BIVARIANT K-THEORY FOR SMOOTH MANIFOLDS I

LECTURES BY: HEATH EMERSON (JOINT WORK WITH RALF MEYER)
NOTES BY: ROBIN DEELEY AND ELKAIOUM MOUTUOU

Abstract. The aim of this talk and the next is to explain two important aspects of equivariant Kasparov theory, especially for smooth manifolds: duality, and the topological description of equivariant KK-groups using equivariant correspondences. In the first talk we will review the basic definitions of KK-theory, including the Thom isomorphism. We then explain duality, which gives a way of reducing KK groups to K-theory groups with support conditions. This will be used in the second talk to prove that equivariant KK-groups for smooth manifolds can be described in topological terms.

1. Duality correspondances and KK-theory

Goal: Give a geometric description of $KK^{\mathcal{G}}(C_0(X), C_0(Y))$ where $\mathcal{G}$ is a proper groupoid, and X and Y are smooth $\mathcal{G}$-manifolds.

Remark 1.1. *A priori the condition that $\mathcal{G}$ is proper is restrictive. However, the Baum-Connes conjecture implies that*

$$KK^{\mathcal{G}}(C_0(X), C_0(Y)) \xrightarrow{\cong} KK^{\mathcal{G}\rtimes\mathcal{E}\mathcal{G}}(C_0(X\times\mathcal{E}\mathcal{G}), C_0(Y\times\mathcal{E}\mathcal{G}))$$

so long as $\mathcal{G}$ acts amenably on X. Moreover, $\mathcal{G}\rtimes\mathcal{E}\mathcal{G}$ is proper so we study a non-proper groupoid, $\mathcal{G}$, by replacing it with $\mathcal{G}\rtimes\mathcal{E}\mathcal{G}$. A protypical example is the case when $\mathcal{G}$ is an infinite discrete group. This example implies that even if we are only interested in group actions, we must study groupoids.
One may also wonder why smoothness is required. The point is that smoothness leads to duality (see Definitions 5.2 and 5.3 for the precise definitions of the duality we will be studying).

2. Preliminaries

Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{G}^{(0)}$ be a locally compact, second countable and Hausdorff groupoid (not necessary proper). Let Z denote the base (i.e., unit space, $\mathcal{G}^{(0)}$) of $\mathcal{G}$. We require our groupoids to have a Haar system μ_x; recall that by this we mean a family of Haar measures with the property that $\mathrm{Supp}\mu_x = r^{-1}(x)$ and $x \longmapsto \int_{\mathcal{G}^x} f d\mu_x$ is continuous for any $f \in C_c(\mathcal{G})$.

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In addition, if $\gamma \in \mathcal{G}, s(\gamma) = x, r(\gamma) = x'$ and we denote the translation by γ by $L_\gamma : \mathcal{G}^x \to \mathcal{G}^{x'}$, then we require that $(L_\gamma)_*(\mu_x) = \mu_{x'}$.

Definition 2.1. A space over Z is a space X together with a map $p_X : X \longrightarrow Z$.

We note that a vector bundle over Z is an example of a space over Z.

Functoriality: $f : Z' \longrightarrow Z$, and X a space over Z then put

$$f^*X = \{(z', x) | f(z') = P_X(x)\}$$

It is a space over Z'. This is a generalization of the pull-back construction from vector bundle theory.

Definition 2.2. A $\mathcal{G}$-space X is a space X over Z together with a homeomorphism of spaces over $\mathcal{G}$:

$$s^*(X) \longrightarrow r^*(X), (g, s) \longmapsto (g, gx)$$

satisfying associativity and unitality conditions.

Definition 2.3. A $\mathcal{G}$-space X is proper if the map

$$s^*(X) \longrightarrow X \times X, (g, x) \longmapsto (gx, x)$$

is proper (as a map).
A groupoid $\mathcal{G}$ is proper is $\mathcal{G}$ acts properly on Z, i.e.

$$(r, s) : \mathcal{G} \longrightarrow Z \times Z$$

is proper.

Examples 2.4. If $\mathcal{G}$ acts on X, then form

$$\mathcal{G} \rtimes X := \{(g, x) | s(g) = P_X(x)\}$$

This groupoid has base (i.e. object space) X, and source and range maps given by $s(g, x) = x$ and $r(g, x) = gx$.

Remark 2.5. *If G is a group, then G acts properly on a point if and only if G is compact. However, $G \rtimes \mathcal{E}G$ is always proper. As we discussed in Remark 1.1, the Baum-Connes conjecture allows one to study G via $G \rtimes \mathcal{E}G$.*

Definition 2.6 ($\mathcal{G}$-C^*-algebras)**.** A C^*-algebra over Z is a C^*-algebra A together with an essential C^*-homomorphism $C_0(Z) \longrightarrow \mathcal{ZM}(A)$.

Functoriality: Given $f : Z' \to Z$, then if A is a C^*-algebra over Z, we form a C^*-algebra over Z' given by $f^*(A) := C_0(Z') \otimes_{C_0(Z)} A$. $f^*(A)$ is a C^*-algebra over Z'.

Restriction: Given a C^*-algebra over Z and $S \subseteq Z$ (S is assumed to be closed or open). Then we define the restriction to S via

$$A|_S = \begin{cases} C_0(S) \cdot A & S \text{ open}, \\ A/A|_{S^c} & S \text{ closed} \end{cases}$$

Every C^*-algebra over Z is the C^*-algebra of continuous field of C^*-algebras via the restriction to points. (The fiber at z is given by $A|_z$, for $z \in Z$).

Definition 2.7. A $\mathcal{G}$ C^*-algebra is a C^*-algebra over Z and an isomorphism (of C^*-algebras over $\mathcal{G}$) between $s^*(A)$ and $r^*(A)$. We note that this isomorphism induces an isomorphism between the fibers of $s^*(A)$ and $r^*(A)$.

2.1. **Tensor Products.** Let A and B be $\mathcal{G}$-C^*-algebras. Then their tensor product is defined by

$$A \otimes_Z B := (A \otimes B)|_{\Delta(Z \times Z)}$$

where $\Delta(Z \times Z)$ is the diagonal in $Z \times Z$ and $\mathcal{G}$ acts diagonally on $A \otimes_Z B$.

Example 2.8. Take $A = C_0(U)$ and $B = C_0(V)$; then one can show that $A \otimes_Z B = C_0(U \times_Z V)$, where $U \times_Z V := \{(u, v) | p_U(u) = p_V(v)\}$.

Definition 2.9. A symmetric monoidal category is a category with tensor product operation $\otimes$ such that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $A \otimes B \cong B \otimes A$, and there is an element $\mathbf{1}$ (the identity) such that $\mathbf{1} \otimes A \cong A \otimes \mathbf{1}$.

Proposition 2.10. *The category of $\mathcal{G}$-C^*-algebras is a symmetric monoidal category with identity $\mathbf{1} = C_0(Z)$.*

Definition 2.11. Let A and B be $\mathcal{G}$-algebras. Then we define $KK^{\mathcal{G}}(A, B)$ to be the quotient of $E^{\mathcal{G}}(A, B)$ by homotopy; where $E^{\mathcal{G}}(A, B)$ is the set of pairs $(\mathcal{E}, F)$, where $\mathcal{E}$ is a $\mathbb{Z}_2$-graded $\mathcal{G}$-equivariant Hilbert B-module with a $\mathcal{G}$-equivariant $*$-homomorphism, $\phi : A \to \mathcal{B}(\mathcal{E})$, and $F \in \mathcal{B}(\mathcal{E})$ is odd and each of the following are compact: $[F, \phi(a)]$, $(F^2 - I)\phi(a)$, $(F^* - F)\phi(a)$, and $(g \cdot F - F)\phi(a)$ (This last this condition is vague; it is an exercise for the reader to precisely define it).

Theorem 2.12 (Kasparov). *$KK^{\mathcal{G}}$ is a symmetric monoidal category. We note that the tensor product operation required is given by the exterior product in $KK^{\mathcal{G}}$.*
Moreover, we have a functor (of symmetric monoidal categories):

$$\mathcal{G}\text{-algebras} \longrightarrow KK^{\mathcal{G}}$$

The composition (i.e. the Kasparov product) in $KK^{\mathcal{G}}$ will be denote by:

$$KK^{\mathcal{G}}(A, B) \times KK^{\mathcal{G}}(B, C) \longrightarrow KK^{\mathcal{G}}(A, C), (f, h) \longmapsto f \otimes_B h.$$

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3. Equivariant K-Theory

We now consider $KK^{\mathcal{G}}(C_0(Z), C_0(X))$ under the condition that $\mathcal{G}$ is proper.

Remark 3.1. *Properness of the groupoid is required to ensure topological dependence. For example, if G is an infinite group and X a G-space, then $KK^G(\mathbb{C}, C_0(X))$ is* **NOT** *topological in nature. It is related to coarse geometry.*

Kasparov cycles $(\mathcal{E}, F)$ in this case are of the form: $\mathcal{E}$ is a field of Hilbert spaces $\{\mathcal{H}_x | x \in X\}$ over X vanishing at infinity; and F corresponds to a field of operators $\{F_x | x \in X\}$ such that certain Fredholm conditions hold, which we now discuss.

Fredholm condition: $f \in C_0(Z), f \cdot (F^2 - 1) \in \mathcal{K}(\mathcal{E})$ implies that for each $x \in X$, F_x is a Fredholm. Moreover, the $\{F_x\}$ is a norm continuous family, which vanishes at infinity (in the sense that, $(\rho \circ p_X) \cdot (F_x^2 - 1)$ is required to be a norm continuous compact operator valued function that vanishes at infinity). This last condition should to be thought of as a condition on the support (of $\{F_x\}$) in the vertical direction (i.e., in the direction of the fibers of X over Z).
We note that, in this case, we can assume that F is $\mathcal{G}$-equivariant (rather than just equivariant up to compacts).

Theorem 3.2. *Let $\mathcal{H}_{\mathcal{G}}$ be a bundle of Hilbert spaces $x \longmapsto L^2(\mathcal{G}^x) \otimes l^2(\mathbb{N})$. Then $KK^{\mathcal{G}}(C_0(Z), C_0(X))$ is isomorphic to homotopy classes of $\mathcal{G}$-equivariant maps $X \longrightarrow \coprod_{x \in X} Fred(\mathcal{H}_{\mathcal{G}^x})$. (The topology on the space $\coprod_{x \in X} Fred(\mathcal{H}_{\mathcal{G}^x})$ is defined in* [1]*).*

Definition 3.3. Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathcal{G}^{(0)}$ be a groupoid, X a proper $\mathcal{G}$-space. Then the $\mathcal{G}$-equivariant representable K-theory of X is

$$\mathcal{R}K_{\mathcal{G}}(X) := KK^{\mathcal{G} \rtimes X}(C_0(X), C_0(X)).$$

The $\mathcal{G}$-equivariant representable K-theory of Y, where Y is a space over X with X-compact supports is

$$\mathcal{R}K_{\mathcal{G}}(X) := KK^{\mathcal{G} \rtimes X}(C_0(X), C_0(Y)).$$

Example 3.4. We now consider an example of a class in representable K-theory. Let X be a proper $\mathcal{G}$-space, V a $\mathcal{G}$-equivariant vector bundle over X, which is K-oriented. Fix a $\mathcal{G}$-equivariant Euclidean metric on V and form the Clifford bundle $\mathbb{C}l(V)$ over X (i.e. the family $\{\mathbb{C}l(V_x) | x \in X\}$). The K-oriented condition implies that there exists a $\mathcal{G}$-vector bundle S over X such that $c_x : \mathbb{C}l(V_x) \longrightarrow \mathrm{End}(S_x)$ are $*$-isomorphisms for each x (i.e. we have $c : \mathbb{C}l(V) \cong \mathrm{End}(S)$). This is a global condition. Locally, we always have such an S and K-orientablity allows us to "paste" these locally defined objects together.

Next, we let $\mathcal{E}$ be the sections of the bundle $\pi_V^*(S)$ and note that it forms a $C_0(X)$-module. We define $F \in \mathcal{B}(\mathcal{H})$ as follows: If $x \in X$ and $v \in V_x$, then F_v is define to act as Clifford multiplication by v (i.e. acts by $c(v) \in \mathrm{End}(S)$). One can can check that $(\mathcal{E}, F)$ forms a cycle in $KK^{\mathcal{G}\rtimes X}(C_0(X), C_0(V))$ (i.e. $RK^{*}_{\mathcal{G},X}(V)$). The class of this cycle is call the Thom class. We note that it is invertible and induces the Thom isomorphism (in representable K-theory).

4. K-homology

Suppose (X, p_X) is a proper $\mathcal{G}$-equivariant K-oriented manifold. We now construct the (fiberwise) Dirac operator D_X of X. To begin, fix a cover of X by charts containing open sets of the form $L \times \mathbb{R}^n$ where $L \subseteq Z$ (recall Z is the base space of $\mathcal{G}$). Moreover, we assume that given any chart homeomorphism, $\phi : U \to L \times \mathbb{R}^n$, we have that, (for $x \in U$), $p_X(x) = (proj_1 \circ \phi)(x)$. We then define the fiberwise Dirac operator D_X to be the pullback of the Dirac operators on the $\mathbb{R}^n$ fibers of the open cover described above. We denote this by D_X and denote its class by $[D_X] \in KK^{\mathcal{G}}(C_0(X), C_0(Z))$.

Example 4.1. Let V be a K-oriented $\mathcal{G}$-equivariant vector bundle over X. Then $[D_V] \in KK^{\mathcal{G}\rtimes X}(C_0(V), C_0(X))$ is the inverse to $[\xi_V]$ (where $[\xi_V]$ is the Thom class which was defined in the previous section).

5. Duality

Theorem 5.1 (Kasparov)**.** *For a compact manifold X, there exists a canonical isomorphism*

$$KK(C(X), \mathbb{C}) \longrightarrow KK(\mathbb{C}, C_0(TX))$$

which maps the class $[D]$ *of all elliptic operator on X to the class* $[\sigma_D]$ *of its symbol* $[\sigma_D] \in KK(\mathbb{C}, C_0(TX))$.

Definition 5.2. Let X be a $\mathcal{G}$-space. An *abstract dual* for X is a pair (P, θ) where P is a $\mathcal{G}$-algebra, and $\theta \in KK^{\mathcal{G}\rtimes X}(C_0(X), C_0(X) \otimes P)$ such that the map

$$KK^{\mathcal{G}}(P\otimes A, B) \longrightarrow KK^{\mathcal{G}\rtimes X}(C_0(X)\otimes P\otimes A, C_0(X)\otimes B) \longrightarrow^{G\otimes} KK^{\mathcal{G}\rtimes X}(C_0(X)\times A, C_0(X)\otimes B)$$

is an isomorphism for all A and B.

Definition 5.3. A *Kasparov dual* for X is a triple (P, D, θ) where P is a $\mathcal{G}\rtimes X$-algebra, $\theta \in KK^{\mathcal{G}\rtimes X}(C_0(X), C_0(X) \otimes P)$ and $D \in KK^{\mathcal{G}}(P, C_0(Z)$ with the properties

(1) $\theta \otimes_P D = \mathbf{1}_X \in KK^{\mathcal{G}\rtimes X}(C_0(X), C_0(X))$;
(2) $\theta \otimes_X f = \theta \otimes_P T_P(f) \in KK^{\mathcal{G}\rtimes X}(C_0(X)\otimes A, C_0(X)\otimes B\otimes P)$ for $f \in KK^{\mathcal{G}\rtimes X}(C_0(X)\otimes A, C_0(X) \otimes B)$;
(3) $T_P(\theta) \otimes_{P\otimes P} f_P = T_P(\theta)$.

We note that T_P is defined by the following sequence of maps:

$$KK^{\mathcal{G}\rtimes X}(C_0(X)\otimes A, C_0(X)\otimes B) \to KK^{\mathcal{G}\rtimes X}(C_0(X)\otimes A\otimes P, C_0(X)\otimes B\otimes P) \to KK^{\mathcal{G}\rtimes X}(A\otimes P, B\otimes P)$$

where the first map is given by exterior product with $[id_P] \in KK^{\mathcal{G}}(P, P)$ and the second forgets the $C_0(X)$-structure.

Theorem 5.4. *If (P, D, θ) is a Kasparov dual for X, then (P, θ) is an abstract dual and the inverse of the duality map in Definition 5.2 is given by*

$$(\bullet \otimes [D]) \circ T_p : KK^{\mathcal{G}\rtimes X}(C_0(X) \otimes A, C_0(X) \otimes B) \to KK^{\mathcal{G}}(P \otimes A, B)$$

Remark 5.5. *We note that this form of duality occurs naturally in certain nonsmooth cases.*

Theorem 5.6. *Let X be a smooth proper $\mathcal{G}$-manifold. Then there exists natural isomorphisms*

$$\begin{aligned} KK^{\mathcal{G}}(C_0(TX) \otimes A, B) &\cong KK^{\mathcal{G}\rtimes X}(C_0(X) \otimes A, C_0(X) \otimes B) \\ KK^{\mathcal{G}X}(C_0(X) \otimes A, B) &\cong KK^{\mathcal{G}\rtimes X}(C_0(X) \otimes A, C_0(TX) \otimes B) \end{aligned}$$

References

[1] H. Emerson and R. Meyer *Equivariant representable K-theory*, J. Topol. 2 (2009), no. 1, 123-156.