BIVARIANT K-THEORY FOR SMOOTH MANIFOLDS II

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Abstract. Building on the first talk, we describe equivariant Kasparov theory for smooth manifolds in purely topological terms, using an appropriate theory of equivariant correspondences. There is always a map from the topological theory to the analytic theory, and it is an isomorphism under some hypotheses related to the existence of certain equivariant vector bundles. We conclude by listing various geometric situations under which the hypotheses hold, and by showing how and why these considerations are important for equivariant index theory.

1. Duality correspondances and KK-theory

Goal: Give a geometric description of $KK^{\mathcal{G}}(C_0(X), C_0(Y))$ where $\mathcal{G}$ is a proper groupoid, and X and Y are smooth $\mathcal{G}$-manifolds.

Recall from last time: We denote by $\mathcal{G}$ a proper groupoid with base space Z and X denotes a $\mathcal{G}$-space (see Section 2 of the previous lectures notes for the definitions of these terms).

Theorem 1.1. *Let X be a smooth proper $\mathcal{G}$-manifold. Then there exists natural (duality) isomorphisms*

$$\begin{aligned} KK^{\mathcal{G}}(C_0(TX) \otimes A, B) &\cong KK^{\mathcal{G} \ltimes X}(C_0(X) \otimes A, C_0(X) \otimes B) \\ KK^{\mathcal{G}X}(C_0(X) \otimes A, B) &\cong KK^{\mathcal{G} \ltimes X}(C_0(X) \otimes A, C_0(TX) \otimes B) \end{aligned}$$

Today: Describe index theory topologically. (The main reference for the material contained in these notes is [2])

2. Wrong-way maps and index theory

We begin by discussing some index problems.

Example 2.1. We take $\mathcal{G}$ to be trivial and X to be a K-oriented smooth manifold (note: K-orientablity in the classical case is equivalent to being *spin*^{c}). For now we assume that X is compact so that we are in the classical case of the Atiyah-Singer Index Theorem. We denote the Dirac operator of X by D_X and its class in $KK(C(X), \mathbb{C})$ by $[D_X]$.

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Problem: For $E \in K^*(X)$ compute the index of the twisted Dirac operator (i.e. $ind(D_X^E)$).
Solution: Fix a smooth embedding $i : X \hookrightarrow \mathbb{R}^n$. Denote the normal bundle of $i(X)$ by N and use the tubular neighborhood theorem to get $\hat{\phi} : N \hookrightarrow \mathbb{R}^n$ an open embedding. (Note that N is K-oriented since both the trivial bundle $X \times \mathbb{R}^n$ and TX are).

The above data produces a number of KK-theory elements via the construction in Section 3 of the previous lectures notes. This construction took a $\mathcal{G}$-equivariantly K-oriented vector bundle, V, over a space Y (where $\mathcal{G}$ is a proper groupoid) and produced an element in $KK^{\mathcal{G} \ltimes Y}(C_0(Y), C_0(V))$. In our current setup we have two bundles: N over the zero section and $\mathbb{R}^n$ considered as vector bundle over a point. Hence, we can form

$$\begin{aligned} \xi_N! &\in KK^X(C(X), C_0(N)) \\ \pi_{\mathbb{R}^n}! &\in KK(C(X), \mathbb{C}) \end{aligned}$$

We also have $\hat{\phi}! \in KK(C_0(N), C_0(\mathbb{R}^n))$.

Theorem 2.2 (Atiyah-Singer). *Using the notation we have defined above we have that*

$$[D_X] = \xi_N! \otimes_{C_0(N)} \hat{\phi}! \otimes_{C_0(\mathbb{R}^n)} \pi_{\mathbb{R}^n}!$$

Exercise: Use this theorem to produce a cohomology formula for $ind(D_X^E)$.

Remark. Our construction in the previous example still works if X is noncompact.

Example 2.3. Let X be a smooth manifold with a smooth action of a compact group, G. Moreover, assume that the action preserves a fixed K-orientation on X. We can again form the Dirac operator, D_X, and its class, $[D_X] \in KK^G(C_0(X), \mathbb{C})$.

We produce an equivariant of the previous example. To begin, fix an equivariant embedding $i \colon X \hookrightarrow E$, where E is a Euclidean space with a linear G action. That such an embedding exists is due to Mostow – as long as X is *compact*. We let N denote the normal bundle and again form KK-elements:

$$\begin{aligned} \xi_N! &\in KK^G(C(X), C_0(N)) \\ \pi_{\mathbb{R}^n}! &\in KK^G(C(X), \mathbb{C}) \\ \hat{\phi}! &\in KK^G(C_0(N), C_0(\mathbb{R}^n)) \end{aligned}$$

Theorem 2.4. *With the notation defined above, we have that*

$$[D_X] = \xi_N! \otimes_{C_0(N)} \hat{\phi}! \otimes_{C_0(E)} \pi_{\mathbb{R}^n}!$$

Remark 2.5. It turns out that if X is not compact there may not be an embedding of the required form. The next theorem makes this precise. For purposes of the theorem, "finite orbit type" means that there are at most finitely many subgroups (up to conjugacy) which are stabilizers of points in X.

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Theorem 2.6 (Mostow)**.** *A smooth G-manifold, X, embeds in a linear representation of G if and only if X has finite orbit type.*

Example 2.7. Let $G = S^1$ and X equal the disjoint union of countable copies of S^1. We define $z \in G = S^1$ to act on the n^{th}-copy of $S^1 \subseteq X$ by z^n. The roots of unity are stabilizers. It follows that this action is not of finite orbit type and hence (by the previous Theorem) that the "required" embedding does not exist.

Therefore, in this case, the analogue of (2.4) does not exist, of course.

We now return to the general setting. That is, *$\mathcal{G}$ a proper groupoid and X a smooth equivariantly K-oriented $\mathcal{G}$-manifold* – thus a bundle of smooth, K-oriented manifolds over the base of $\mathcal{G}$, with elements in $\mathcal{G}$ acting diffeomorphically between fibres and preserving K-orientations.

General Problem: Describe the class of the fibrewise Dirac operator $[D_X] \in KK^{\mathcal{G}}(C_0(X), C_0(Z))$ topologically – if possible (see Section 4 of the notes from the first lecture.) We first need an embedding result. A natural choice in the group*oid* case is a embedding of the form:

$$i : X \hookrightarrow E$$

where E is a $\mathcal{G}$-equivariant vector bundle over Z.

Example 2.8. We define a groupoid as follows. First we consider $\mathbb{R} \times_{\mathbb{Z}} T^2$ where T^2 is the torus and the $\mathbb{Z}$ action on $\mathbb{R}$ is by translation and the action on T^2 is determined by $A \in GL_2(\mathbb{Z})$. We then have $p : \mathbb{R} \times_{\mathbb{Z}} T^2 \to S^1$. We denote this groupoid by $\mathcal{G}_A$ and note that its base (which we denote by Z_A) is S^1.

Lemma 2.9. *The $\mathcal{G}_A$-equivariant vector bundles over S^1 are in one-to-one correspondance with representations $\pi \in Rep(T^2)$, which are fixed by the action of A on the dual, $\hat{T}^2$ (i.e., $\pi \circ A \sim \pi$).*

We now assume that the action on T^2 is ergodic. For example, this is the case if $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It then follows that there are no fixed points in $Rep(T^2)$ except for multiples of the trivial representation.

Corollary 2.10. *If A induces an ergodic action on T^2, then the only $\mathcal{G}_A$-equivariant vector bundles over $S^1(= Z_A)$ carry the trivial action of $\mathcal{G}_A$.*

If we let $\mathcal{G}_A$ act on itself by translation, then it can not be imbedded into a $\mathcal{G}_A$-equivariant vector bundle (since if it could it would carry the trivial action). Thus, for this *compact* example, the immediate analogue of (2.4) does not exist.

3. An embedding theorem for certian groupoids

We have seen that in order to get index theorems, we need embeddings. Our goal in this section is to address the question of when such embeddings exist (see Theorem 3.6). To begin, we have a number of definitions.

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Definition 3.1. Let X be a $\mathcal{G}$-space. Then X has enough $\mathcal{G}$-equivariant vector bundles, if for each $x \in X$ and each finite-dimensional representation of the compact isotropy group, $\mathcal{G}_x^x$, there exists $\mathcal{G}$-equivariant vector bundle (over X) whose restriction to x contains the given representation of $\mathcal{G}_x^x$.

Definition 3.2. Again let X be a $\mathcal{G}$-space. Then X has a full vector bundle if there exists $\mathcal{G}$-equivariant vector bundle, V, (such a V will be called a full vector bundle) such that for any $x \in X$ and any irreducible representation of $\mathcal{G}_x^x$ is contained in the representation of $\mathcal{G}_x^x$ on V_x.

Definition 3.3. A $\mathcal{G}$-equivariant vector bundle over a $\mathcal{G}$-space is trivial if it has the form $p_X^*(E)$ for some E over Z. Note that p_X is the map X to Z given in the definition of $\mathcal{G}$-space. A $\mathcal{G}$-equivariant vector bundle is called sub-trivial if it is a direct summand of a trivial bundle.

Remark 3.4. In the case of a group, G, the base Z is a point, so that G-trivial bundles on X are trivial in the classical sense. If $\mathcal{G}$ has non-trivial base Z but no morphisms, then $\mathcal{G}$-equivariant vector bundles on Z are just the same as vector bundles on Z, $\mathcal{G}$ spaces are just spaces which fibre over Z, and trivial bundles on $\mathcal{G}$-spaces X are bundles which are pulled back from Z.

The following gives an example of a $\mathcal{G}$-equivariant vector bundle which is not even *sub*-trivial.

Example 3.5. Let X be $\mathbb{Z}$ as a S^1-space (using the trivial action). We consider the vector bundle $\mathbb{Z} \times \mathbb{C}$ with the action of S^1 defined via $z \cdot (n, \lambda) = (n, z^n \cdot \lambda)$. This vector bundle is not sub-trivial (since it contains infinitely many irreducible representations of S^1).

Theorem 3.6. *Let $\mathcal{G}$ be a proper groupoid and X be a smooth $\mathcal{G}$-manifold. If either*
A: *$X/\mathcal{G}$ is compact and Z has enough vector bundles.*
B: *The covering dimension of $X/\mathcal{G}$ is finite and Z has a full vector bundle.*
Then, every $\mathcal{G}$-equivariant vector bundle over X is subtrivial and there exists a smooth embedding of X into a trivial vector bundle over Z.

Example 3.7. Lück and Oliver have proved that if Γ is a discrete group with $\mathcal{E}\Gamma/\Gamma$ compact, then, letting $\mathcal{G} = \Gamma \rtimes \mathcal{E}\Gamma$, we have the $Z(= \mathcal{E}\Gamma)$ has enough vector bundles. Therefore, our embedding theorem applies to this groupoid.

Example 3.8. If G is a closed subgroup of an almost connected group, then $\mathcal{E}G$ (as the base of $\mathcal{G} = G \rtimes \mathcal{E}G$) has enough vector bundles. This follows from the Morita invariance of the condition of having enough vector bundles, and the corresponding fact about compact groups.

4. Normal Maps

We now have the required embedding theorem (Theorem 3.6). Looking at Example 2.1, we see that we need equivariant versions of normal bundles and

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the tubular neighborhood theorem. To this end, we define normal maps, which roughly speaking are embeddings along with a tubular neighborhood.

Definition 4.1. Let X and Y be $\mathcal{G}$-spaces. A $\mathcal{G}$-equivariant normal map from X to Y is a triple, $\Phi = (V, E, \hat{f})$, where V is a subtrivial $\mathcal{G}$-equivariant vector bundle over X, E is a $\mathcal{G}$-equivariant bundle over Z, and $\hat{f} : V \hookrightarrow E^Y (:= p_Y^*(E))$ is an open $\mathcal{G}$-equivariant embedding.

We define the degree of Φ to be $dim(V) - dim(E)$. The stable normal bundle of Φ is defined to be $[V] - [E^X] \in VKO_{\mathcal{G}}(X)$, where $VKO_{\mathcal{G}}(X)$ denotes the Grothendieck group of the monoid of isomorphism classes of real $\mathcal{G}$-equivariant vector bundles over X. Finally, we define the trace of Φ via

$$tr(\Phi) := f = \pi_{E^Y} \circ \hat{f} \circ \xi_V$$

We note that ξ_V maps the zero section (i.e. X) into V. Hence, f is a map from X to Y.

We say that a normal map, Φ, is K-oriented (resp. smooth) if the vector bundles occurring in the triple (e.g., V, E, etc) are K-oriented (resp. smooth).

Example 4.2. Let $\mathcal{G}$ be trivial. Then normal maps from X to a point are "equivalent" to smooth structures on $X \times \mathbb{R}^n$. (The quotes are needed, since we need a notion of equivalence on the set of smooth structures to make this statement precise).

The next theorem is a reformulation of Theorem 3.6 into the language of normal maps.

Theorem 4.3. *Let $\mathcal{G}$ be proper. Then, any smooth $\mathcal{G}$-manifold X, such that either condition A) or B) of Theorem 3.6 hold, admits a smooth normal map to Z.*

Definition 4.4. We define an equivalence relation on the set of isomorphism classes of normal maps via the following (throughout, $\Phi = (V, E, \hat{f})$ is a normal map)

1. **Lifting:** For any vector bundle over Z, W, we define
$$(V, E, \hat{f}) \sim (V \oplus W^X, E \oplus W, \hat{f} \times_Z id_W)$$
and call the second of these two the lift of Φ by W. Two normal maps, Φ_0 and Φ_1, are stably isomorphic if there are lifts Φ_0 and Φ_1 which are isomorphic.
2. **Isotopy:** An isotopy between two normal maps, Φ_0 and Φ_1, is a family (parameterized by $[0, 1]$) of normal maps such that the end points are isomorphic to Φ_0 and Φ_1 respectively.

We then define an equivalence relation by taking the one generated by the lifting and isotopy relations above. Moreover, if we restricted to smooth normal maps, then we can define a "smooth" equivalence relation by requiring the isomorphisms, lifts and isotopies be smooth.

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Theorem 4.5. *Let X and Y be smooth $\mathcal{G}$-manifolds. We assume that*

(1) *TY is subtrivial;*
(2) *X admits a smooth $\mathcal{G}$-equivariant normal map to Z*

Then smooth equivalence class of K-oriented smooth normal maps from X to Y are in one-to-one correspondance with pairs (f, τ), where f is a smooth homotopy class of a smooth map X to Y and τ is K-orientation on $[TX] - f^*([TY])$.

Proof. We give the main idea of the proof. The idea is to construct a normal map that has trace equal to a given $f : X \to Y$. Assuming that $\mathcal{G}$ is trivial, we have embedding $i : X \hookrightarrow \mathbb{R}^n$ (in general, we have an embedding of the type in Theorem 3.6). We then have embedding $f \times i : X \to Y \times \mathbb{R}^n$ and projection map $Y \times \mathbb{R}^n \to Y$. Moreover, f is just the composition of these two maps. The next step in the proof is to produce normal maps maps for smooth embeddings and projections. The previous argument shows that once we have this result we are done. □

Remark 4.6. We have not discussed the composition of normal maps. Composition is defined up to isotopy and, moreover, the trace of a composition is the composiiton of the traces.

Definition 4.7. We denote the category of equivalence classes of $\mathcal{G}$-equivarant normal maps by *Mor*($\mathcal{G}$).

Proposition 4.8. *The following defines a functor:*

$$Mor(\mathcal{G}) \to KK^{\mathcal{G}} \tag{4.1}$$

$$(V, E, f) \mapsto \xi_V! \otimes \hat{f}! \otimes \pi_{E^Y}! \tag{4.2}$$

5. Correspondances

We now define the notion of a correspondence. Correspondences are the topological objects which in a natural way produce elements of *KK*-theory. They were first introduced in [1]. The definition here is different (see Remark 5.2).

Definition 5.1. Let X and Y be $\mathcal{G}$-spaces. Then a ($\mathcal{G}$-equivariant K-oriented) correspondance from X to Y is a quadruple, $(M, b, \hat{f}, \xi)$, where
1) M is a $\mathcal{G}$-space,
2) $b : M \to X$ is a $\mathcal{G}$-map,
3) $\hat{f} : M \to Y$ is an K-oriented normal $\mathcal{G}$-map,
4) $\xi \in RK^*_{\mathcal{G},X}(M)$ (see the notes from the first lecture for the definition of $RK^*_{\mathcal{G},X}(M)$).

Remark 5.2. There are three main differences between this definition of correspondances and the original one in [1]. Firstly, we do not assume that the map b is proper (the "support" conditions have been passed to the K-theory part; we

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note that in [1] vector bundles are used rather than actual classes in K-theory). Secondly, because we are working equivariantly, we need $\hat{f}$ to be a normal map. Finally, the spaces X and Y need not be smooth manifolds.

Definition 5.3. We define an equivalence relation on correspondances by letting it be the equivalence relation generated by

(1) Bordism;
(2) Equivalence of normal maps;
(3) Thom modification.

We discuss the last of these relations (i.e. Thom modification) in more detail. Let $(M, b, \hat{f}, \xi)$ be a correspondance and V a $\mathcal{G}$-equivariant K-oriented vector bundle over M, then the Thom modification of $(M, b, \hat{f}, \xi)$ by V is the correspondance

$$(V, b \circ \pi, \hat{f} \circ \pi, \pi^*(\xi) \cdot \xi_V)$$

where π denotes the projection map $V \to M$ and ξ_V is the Thom class in $RK^*_{\mathcal{G},M}(V)$. We note that the Thom class was defined in the previous lecture (see either the notes from that lecture or [3]).

Theorem 5.4. *We denote the equivalence classes of correspondances from X to Y by $\hat{kk}_{\mathcal{G}}(X, Y)$. Then $\hat{kk}_{\mathcal{G}}(X, Y)$ is a category.*

Theorem 5.5. *Let $\mathcal{G}$ be a proper groupoid and X be a smooth $\mathcal{G}$-manifold. Assume that X admits a smooth normal map to the base of $\mathcal{G}$ (i.e. Z) and that all vector bundles over X are subtrivial. Then, for each G-space Y, the map $\hat{K}K_G \to KK^G$ obtained by combining ordinary functoriality with respect to the map $b\colon M \to X$, the wrong-way functoriality of the map f (see Equation 4.2), and the module structure of $KK^G(C_0(M), C_0(Y))$ over $RK^*_{G,X}(M)$, determines a canonical isomorphism $\hat{K}K_G(X, Y) \to KK^G(C_0(X), C_0(Y))$.*

This result describes equivariant Kasparov groups for commutative C*-algebras in purely topological terms. But it requires some hypotheses. One is that we only consider Kasparov morphisms with source a smooth $\mathcal{G}$-manifold. Another hypothesis has to do with an ample supply of $\mathcal{G}$-equivariant vector bundles on X.

References

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