=1$)

Alternative : $F_n \longrightarrow F_n^{ab} = F_n / [F_n, F_n] = \mathbb{Z}^n$

$$C^*F_n \longrightarrow C^*\mathbb{Z}^n \underset{\mathcal{F}}{\simeq} C(\mathbb{T}^n) \longrightarrow C(X_n)$$

$$X_n \subset \mathbb{T}^n$$


$X_2 \subset \mathbb{T}^2$

$$
\begin{aligned}
j \circ \pi : C^*F_2 &\longrightarrow M_2(C^*F_2) \\
a &\longmapsto \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \\
b &\longmapsto \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \sim \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

by a rotation trick

$$
\left.\begin{aligned}
C^*F_2 &\longrightarrow M_2(C^*F_2) \\
g &\longmapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}\right\} = \begin{bmatrix} id & 0 \\ 0 & \tau_{C^*} \end{bmatrix}.
$$

This gives $j_* \circ \pi_*$.

Sums of morphisms don't make sense at the level of algebras, but do at the level of K-theory. (The category of abelian groups is an abelian category)

$$\begin{tikzcd}
\hat{\tau} : C(X_2) \arrow[rr] \arrow[dr] & & C^*F_2 \\
& \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

$$(j_* - \hat{\tau}_*) \circ \pi_* = id$$

$$\pi_* \circ (j_* - \hat{\tau}_*) = id$$

This proves that π induces an isomorphism in K-theory.

Th

$$\begin{tikzcd} K(C^*F_n) \arrow[r, "\sim", "\pi_*"'] \arrow[d, "\lambda_*", "\wr"'] & K(C(X_n)) \\ K(C_r^*F_n) & \end{tikzcd}$$

n = 2 $\quad \pi : \underset{(a,b)}{C^*F_2} \longrightarrow C(X_2) \quad , \quad \pi_* : K(C^*F_2) \longrightarrow K(C(X_2))$

Construct $j : C(X_2) \longrightarrow M_2(C^*F_2)$

"Dual" maps to π_1, π_2 :

$$j_1 : C(S^1) \xrightarrow{\sim} C^*\mathbb{Z} \longrightarrow C^*F_2$$
$$1 \longmapsto a$$
$$f = \sum c_n e^{in\theta} \longmapsto \sum c_n a^n$$

$$j_2 : C(S^1) \longrightarrow C^*F_2$$
$$f = \sum c_n e^{in\theta} \longmapsto \sum c_n b^n$$

Define $j : (f_1, f_2) \longmapsto \begin{bmatrix} j_1(f_1) & 0 \\ 0 & j_2(f_2) \end{bmatrix}$

$$j_* : K(C(X_2)) \longrightarrow K(M_2(C^*F_2)) = K(C^*F_2)$$

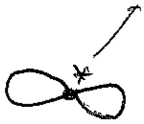
Is j_* an inverse for π_* ?

Lemma :

$$j_* \circ \pi_* = id_{K(C^*F_2)} + (\tau_{C^*})_*$$
$$\pi_* \circ j_* = id_{K(C(X_2))} + (\tau_X)_*$$

where

$$\begin{tikzcd} \tau_{C^*} : C^*F_2 \arrow[rr] \arrow[dr, "1_{F_n}"'] & & C^*F_2 \\ & \mathbb{C} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} C(X_2) \arrow[rr] \arrow[dr, "w_*"'] & & C(X_2) \\ & \mathbb{C} \arrow[ur] & \end{tikzcd} \quad \tau_X .$$



Aim: prove $K(C^*F_2) \xrightarrow{\lambda_*} K(C^*_r F_2)$ is an isomorphism.

Tools (Cuntz): $KK^G(A,B)$ (objects are not really equivariant, but up to compacts).

The key is the construction of an element in the commutative ring $KK^G(\mathbb{C},\mathbb{C})$, with unit $1 = [\mathbb{C} \xrightarrow{0} 0]$ with trivial rep on $\mathbb{C}$.

(If G is compact, $KK^G(\mathbb{C},\mathbb{C}) \simeq \mathcal{R}(G)$)

Lemma: $G = F_2$. There exist two unitary representations (π_0, H_0) and (π_1, H_1) of G and $T : H_0 \longrightarrow H_1$ such that:

1/ $\left(\underset{\pi_0}{H_0} \xrightarrow{T} \underset{\pi_1}{H_1}\right)$ defines a class in $KK^G(\mathbb{C},\mathbb{C})$, that is

$1 - T^*T$, $1 - TT^*$ and $T\pi_0(g) - \pi_1(g)T$ are compact

2/ π_0 and π_1 are weakly contained in λ. ($\pi_i : C^*(G) \xrightarrow[G]{} \mathcal{L}(H_i)$)

$$\begin{tikzcd} C^*(G) \arrow[r, "G"'] \arrow[dr] & \mathcal{L}(H_i) \\ & C^*_r(G) \arrow[u, dashed] \end{tikzcd}$$

3/ In $KK^G(\mathbb{C},\mathbb{C})$, $[H_0 \xrightarrow{T} H_1] \simeq \underset{\underset{[\mathbb{C}\xrightarrow{0}0]}{\shortparallel}}{1_G}$ (homotopy).

This lemma implies that λ_* is an isomorphism:

$j^G : KK^G(\mathbb{C},\mathbb{C}) \longrightarrow KK^G(C^*G, C^*G)$

$j^G_r : KK^G(\mathbb{C},\mathbb{C}) \longrightarrow KK^G(C^*_rG, C^*_rG)$

$\lambda : C^*G \longrightarrow C^*_rG$, $\lambda_* \in KK(C^*G, C^*_rG)$.

Let $\gamma \in KK^G(\mathbb{C},\mathbb{C})$ be the class $[H_0 \xrightarrow{T} H_1]$.

$j^G(\gamma) \in KK(C^*G, C^*G)$

$j^G_r(\gamma) \in KK(C^*_rG, C^*_rG)$

$H_i \otimes C^*G$ with left action

$$\begin{tikzcd} C^*G \arrow[r] \arrow[dd] & \mathcal{L}(H_i \otimes C^*G) \arrow[dd] \\ g \arrow[r, mapsto, "G"'] & \pi_i(g) \otimes U_g \\ C^*_rG \arrow[r, "\exists"'] & \mathcal{L}(H_i \otimes C^*_r(G)) \end{tikzcd}$$

Condition 2/ gives that $\pi_1 \propto \lambda$. Then

$$\begin{tikzcd} C^*G \arrow[r] \arrow[d] & \mathcal{L}(H \otimes C^*G) \arrow[d] \\ C^*_r G \arrow[r] \arrow[ur] & \mathcal{L}(H \otimes C^*_r G) \end{tikzcd}$$

$\Rightarrow \exists\, \mu \in KK(C^*_r G, C^*G)$

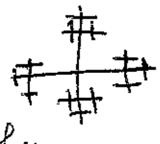
Recall $\lambda_* \in KK(C^*G, C^*_r G)$

One has $\left| \begin{array}{l} \lambda_* \otimes \mu = j^G(\gamma) \in KK(C^*G, C^*G) \\ \mu \otimes \lambda_* = j^G_r(\gamma) \in KK(C^*_r G, C^*_r G) \end{array} \right.$

Conditions 3/ gives that $\gamma = 1$, so $j^G(\gamma)$ and $j^G_r(\gamma) = 1$.

$$\text{then} \left. \begin{array}{l} \lambda_* \otimes \mu = 1_{KK(C^*G, C^*G)} \\ \mu \otimes \lambda_* = 1_{KK(C^*_r G, C^*_r G)} \end{array} \right\} \Rightarrow \lambda^* \text{ isomorphism.}$$

<u>Proof of the lemma</u>: G is a group acting on a tree | here action is free



Tree: $\Delta^0 = \{\text{vertices}\}$, $\Delta^1 = \{\text{edges}\}$

$\Delta^1 \xrightarrow{s} \Delta^0$ source map

$\Delta^1 \xrightarrow{t} \Delta^0$ target map

$\Delta^1 \xrightarrow{op} \Delta^1$: 

+ no <u>loops</u>.

$H_0 := \ell^2(\Delta^0) = \left\{ f : \Delta^0 \to \mathbb{C} \;\middle|\; \sum_{x \in \Delta^0} |f(x)|^2 < \infty \right\}$, π_0 unit rep on H_0.

Action of G defined on vertices. Then on edges: 

<u>Proof of condition 1/</u>

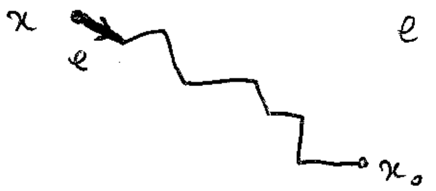
$H_1 := \ell^2(\Delta^1) = \left\{ \underset{f(e^{op}) = -f(e)}{f : \Delta^1 \to \mathbb{C}} \;\middle|\; \frac{1}{2} \sum_{e \in \Delta^1} |f(e)|^2 < \infty \right\}$

π_1 unitary rep of G on H_1.

" Think of H_0 as functions and H_1 as 1-forms. "

$T_{x_0} : \ell^2\Delta^0 \longrightarrow \ell^2\Delta^1$ analogue of Bott element.

Chose $x_0 \in \Delta^0$. For all $x \in \Delta^0$, $\exists!$ geodesic $x \to x_0$



e is well-defined for all $x \neq x_0$.

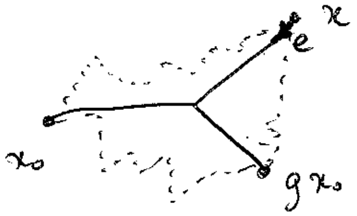
Define T_{x_0} by $\begin{cases} T_{x_0}\delta_x = \delta_e & \text{for } x \neq x_0 \\ T_{x_0}\delta_{x_0} = 0 \end{cases}$, where $\begin{cases} \delta_e(e) = 1 \\ \delta_e(e^{op}) = -1 \\ \delta_e(e') = 0 \text{ if } e' \notin \{e, e^{op} \end{cases}$

$T^*T = 1 - p$, with $p =$ orthogonal proj. on $\mathbb{C}\delta_{x_0}$

$TT^* = 1$

<u>What is $T_{x_0} - \pi_1(g) T_{x_0} \pi_0(g)^{-1}$?</u>

$$T_{x_0} - \pi_1(g)\, T_{x_0}\, \pi_0(g)^{-1} = T_{x_0} - T_{gx_0}$$



for there are no loops.

$\Rightarrow$ $T_{x_0} - T_{gx_0}$ is of finite rank, hence compact.

<u>Prove condition 2/.</u> $G = F_2$, $\Delta^0 = G$ $\pi_0 = \lambda$ on $\ell^2 G$

$\Delta^1 = \underbrace{G \cup G} \underset{\overset{\longleftrightarrow}{op}}{\cup} \underbrace{G \cup G}$ $\pi_1 = \lambda \oplus \lambda$

Proof of condition 3/

$$\ell^2\Delta^0 \xrightarrow{T} \ell^2\Delta^1$$

π_0 π_1

Family of representations $\rho_t(g) = V_t^{-1}\pi_0(g)V_t$ $t\in[0,1]$ st:

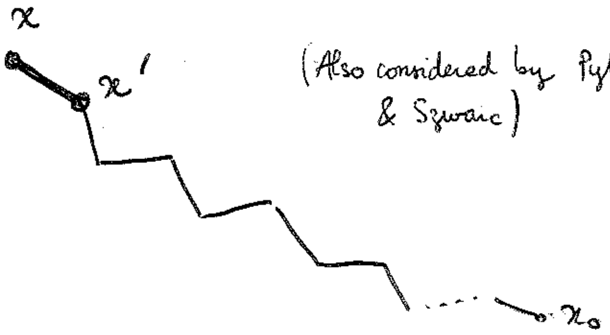
V_t : bounded operator $\ell^2\Delta^0 \longrightarrow \ell^2\Delta^0$, $V_0 = id$ $\rho_0 = \pi_0$

$$V_t^{-1} : \begin{cases} \text{bounded if } t\in[0,t_{\text{critical}}[\\ \text{unbounded (densely defined) if } t\in[t_{\text{critical}},1] \end{cases}$$

Notations: • $P : \ell^2\Delta^0 \longrightarrow \ell^2\Delta^0$

$\delta_x \longmapsto \delta_{x'}$

$\delta_{x_0} \longmapsto 0$



(Also considered by Pytlik & Szwarc)

• p: projection on $\mathbb{C}\delta_{x_0}$.

Consider : $V_t = 1 - tP + \left(\sqrt{1-t^2} - 1\right)p$

Lemma: | $V_t^*V_t$ commutes with $\pi_0(g)$

"Proof": $P+P^*$ turns out to be the operator S sendind δ_x to the sum of its neighbours: $P+P^* = S : \delta_x \longmapsto \sum_{y \text{ neigh. of } x} \delta_y$,

whereas $P^*P = Q + p$, where $Q\delta_x = \#(\text{neighbours of } x)\delta_x$.

Then $V_t^*V_t = 1 - tS + t^2Q$. S, Q are G-invariants.

Consequence: $V_t^{-1}\pi_0(g)V_t$ extends to a unitary.

$$\underset{t\to 1}{s\text{-}\lim}\, V_t^{-1}\pi_0(g)V_t = T^*\pi_1(g)T + p \qquad (T^*T = 1)$$