DIRAC AND DUAL DIRAC METHOD

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1. Dirac and dual Dirac element on $\mathbb{R}^n$

1.1. **Bott (dual Dirac) element** $[\beta]$**.** Let $C_n = \mathrm{Cliff}(\mathbb{R}^n)\otimes\mathbb{C}$ be complex Clifford algebra generated by bases of $\mathbb{R}^n$: $\varepsilon_1, \cdots, \varepsilon_n$ under the relation $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 2\delta_{ij}$ and $\varepsilon_i^* = \varepsilon_i$.($\mathbb{R}$ is viewed as a subalgebra of C_n)

Define $C_\tau(\mathbb{R}^n) = C_0(\mathbb{R}^n) \otimes C_n$, its multiplier $\mathcal{M}(C_\tau(\mathbb{R}^n))$ can be identified as $C_b(\mathbb{R}^n, C_n)$, the bounded map on $\mathbb{R}^n$ with coefficient in C_n. (Recall that multiplier of $C_\tau(\mathbb{R}^n)$ is the same as $\mathcal{L}(C_\tau(\mathbb{R}^n))$).

For $x \in \mathbb{R}^n$ we define Bott element $\beta : \mathbb{R}^n \to C_n$ by

$$\beta(x) = \frac{x}{\sqrt{x^2+1}},$$

and so $\beta \in C_b(\mathbb{R}^n, C_n) = \mathcal{L}(C_\tau(\mathbb{R}^n))$.

View $\mathcal{E} = C_\tau(\mathbb{R}^n)$ as Hilbert module over itself and β as above, then it is easy to check the class $(\mathcal{E}, \beta)$ defines an element $[\beta]$ in $K_0(C_\tau(\mathbb{R}^n)) = KK_0(\mathbb{C}, C_\tau(\mathbb{R}^n))$ (K-theory for graded algebra), we will call $[\beta]$ Bott element or dual Dirac element.

1.2. **Dirac element** $[D]$**.** We will construct Dirac element $[D] \in K^0(C_\tau(\mathbb{R}^n)) = KK_0(C_\tau(\mathbb{R}^n), \mathbb{C})$.

We view $\Lambda^*(\mathbb{R}^n)$ as a finite dimensional Hilbert space and on it we define a representation of $C_n \to \mathcal{L}(\Lambda^*(\mathbb{R}^n))$ by

$$(\xi \in \mathbb{R}^n) \mapsto c_-(\xi) = \mathrm{ext}(\xi) - \mathrm{int}(\xi)$$

where $c(\xi)$ means clifford multiplication by ξ, ext means the exterior multiplication by ξ and int means the adjoint of ext.

We also take Hilbert space $L^2(\mathbb{R}^n)$ and on it we have a representation of $C_0(\mathbb{R}^n)$ on it by multiplication.

Combining the two we get a representation of $\varphi : C_\tau(\mathbb{R}^n) = C_0(\mathbb{R}^n) \otimes C_n$ on $L^2(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^n) = L^2(\Lambda^*(\mathbb{R}^n))$.

Denote $H = L^2(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^n) = L^2(\Lambda^*(\mathbb{R}^n))$ and $\partial_i = \frac{\partial}{\partial x_i}$ define operator D on H by

$$D = \sum_i c_-(e_i)\partial_i$$

where $e_i \in \mathbb{R}^n \subset C_n$ are generators for $\mathbb{R}^n$.

Remark 1.1. Using the fact that $c_-(\xi^*) = -c_-(\xi), c_-(\xi)^2 = -\|\xi\|^2, \partial_i^* = -\partial_i$ we know D self-adjoint.

Proposition 1.2. $(H, \phi, D) = [D] \in K^0(C_\tau(\mathbb{R}^n))$, $[D]$ *or its bounded version* $[F] = [D(1+D^2)^{-\frac{1}{2}}]$ *is called the Dirac element of* $\mathbb{R}^n$.

Proof. (1)$\varphi(a)(F^2-1)\in\mathcal{K}(H),\forall a\in C_\tau(\mathbb{R}^n)$.

We may assume $a\in C_c(\mathbb{R}^n,C_n)$, then $\varphi(a)(1-F^2)\in\mathcal{K}(H)$ follows from Rellich lemma (i.e. the inverse of an elliptic differential operator multiplied by a function with compact support is a compact operator).

(2) $[\varphi(a),F]\in\mathcal{K}(H)$.

$$F=\frac{D}{\sqrt{D^2+1}}=\frac{2}{\pi}\int_0^\infty\frac{D}{1+D^2+\lambda^2}\mathrm{d}\lambda$$

is strongly convergent. From this we have

$$[\varphi(a),F]=\frac{2}{\pi}\int_0^\infty(1+D^2+\lambda^2)^{-1}([\varphi(a),D](1+\lambda^2)+D[\varphi(a),D]D)(1+D^2+\lambda^2)^{-1}\mathrm{d}\lambda$$

The integrant is compact and the integral converges in norm. Assume $a\in C_c(\mathbb{R}^n,C_n)$ and still use Rellich lemma we get the claim. □

1.3. **Product of Dirac and dual Dirac.** What is the intersection product of $[\beta]$ and $[D]$ in $K_0(C_\tau(\mathbb{R}^n))\otimes K^0(C_\tau(\mathbb{R}^n))\to\mathbb{Z}$?

Recall that Bott element acts on $\mathcal{E}=C_\tau(\mathbb{R}^n)$, then the product should act on Hilbert space $\mathcal{E}\otimes_{C_\tau(\mathbb{R}^n)}L^2(\Lambda^*(\mathbb{R}^n))=L^2(\Lambda^*(\mathbb{R}^n))$.

We now want to find an operator Φ corresponds to the product,i.e.

$$\Phi\in\frac{x}{\sqrt{x^2+1}}\sharp\frac{D}{\sqrt{1+D^2}}.$$

Let $c_+:C_n\to\mathcal{L}(L^2(\Lambda^*(\mathbb{R}^n)))$ be the Clifford multiplication operator defined by $(\xi\in\mathbb{R}^n)\mapsto c_+(\xi)=\mathrm{ext}(\xi)+\mathrm{int}(\xi)$ (property: $c_+(\xi)^2=\|\xi\|^2,c_+(\xi)^*=c_+(\xi)$). And we claim

$$\Phi=\frac{D+c_+(x)}{\sqrt{1+(D+c_+(x))^2}}.$$

We denote $B=D+c_+(x):L^2(\Lambda^*\mathbb{R}^n)\to L^2(\Lambda^*\mathbb{R}^n)$ as the unbounded version of Φ, it is related to harmonic oscillator theory. In fact we have

$$B^2=\sum_{k=1}^n(-\frac{\partial}{\partial x_k^2}+x_k^2)+2N-n,N=\deg(\omega),\omega\in L^2(\Lambda^*\mathbb{R}^n),$$

a harmonic oscillator operator.

Remark 1.3. B is an elliptic differential operator on $\mathbb{R}^n$. In fact, it is obtained by the (exterior) intersection product of $B_i=\begin{pmatrix}0 & D_i+c_+(x_i)\\ (D_i+c_+(x_i))^* & 0\end{pmatrix}$ on $L^2(\Lambda^{odd}\mathbb{R})\oplus L^2(\Lambda^{even}\mathbb{R})$: $B=B_1\hat{\otimes}1\hat{\otimes}\cdots\hat{\otimes}1+\cdots+1\hat{\otimes}\cdots\hat{\otimes}1\hat{\otimes}B_n$. For example,

$$B_1^2|_{L^2(\Lambda^{even}\mathbb{R})}=-\frac{\partial}{\partial x_1^2}+x_1^2-1=(-\frac{\partial}{\partial x_1}+x_1)(\frac{\partial}{\partial x_1}+x_1),$$

It is a positive elliptic operator, $\ker(D_1+c_+(x_i))=\{e^{-\frac{x^2}{2}}\}$ and one can find its eigenvalues are $0,2,4,\cdots$.

We need to verify that Φ satisfy the property to be the product of Bott and Dirac element, and that are done in the following lemmas.

Lemma 1.4. Φ *is Fredholm.*

Proof. $\Phi = \frac{B}{\sqrt{B^2+1}}$, where B is an unbounded elliptic operator with eigenvalues $0, 2, 4, \cdots$, then $1-(\Phi)^2 = \frac{1}{1+B^2}$ has eigenvalues $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots$ and therefore implied compactness. □

Lemma 1.5. *$\Phi = \frac{B}{\sqrt{B^2+1}}$ is a $F = \frac{D}{\sqrt{1+D^2}}$-connection.*

Proof. Observe that F can be viewed as a F-connection, so it is enough to show $\varphi(a) \otimes 1(\Phi - F) \in \mathcal{K}(L^2(\Lambda^*\mathbb{R}^n)), \forall a \in C_\tau(\mathbb{R}^n)$. But it is easy if we express Φ and F in integrals and calculate and then apply Rellich lemma. □

Lemma 1.6. *$\forall a \in C_\tau(\mathbb{R}^n), \varphi(a) \otimes 1[\frac{x}{\sqrt{1+x^2}}, B]\varphi(a) \otimes 1 \geq 0$ up to compact operators.*

Remark 1.7. B has one dimensional kernel and surjective, $\mathrm{ind}(\Phi) = \mathrm{ind}(B) = 1$.

In fact both the Bott and Dirac element are invariant under rotation, so Φ can be viewed as the multiplicative identity in the representation ring of $O(n)$.

In general, If $\mathbb{R}^n$ admit an action from a local compact group G, the orient-preserving action on $\mathbb{R}^n$ is the composition of translation and rotation.

If $g \in G, g : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x - a$, $\frac{x}{\sqrt{x^2+1}} - \frac{x-a}{\sqrt{(x-a)^2+1}} \in C_0(\mathbb{R}^n, C_n)$, and $a(g(F)-F) \in \mathcal{K}(L^2(\Lambda^*\mathbb{R}^n))$. The Bott and Dirac element can be viewed as elements of equivariant KK-theory and there paring is in $KK^G(\mathbb{C}, \mathbb{C})$.

1.4. **Bott periodicity.** Now $[\beta] \in KK(\mathbb{C}, C_\tau(\mathbb{R}^n)), [D] \in KK(C_\tau(\mathbb{R}^n), \mathbb{C})$, and we know that $[\beta] \otimes_{C_\tau(\mathbb{R}^n)} [D] = 1 \in \mathbb{Z}$. By Atiyah's rotation trick $[D] \otimes_{\mathbb{C}} [\beta] = 1_{C_\tau(\mathbb{R}^n)} \in KK(C_\tau(\mathbb{R}^n), C_\tau(\mathbb{R}^n))$. Then we get Bott isomorphism

$$\otimes[\beta] : K_0(A) \to K_0(A \hat{\otimes} C_\tau(\mathbb{R}^n)).$$

2. Dirac and dual Dirac element on complete Riemannian manifold

2.1. **Dirac element.** Let X be a complete Riemannian manifold G-manifold, τ is the cotangent vector bundle of X equipped with Riemannian metric and involution. Consider the Clifford bundle $Cliff(\tau, \mathbb{Q})$ associated to the cotangent bundle. Let $C_\tau(X)$ denote the sections of bundle of Clifford algebras vanish at infinity on$X$.

The graded Hilbert space is $H = L^2(\Lambda^*(X))$, L^2-forms on X. On it there is a homomorphism $\varphi : C_\tau(X) \to \mathcal{L}(H)$ by Clifford multiplication of $\mathrm{ext}(\xi) + \mathrm{int}(\xi)$ on the forms.

Let d be exterior derivative and d^* be its adjoint. One can check that $D = \mathrm{d} + \mathrm{d}^*$ is an essential self-adjoint element on H, and $F = D(D^2+1)^{-\frac{1}{2}} \in \mathcal{L}(H)$. Use the similar procedure as in the last section, we get a Dirac element $[D]$ in $K^0_G(C_\tau(X))$.

Lemma 2.1. *$(L^2(\Lambda^*X), \varphi, F)$ is a $(C_\tau(X), \mathbb{C})$-bimodule and its class gives an element in $KK^G(C_\tau(X), \mathbb{C}) = K^0_G(C_\tau(X))$ and denoted by $[D]$, which will be called the Dirac element of the manifold X.*

2.2. **construction of Bott (dual Dirac) element.** While Dirac element exists globally for a Riemannian manifold, there is generally no global Bott(dual Dirac) element. However, we can always construct a local dual Dirac element $[\Theta]$ for a complete Riemannian manifold, as an element in $RKK^G(X, \mathbb{C}, C_\tau(X))$.

$\forall x \in X, \exists$ a neighborhood $U_x \subset X$, the image of a small open ball in T_xX under the exponential map, such that for any two points in U_x there is unique geodesic connecting them. Let r_x be the radius of U_x and $s(x, y)$ be the distance between x and y.

In analogy with the definition of the Bott element on $\mathbb{R}^n$ we have the same definition on U_x under the homeomorphism of it with the euclidean space. Precisely, The one form Θ_x defined by

$$\Theta_x(y) = \frac{s(x,y)}{r_x}(d_y s)(x,y)$$

on U_x is the Bott (dual Dirac) element in local sense. (Note that $\Theta_x \in \mathcal{L}(C_\tau(U_x))$ and $\Theta_x^2 - 1$ vanish at the boundary of U_x, so it is compact when consider $C_\tau(U_x) = C_0(U_x) \otimes C_n$ as a Hilbert module over $C_\tau(X)$.)

Definition 2.2. The field of pairs $\{(C_\tau(U_x), \Theta_x)_{x\in X}\}$ gives an element of $RKK^G(X; \mathbb{C}, C_\tau(X))$, and we denote the element by $[\Theta]$, this is the dual Dirac element we are looking for.

Remark 2.3. $[\Theta]$ can be viewed as an element in $KK^G(C_0(X), C_0(X) \otimes C_\tau(X))$ with the special requirement that the action of $C_0(X)$ is represent on the Hilbert $C_0(X) \otimes C_\tau(X)$ by multiplication and operator Θ_x always commute with the representation.

Recall that for any σ-unital $G - C_0(X)$-algebra D, there is a natural homomorphism

$$\sigma_{X,D} : RKK^G(X; A, B) \to KK^G(A \hat{\otimes}_{C_0(X)} D, B \hat{\otimes}_{C_0(X)} D).$$

Lemma 2.4.

$$\sigma_{X,C_\tau(X)}(\Theta) \in KK^G(C_\tau(X), C_\tau(X) \hat{\otimes} C_\tau(X))$$

and the element is invariant under the flip automorphism of $C_\tau(X) \hat{\otimes} C_\tau(X)$.

2.3. **Duality.**

Theorem 2.5. *Let X be a G-manifold, then*

$$[\Theta] \otimes_{C_\tau(X)} [D] = 1_X \in RK_G^0(X) and$$

$$\sigma_{X,C_\tau(X)}([\Theta]) \otimes_{C_\tau(X)} [D] = 1_{C_\tau(X)} \in KK^G(C_\tau(X), C_\tau(X)).$$

Proof. For the second statement, there are two possibilities to take the product with Dirac element, but the last lemma shows that they lead to the same result. The second statement follows from the first one.

The product of the first statement is a family of pairs $(H_x, S_x), x \in X$, where

$$H_x = L^2(\Lambda^*(U_x)), S_x = \Theta_x + (1 - \Theta_x^2)^{\frac{1}{4}} F (1 - \Theta_x^2)^{\frac{1}{4}}.$$

One need to show this element is 1 in $RK_G^0(X)$ use a homotopy. For detail see [1]4.8 page 181. □

Theorem 2.6 (Poincare Duality)**.** *Let X be G-manifold and A, B be G-algebra, then*

$$RKK^G(X; A, B) \cong KK^G(C_\tau(X) \hat{\otimes} A, B).$$

Proof. "$\Rightarrow$" is given by $\sigma_{X,C_\tau(X)} : RKK^G(X; A, B) \to KK^G(C_\tau(X) \hat{\otimes} A, C_\tau(X) \hat{\otimes} B)$ and then take intersection product with $[D] \in KK^G(C_\tau(C), \mathbb{C})$.

" $\Leftarrow$ " is essentially taking intersection product of $[\Theta]$ with elements in $KK^G(C_\tau(X) \hat{\otimes} A, B)$. □

3. SPECIAL MANIFOLD AND γ ELEMENT

3.1. **special manifold.** Special manifold a manifold where there is a global dual Dirac operator on X.

Definition 3.1. A G-manifold Xwill be called special if there is an element $\eta \in K_0^G(C_\tau(X))$ satisfying one of the following equivalent conditions:
(1) $p^*(\eta) = [\Theta] \in RKK^G(X; \mathbb{C}, C_\tau(X))$ where $p : X \to pt$.
(2) $[D] \otimes \eta = 1_{C_\tau(X)} \in KK^G(C_\tau(X), C_\tau(X))$.

Remark 3.2. The element η is defined uniquely. It is the global dual Dirac element. The intersection product $\eta \otimes [D] \in R(G)$ is denoted by γ.

Remark 3.3. property (2) follows from theorem 2.6 and implies the surjectivity part for the Bott periodicity when X is a euclidean space.

Example 3.4. Let X be a simply connected G-manifold of non-positive sectional curvature, then X is special.

To see that, consider $C_\tau(X)$ as a Hilbert module over itself and for a fixed $x \in X$ define $\xi_x(y) = \mathrm{d}_y(1 + s(x,y)^2)^{\frac{1}{2}}$, an operator of left multiplication by the 1-form ξ_x as an element $\xi_x \in \mathcal{L}(C_\tau(X))$. We claim that $(C_\tau(X), \xi_x) \in K_0^G(C_\tau(X))$, is the global dual Dirac element.

(Use the cosine inequality on non-positive curvature manifold $a^2 + b^2 - c^2 \leq 2ab\cos\alpha$ one can show the crucial property ξ_x is G-continuous, $g(\xi_x) - \xi_x \in \mathcal{K}(C_\tau(X))$, and $\xi_x - \xi_{x'} \in \mathcal{K}(C_\tau(X)), \forall x' \in X$.)

Theorem 3.5. *Let $P \to Z$ be a principal fiber bundle with fiber Γ. Assume P has a G action which commutes with Γ action and the projection. Let Y be a Γ-manifold and form G-manifold $X = P \times_\Gamma Y$, a bundle over Z $(q : X \to Z)$.*

We choose a Riemannian metric which compatible with the metric on the fiber and such that $q^* : T^*_{q(x)}Z \to T^*_x X$ is isometric for all $x \in X$.

If Z is a special G-manifold and Y is a special Γ-manifold, X is a special G-manifold.

Proof. Construction of the dual Dirac element on X will be given here, for detailed proof see [1] 5.4 on page 186.

Since Y and Z are special, there are element $\theta \in K_0^\Gamma(C_\tau(Y))$ and $\xi \in K_0^G(C_\tau(Z))$.

Let E be the algebra of all bounded Γ-equivariant continuous maps $P \to C_\tau(Y)$ and $E_0 = C_0(X) \cdot E$.

We can define an operator on E_0 (considered as a Hilbert module over itself) by averaging:

$$\tilde{\theta}(x) = \int_\Gamma c(g^{-1}x) g(\theta(x)) \mathrm{d}g, x \in Y,$$

where $c : Y \to \mathbb{R}_+$ is a cut off function on Y(i.e. $\int_\Gamma c(g^{-1}x)\mathrm{d}g = 1, \forall x \in Y$).

This defines an element of the group $KK^G(C(Z), E_0)$ with the additional property that the action of $C_0(Z)$ on E_0 is by multiplication via the homomorphism q^*. Also, $\sigma_{C_\tau(Z)}(\tilde{\theta}) \in KK^G(C_\tau(Z), C_\tau(X))$, then

$$\xi \otimes_{C_\tau(Z)} \sigma_{C_\tau(Z)}(\theta) \in K_0^G(C_\tau(X))$$

is the global dual Dirac element of X. Property (1) of the definition 3.1 is easy to verify. $\square$

3.2. γ **element.** Let G_0 be the connected component of identity of the group G, G is called almost connected if G/G_0 is compact. We are now going to prove that all homogeneous space G/K where G is almost connected and K is its maximal compact subgroup, are special G-manifold.

Theorem 3.6. *Let G be an almost connected group and K is its maximal compact subgroup, then $X = G/K$ is a special G. The element $\gamma \in R(G)$ does not depend on the choice of K and will denoted by γ_G.*

The following structure lemma is to be used to prove the theorem:

Lemma 3.7. *Let G be almost connected, then there is a series of normal subgroups of G: $\{1\} = N_0 \subset N_1 \subset \cdots \subset N_m \subset G$, so that N_{k+1}/N_k are either compact or Euclidean $\mathbb{R}^n$, and G/N_m is a semisimple Lie group.*

Idea of proof of the last theorem:

Consider G/N_{m-i} and take induction on i:

When $i = 0$, we need to prove a semisimple group G quotient its maximal compact K is G special. Denote π be the center of G and Γ be the inverse of the maximal compact subgroup of $G' = G/\pi$ in G,then K acts on Γ and one can show that $Y = \Gamma/K$ is homeomorphic to $\mathbb{R}^n$ and $G/K = G \times_\Gamma Y$, observe that $G/\Gamma = G'/K'(K'$ is maximal compact subgroup of $G')$ is a Riemannian symmetric space of noncompact type, hence have non-positive sectional curvature. Since G/Γ and Y are G and Γ special respectively, G/K is G-special by theorem 3.4.

Now assume G/N_{k+1} have the property and we want to prove G/N_k also have the property. Replace N_{k+1}/N_k by Γ and G/N_k by G, then it is equivalent to prove if Γ is compact or Euclidean as a normal group of G with statement true for G/Γ, then it is true for G. This can be proved by theorem 3.4 again.

γ element does not depend on K because all maximal compact subgroup of G are $\mathrm{Ad}(G_0)$-conjugate. (G_0 is connected component of identity in G).

Corollary 3.8. *Let G be an almost connected group and K its maximal compact subgroup. Then for any σ-compact G-space Y, any separable G-algebra A and any G-algebra B, the restriction homomorphism*

$$\mathrm{res}_K^G : RKK^G(Y; A, B) \to RKK^K(Y; A, B)$$

maps the subgroup $\gamma_G \cdot RKK^G(Y; A, B)$ isomorphically onto $RKK^K(Y; A, B)$.

In particular, $\gamma_G \cdot R(G) \cong R(K)$.

Proof. [1] page 189 □

Theorem 3.9. *Let $f : G_1 \to G_2$ be a homomorphism between almost connected groups with $\mathrm{Ker}\, f$ amenable and $\mathrm{Im} f$ closed. Then $\mathrm{res}_{G_1}^{G_2}(\gamma_{(G_2)}) = \gamma_{(G_1)}$. In particular, $\gamma_{(G)} = 1$ for amenable almost connected group G.*

Proof. [1] theorem 5.9. □

Theorem 3.10. *If $\gamma_G = 1$, then the Baum-Connes conjecture holds for G with arbitrary coefficient.*

For example, $\gamma_G = 1$ for $SU(n,1), SO(n,1)$. But for $Sp(n,1)$, γ_G is not 1.

Reference:

[1] Kasparov, G.G., Equivariant KK-theory and the Novikov conjecture, Inventiones mathematicae, Volume 91,1988.