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Dimension Functions on Simple C^*-Algebras

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In order to make available for C^*-algebras the results of Goodearl and Handelman [5] on existence and uniqueness of rank functions on regular rings, we associate in the present note with every C^*-algebra A an abelian group $K_0^*(A)$. The construction of this group is analogous to the construction of the Grothendieck group $K_0(A)$ (which recently has been applied successfully to the classification of AF-algebras by G. Elliott [4]) but the group $K_0^*(A)$ itself is in general quite different from $K_0(A)$. We are mainly interested in the case where A is a simple C^*-algebra with unit. If A is such an algebra we call A finite, if $x^*x=1$ implies $xx^*=1$ ($x\in A$), and we call A stably finite if $M_n\otimes A$ ($M_n = C^*$-algebra of $n\times n$ complex matrices) is finite for all $n\in\mathbb{N}$. Then $K_0^*(A)$ is non-trivial if and only if A is stably finite. We note, incidentally, that it was shown in [3, 2.4] that a simple C^*-algebra A with unit is stably finite if and only if $\mathcal{K}\otimes A$ ($\mathcal{K}=C^*$-algebra of compact operators on a separable infinite-dimensional Hilbert space) contains at least one non-trivial ideal (which can of course not be closed). It is an open problem which is probably difficult to decide if every finite simple C^*-algebra with unit is stably finite. However, if one studies traces on A, one may restrict attention to stably finite algebras. In fact, if there is a trace on A, then A is stably finite.

In the classification of factors by Murray and von Neumann [7] a decisive role is played by dimension functions. These functions are closely related to traces on the factors. In the case of simple C^*-algebras, we define, for every finite trace τ on A, a dimension function $\bar{R}_\tau$ which is related in a similar way to τ. In particular, τ can be recovered from $\bar{R}_\tau$ by integration. Thus, results on dimension functions will in particular yield information about traces on A.

We equip $K_0^*(A)$ with a partial order in the same way in which this is done for K_0 in [4] and [5]. Then order preserving homomorphisms from $K_0^*(A)$ to $\mathbb{R}$ correspond in a one to one fashion to dimension functions on A. This allows us to apply the abstract results on ordered abelian groups in [5] to the study of dimension functions on C^*-algebras. In particular, we get lower and upper bounds for all dimension functions on A and we get the existence of sufficiently many dimension functions on every stably finite simple C^*-algebra with unit. As a

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consequence, a simple C^*-algebra with unit is stably finite if and only if it has a dimension function.

The ideas and techniques of [2] are basic for the present paper. However, instead of the relation $\precsim$ of [2], we use here a relation $\lesssim$ which takes into account the topological structure of the algebra. The main technical advantage of $\lesssim$ lies in the validity of Proposition 1.1. In Sections 5 and 6 we indicate without proofs some connections between the present ideas and the results in [2] and [4].

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1. Let B be a C^*-algebra (or an involutive subalgebra of a C^*-algebra admitting functional calculus) and $x, y \in B$. We write $x \precsim y$ if there are a, b in the C^*-algebra $\tilde{B}$, obtained from B by adjoining a unit, such that $x = ayb$.

Further we define a relation $\lesssim$ on B by $x \lesssim y$ if there is a sequence $\{x_n\}$ in B such that $x_n \precsim y$ and $x_n \to x$. This relation is clearly transitive and reflexive. We write $x \equiv y$ if $x \lesssim y$ and $y \lesssim x$. We have $x \equiv x^*$ and if $|x|$ denotes the absolute value $|x| = x^*x^{1/2}$ of x then for any element h of the C^*-algebra generated by $|x|$ we have $h \lesssim x$.

Denote by $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ the continuous function defined by $f_\varepsilon(t) = 0 \; (-\infty < t \leqq \varepsilon/2)$, f_ε is linear on $[\varepsilon/2, \varepsilon]$, $f_\varepsilon(t) = 1 \; (\varepsilon \leqq t < \infty)$. Then $yf_\varepsilon(|y|) \to y (\varepsilon \to 0)$ for every $y \in B$. If $x \lesssim y$ and $x_n = a_n y b_n \to x$ $(a_n, b_n \in \tilde{B})$, then $a_n y f_{\varepsilon_n}(|y|) b_n \to x$ if ε_n converges sufficiently fast to 0. Therefore we may assume that for each $n \in \mathbb{N}$ there is $\varepsilon > 0$ such that $x_n \precsim f_\varepsilon(|y|)$. We say that x is orthogonal to y $(x \perp y)$ if $xy = yx = x^*y = yx^* = 0$.

1.1. Proposition. *Let a, b, $\bar{a}$, $\bar{b} \in B$. If $\bar{a} \perp \bar{b}$ and $a \lesssim \bar{a}$, $b \lesssim \bar{b}$ then $a + b \lesssim \bar{a} + \bar{b}$.*

Proof. Let $a_n = x_n f_{\varepsilon_n}(|\bar{a}|) y_n \to a$ and $b_n = u_n f_{\varepsilon_n}(|\bar{b}|) v_n \to b$. Since $f_\varepsilon(|\bar{a} + \bar{b}|) = f_\varepsilon(|\bar{a}|) + f_\varepsilon(|\bar{b}|)$ and since

$$f_{\varepsilon/2}(|\bar{a}|) f_\varepsilon(|\bar{a}|) = f_\varepsilon(|\bar{a}|), \qquad f_{\varepsilon/2}(|\bar{a}|) f_\varepsilon(|\bar{b}|) = 0,$$

we get

$$a_n + b_n = (x_n f_{\varepsilon_n}(|\bar{a}|)^{1/2} + u_n f_{\varepsilon_n}(|\bar{b}|)^{1/2}) (f_{\varepsilon_n/2}(|\bar{a} + \bar{b}|)) (f_{\varepsilon_n}(|\bar{a}|)^{1/2} y_n + f_{\varepsilon_n}(|\bar{b}|)^{1/2} v_n).$$

Since $f_\varepsilon(|\bar{a} + \bar{b}|) \precsim \bar{a} + \bar{b}$ we have $a_n + b_n \lesssim \bar{a} + \bar{b}$, and the proof is complete.

2. For the rest of the paper A denotes a C^*-algebra with unit. We call a function $N : A \to [0, 1]$ a rank function if

(a) $N(1) = 1$

(b) $N(x) = 0$ if and only if $x = 0$.

(c) $N(x + y) = N(x) + N(y)$ if x, y are orthogonal elements of A.

(d) $x \lesssim y$ implies $N(x) \leqq N(y)$.

A function N which satisfies (a), (b), (c) and is lower semi-continuous is already a rank function, if it satisfies

(d)$_1$ $N(xy) \leqq N(x), N(y)$

or the equivalent condition

$$(\mathrm{d})_2 \quad x \lessapprox y \quad \text{implies} \quad N(x) \leqq N(y).$$

This follows at once from the definition of $\lesssim$.

We say that a state τ on A is a finite trace if $\tau(xy) = \tau(yx)$ $(x, y \in A)$.

2.1. Proposition. *Let τ be a finite trace on A. Then the limit $R_\tau(x) = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(|x|))$ exists for every $x \in A$ and the function R_τ thus defined is a lower semi-continuous rank function on A.*

Proof. The first assertion is obvious since $f_\varepsilon(|x|)$ is monotone increasing as $\varepsilon \to 0$, and $\leqq 1$.

Let us show that R_τ is lower semi-continuous. If $x_n \to x$, then $f_\varepsilon(|x_n|) \to f_\varepsilon(|x|)$ for every $\varepsilon > 0$ (it suffices to show this for a complex polynomial P in the place of f_ε). Thus by continuity of τ, for every $\alpha > 0$, there are ε_0, n_0 such that

$$R_\tau(x) \leqq \tau(f_\varepsilon(|x_n|)) + \alpha \quad (n \geqq n_0, \varepsilon < \varepsilon_0).$$

Thus $R_\tau(x) \leqq \liminf_{n \to \infty} R_\tau(x_n)$.

In order to prove that R_τ is a rank function, it suffices now to check condition $(\mathrm{d})_1$ [(a), (b), (c) are obviously fulfilled]. Let $x, y \in A$ and set $h = y^* x^* x y$. The obvious equation $\lim_{\varepsilon \to 0} f_\varepsilon(|y|)\, y^* = y^*$ yields

$$\lim_{\varepsilon \to 0} f_\varepsilon(|y|) P(h) = P(h)$$

for any complex polynomial P without constant coefficient. By continuity we may replace in this equation P by any continuous function on $\mathbb{R}$ vanishing at 0. In particular

$$\lim_{\varepsilon \to 0} f_\varepsilon(|y|) f_\delta(h^{1/2}) f_\varepsilon(|y|) = f_\delta(h^{1/2}).$$

Now we have $f_\varepsilon(|y|) f_\delta(h^{1/2}) f_\varepsilon(|y|) \leqq f_\varepsilon(|y|)$ and thus

$$\begin{aligned} R_\tau(y) &= \lim_{\varepsilon \to 0} \tau(f_\varepsilon(|y|)) \\ &\geqq \lim_{\varepsilon \to 0} \tau(f_\varepsilon(|y|) f_\delta(h^{1/2}) f_\varepsilon(|y|)) = \tau(f_\delta(h^{1/2})). \end{aligned}$$

Since finally $R_\tau(xy) = \lim_{\delta \to 0} \tau(f_\delta(h^{1/2}))$, this shows $R_\tau(xy) \leqq R_\tau(y)$. The corresponding relation for x follows from the identities $R_\tau(x) = R_\tau(x^*)$ and $R_\tau(xy) = R_\tau(y^* x^*)$.

2.2. Proposition. *Two finite traces which determine the same rank function, are equal.*

Proof. It was shown in [2, 4.3] that τ can be obtained from R_τ by integration. We include the argument for completeness. Let τ be a finite trace on A and $h = h^* \in A$. Then τ induces a positive measure $d\tau$ on the spectrum $\mathrm{Sp}\, h$ of h through the

relation

$$\int_{\mathrm{Sp}\,h} f\,d\tau = \tau(f(h))$$

for each continuous function f on $\mathrm{Sp}\,h$. Let g be any positive continuous function on $\mathrm{Sp}\,h$ and let $U_g = \{s \in \mathrm{Sp}\,h \mid g(s) > 0\}$. As ε tends to 0, the functions $f_\varepsilon \circ g$ converge pointwise to the characteristic function χ_{U_g} of U_g and

$$\int_{\mathrm{Sp}\,h} \chi_{U_g} d\tau = \lim_{\varepsilon \to 0} \int_{\mathrm{Sp}\,h} f_\varepsilon \circ g\, d\tau = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(g(h))) = R_\tau(g(h)).$$

This situation can be interpreted in the opposite sense. In fact, R_τ induces then a unique measure ϱ on $\mathrm{Sp}\,h$ by $\varrho(U) = R_\tau(g(h))$, where U is any open subset of $\mathrm{Sp}\,h$ and g is a positive continuous function on $\mathrm{Sp}\,h$ chosen in such a way that $U = U_g$ [1, 5.7]. From this point of view we get

$$\tau(g(h)) = \int_{\mathrm{Sp}\,h} g\,d\varrho$$

for all continuous functions g on $\mathrm{Sp}\,h$. In particular τ is uniquely determined by R_τ.

Remark. In the present terminology the function λ constructed in [2] is a rank function rather than a dimension function.

3. In the following we denote by $\mathscr{F}$ the algebra of operators of finite rank on an infinite-dimensional separable Hilbert space and by $\mathscr{F} \otimes A$ the algebraic tensor product of $\mathscr{F}$ and A. We note that any normal $x \in \mathscr{F} \otimes A$ admits functional calculus of continuous functions in $\mathscr{F} \otimes A$, since there is a finite-dimensional C^*-subalgebra M of $\mathscr{F}$ such that $x \in M \otimes A \subset \mathscr{F} \otimes A$, and $M \otimes A$ is a C^*-algebra.

We call a function $D : \mathscr{F} \otimes A \to [0, \infty[$ a dimension function (on A) if

(a) $D(p \otimes 1) = 1$ for any one-dimensional projection p in $\mathscr{F}$.
(b) $D(x) = 0$ if and only if $x = 0$.
(c) $D(x + y) = D(x) + D(y)$ for all orthogonal elements x, y of $\mathscr{F} \otimes A$.
(d) $x \lesssim y$ implies $D(x) \leqq D(y)$.

D is called a lower (upper) dimension function if instead of (c) we have $D(x+y) \leqq D(x) + D(y)$ $(D(x+y) \geqq D(x) + D(y))$ for all orthogonal $x, y \in \mathscr{F} \otimes A$, and if, in addition, $D(x_1 + \ldots + x_n) = nD(x)$, whenever $x_1, \ldots, x_n$ are orthogonal elements of $\mathscr{F} \otimes A$ such that $x_i \equiv x$ $(i = 1, \ldots, n)$.

If D is a dimension function and p is a one-dimensional projection in $\mathscr{F}$ then $p \otimes A$ is isomorphic to A and $N_D = D|_{p \otimes A}$ is a rank function. It does not seem clear that, conversely, every rank function on A should extend to a dimension function.

Remark. If there is a dimension function D on A, then A is stably finite. In fact, if A is not stable finite then we can find in $\mathscr{F} \otimes A$ two projections p and q such that $p \lneqq q$ and $p \equiv q$. Then $D(p) = D(q) = D(p) + D(q - p)$. This implies $D(q - p) = 0$ which is impossible.

3.1. Proposition. *Let D be a lower dimension function on A. Then $D(x+y) \leqq D(x) + D(y)$ for all $x, y \in \mathscr{F} \otimes A$.*

Proof. There are two orthogonal elements $\bar{x}, \bar{y} \in \mathscr{F} \otimes A$ such that $x \equiv \bar{x}$ and $y \equiv \bar{y}$. Using 1.1 we get $x+y \lesssim \bar{x}+\bar{y}$. But then

$$D(x+y) \leqq D(\bar{x}+\bar{y}) \leqq D(\bar{x}) + D(\bar{y}) = D(x) + D(y).$$

Remark. Let $D : \mathscr{F} \otimes A \to \mathbb{R}_+$ be a function satisfying conditions (a), (c), (d). Then by (c) $D(0)=0$, and in view of Proposition 3.1 and condition (d) the set $I = \{x \in \mathscr{F} \otimes A | D(x) = 0\}$ is an ideal in $\mathscr{F} \otimes A$. If we assume that A, hence $\mathscr{F} \otimes A$ is simple, we get $I = \{0\}$ and D is a dimension function.

3.2. Proposition. *Let τ be a finite trace on A. We set $\bar{\tau} = T \otimes \tau$, where T is the canonical trace on $\mathscr{F}$ and $T \otimes \tau$ is the unique linear functional on $\mathscr{F} \otimes A$ such that $T \otimes \tau(a \otimes b) = T(a)\tau(b)$, $(a \in \mathscr{F}, b \in A)$. Then the limit*

$$\bar{R}_\tau(x) = \lim_{\varepsilon \to 0} \bar{\tau}(f_\varepsilon(|x|))$$

exists for every $x \in \mathscr{F} \otimes A$ and the function $\bar{R}_\tau$ thus defined is a lower semi-continuous dimension function on A.

Proof. This follows from 2.1 since we may calculate for finitely many elements $x_1, \ldots, x_n$, the numbers $\bar{R}_\tau(x_i)$ $(i = 1, \ldots, n)$ in $M \otimes A \subset \mathscr{F} \otimes A$, where M is a suitable finite-dimensional subalgebra of $\mathscr{F}$.

Remark. R_τ is obtained from $\bar{R}_\tau$ by restriction to $p \otimes A \cong A$, where p is a one-dimensional projection in $\mathscr{F}$.

3.3. Corollary. *Let τ be a finite trace on A. Then*

$$R_\tau(x+y) \leqq R_\tau(x) + R_\tau(y) \quad \textit{for all} \quad x, y \in A.$$

3.4. Proposition. *Let τ_1, τ_2 be two finite traces on A such that $\bar{R}_{\tau_1} = \bar{R}_{\tau_2}$. Then $\tau_1 = \tau_2$.*

Proof. This follows from 2.2 and the remark above.

4. Throughout this section we let A be a simple C^*-algebra with unit. Given $x \in \mathscr{F} \otimes A$, we denote by $\langle x \rangle$ the $\equiv$-equivalence class of x in $\mathscr{F} \otimes A$. Let F be the free abelian group generated by $\{\langle x \rangle | x \in \mathscr{F} \otimes A\}$ and let R be the subgroup of F generated by all elements of the form $\langle x \rangle + \langle y \rangle - \langle x_1 + y_1 \rangle$ $(x_1 \in \langle x \rangle$, $y_1 \in \langle y \rangle$, $x_1 \perp y_1)$. Note that $\langle x_1 + y_1 \rangle$ does not depend on the choice of x_1, y_1 by Proposition 1.1. We denote by $K_0^*(A)$ the quotient F/R.

Let $[x]$ be the image of $x \in \mathscr{F} \otimes A$ in $K_0^*(A)$. Every element of $K_0^*(A)$ has obviously the form $[x] - [y]$ $(x, y \in \mathscr{F} \otimes A)$. Given $x, y \in \mathscr{F} \otimes A$, we have $[x] = [y]$ if and only if there is $a \in \mathscr{F} \otimes A$, $a \perp x$, $a \perp y$ such that $x + a \equiv y + a$. This follows from the proof of Lemma 1.1 in [6] together with Proposition 1.1.

If p is an infinite projection in $\mathscr{F} \otimes A$, then for every a in $\mathscr{F} \otimes A$ we have $a \lessapprox p$ (cf. [2, 2.2]). Hence $[p] + [x] = [p]$ for every x in $\mathscr{F} \otimes A$. In fact, if $x_1 \in [x]$ is orthogonal to p, then $p + x_1 \lesssim p$, and $p \lesssim p + x_1$ is obvious. Therefore $[x] = 0$ for all x in $\mathscr{F} \otimes A$ and this shows that $K_0^*(A)$ is trivial if A is not stably finite. We show

now in the following lemma that, for stably finite A, we have $[a] \neq 0$ whenever $a \neq 0$ in $\mathscr{F} \otimes A$.

4.1. Lemma. *Let A be stably finite and let a, b be orthogonal elements of $\mathscr{F} \otimes A$. Then $a+b \lesssim a$ implies $b=0$.*

Proof. Given $a, b \in \mathscr{F} \otimes A$ write $n\langle a\rangle \lesssim m\langle b\rangle$ if there are two families of orthogonal elements $a_1, \ldots, a_n \in \langle a\rangle$ and $b_1, \ldots, b_m \in \langle b\rangle$ such that $a_1 + \ldots + a_n \lesssim b_1 + \ldots + b_m$ in $\mathscr{F} \otimes A$. Let q be a one-dimensional projection in $\mathscr{F}$ and $\bar{q} = q \otimes 1 \in \mathscr{F} \otimes A$. Given $a \in \mathscr{F} \otimes A$ and $n \in \mathbb{N}$ denote by $s_n(a)$ the largest natural number s such that $s\langle \bar{q}\rangle \lesssim n\langle a\rangle$. Then $s_n(a)$ has the following properties

(a) If $a, b \in \mathscr{F} \otimes A$ and $a \lesssim b$, then $s_n(a) \leqq s_n(b)$.
(b) If $a, b \in \mathscr{F} \otimes A$ are orthogonal, then $s_n(a+b) \geqq s_n(a) + s_n(b)$.
(c) If $a \in \mathscr{F} \otimes A$ is non-zero, then $s_n(a) > 0$ for sufficiently large n.

For the proof of (a) note that $a \lesssim b$ implies $n\langle a\rangle \lesssim n\langle b\rangle$ by 1.1. Property (b) is also a consequence of 1.1. To prove (c) note that $\mathscr{F} \otimes A$ is algebraically simple and use [2, 1.1 IV] (if $(\bar{q}/a) \leqq n$, then $\langle \bar{q}\rangle \lesssim n\langle a\rangle$).

Since A is stably finite, $k\langle \bar{q}\rangle \lesssim n\langle a\rangle$ ($a \in \mathscr{F} \otimes A$ fixed) can not hold for all $k \in \mathbb{N}$ (note that for a projection p one has $p \lesssim x$ if and only if $p \precapprox x$ [2, 1.9] and use the equivalent definition of finiteness given in [2, 2.1] for $M_k \otimes A$). Thus $s_n(a)$ is finite for every $a \in \mathscr{F} \otimes A$.

Now, if $a, b \in \mathscr{F} \otimes A$ are orthogonal and $a+b \lesssim a$, then $s_n(a) + s_n(b) \leqq s_n(a+b) \leqq s_n(a)$. Thus, $s_n(b) = 0$ for all $n \in \mathbb{N}$. By (c) this implies $b=0$.

We define now an order on $K_0^*(A)$.

Definition. Given $[a]-[b]$ and $[c]-[d]$ in $K_0^*(A)$ we write $[a]-[b] \leqq [c]-[d]$ if there is $x \in \mathscr{F} \otimes A$ such that $a_1 + d_1 + x \lesssim c_1 + b_1 + x$ where $a_1 \in [a]$, $b_1 \in [b]$, $c_1 \in [c]$, $d_1 \in [d]$ are elements of $\mathscr{F} \otimes A$ which are pairwise orthogonal and orthogonal to x.

It is easily checked that $\leqq$ defines a translation-invariant partial order on $K_0^*(A)$, and that $K_0^*(A)$ is upward directed with $\leqq$. Hence $K_0^*(A)$ becomes a directed ordered abelian group. By a strong unit in $K_0^*(A)$ we mean an element $u > 0$ such that, for every x in $K_0^*(A)$, there is $n \in \mathbb{N}$ such that $x \leqq nu$.

4.2. Proposition. *Let A be stably finite.*

(1) $[x] > 0$ *for all non-zero* $x \in \mathscr{F} \otimes A$.

(2) *Every* $[x]$ $(0 \neq x \in \mathscr{F} \otimes A)$ *is a strong unit in* $K_0^*(A)$.

Proof. (1) We have $[x] \geqq 0$ by definition and $[x] \neq 0$ by 4.1. (2) This follows from the simplicity of $\mathscr{F} \otimes A$, if we use [2, 1.1 IV].

Following [5] we call an order preserving homomorphism $f: K_0^*(A) \to \mathbb{R}$ a functional. We say that f is normalized if $f([p \otimes 1]) = 1$ for all one-dimensional projections p in $\mathscr{F}$.

4.3. Proposition. *Let A be stably finite.*

(1) *If f is a normalized functional on $K_0^*(A)$, then the function D_f defined by $D_f(x) = f([x])$ is a dimension function on A.*

(2) *If D is a dimension function on A then there is a unique normalized functional f on $K_0^*(A)$ such that $D=D_f$.*

Proof. Cf. the proof of 2.4 in [5].

All that remains now to do is to translate the results on functionals in [5] to propositions about dimension functions.

4.4. Corollary. *Let A be stably finite. Then there is an upper dimension function D_- and a lower dimension function D_+ on A, such that*

$$D_-(x) \leqq D(x) \leqq D_+(x) \quad (x \in \mathscr{F} \otimes A)$$

for every dimension function D on A.

Proof. Set $D_-(x)=f_*([x])$ and $D_+(x)=f^*([x])$, where

$$f_*(t)=\sup\{m/n \mid m \geqq 0, n>0, m[p\otimes 1] \leqq nt\}$$
$$f^*(t)=\inf\{m/n \mid m, n>0, m[p\otimes 1] \geqq nt\}$$

are defined for $t \geqq 0$ in $K_0^*(A)$ as in [5, 4.1] (p a one-dimensional projection in $\mathscr{F}$).

4.5. Corollary. *Let A be stably finite and let $x \in \mathscr{F} \otimes A$, $r \in \mathrm{R}_+$. If $D_-(x) \leqq r \leqq D_+(x)$, then there is a dimension function D on A such that $D(x)=r$.*

Proof. [5, 4.1(c)].

4.6. Corollary. *Let A be stably finite. The following are equivalent*

(1) *A has a unique dimension function.*

(2) *$D_-=D_+$.*

(3) *D_- is a dimension function.*

(4) *D_+ is a dimension function.*

(5) *Let $x \neq 0 \in \mathscr{F} \otimes A$. There exists a rational number $p \geqq 1$ such that given any $t \in K_0^*(A)^+$ and any $j>0$, there is some $n>0$ for which np is an integer and either $nt \leqq np(j[x])$ or $n(j[x]) \leqq npt$.*

Proof. [5, 4.2 and 4.3].

4.7. Corollary. *Let B be a simple C^*-algebra with unit. Then B is stably finite if and only if there is a dimension function on B.*

Proof. This follows from 4.5 and the first remark in Section 3.

5. Let A be a stably finite simple C^*-algebra with unit. The functions D_-, D_+ give sharp bounds for all possible dimension functions on A. But they are difficult to calculate in concrete examples. We want to indicate (without proofs) in this section, how one can construct upper and lower dimension functions Λ_-, Λ_+ on A which give less sharp bounds for dimension functions on A but which are more closely related to A and more easily computed (they can be computed for simple AF-algebras).

Given $x, y \in \mathscr{F} \otimes A$, as in 4.1 we write $n\langle x\rangle \lesssim m\langle y\rangle$ if there are orthogonal elements $x_1, \ldots, x_n \in \langle x\rangle$ and $y_1, \ldots, y_m \in \langle y\rangle$ such that $x_1 + \ldots + x_n \lesssim y_1 + \ldots + y_m$ in $\mathscr{F} \otimes A$. Let q be a one-dimensional projection in $\mathscr{F}$ and set $\bar{q} = q \otimes 1 \in \mathscr{F} \otimes A$. We put, given $x \in \mathscr{F} \otimes A$

$$s_n(x) = \sup\{s \in \mathbb{N} | s\langle \bar{q}\rangle \lesssim n\langle x\rangle\}$$
$$r_n(x) = \inf\{r \in \mathbb{N} | r\langle \bar{q}\rangle \gtrsim n\langle x\rangle\}.$$

Then the limits

$$\Lambda_-(x) = \lim_{n\to\infty} s_n(x)/n$$

$$\Lambda_+(x) = \lim_{n\to\infty} r_n(x)/n$$

exist and are finite. Λ_- defines an upper dimension function and Λ_+ a lower dimension function such that

$$0 < \Lambda_-(x) \leqq D_-(x) \leqq D_+(x) \leqq \Lambda_+(x)$$

for all $0 \neq x \in \mathscr{F} \otimes A$. Thus in particular $\Lambda_-(x) \leqq \bar{R}_\tau(x) \leqq \Lambda_+(x)$ $(x \in \mathscr{F} \otimes A)$ for every finite trace τ on A.

Furthermore Λ_- is lower semi-continuous and if A is factorial in the sense of [2] then the restrictions of Λ_- and Λ_+ to A coincide with the function λ constructed in [2]. This shows that λ is a lower semi-continuous rank function.

6. We used the group $K_0^*(A)$ mainly as a technical device in order to prove the existence of dimension functions on stably finite algebras. On the other hand $K_0^*(A)$ is of course an algebraic invariant for A. We do not know if $K_0^*(A)$ will turn out to be as useful for the classification problem for simple C^*-algebras, as K_0 is in fact for the special class of AF-algebras [4].

In the following we discuss some relations between K_0 and K_0^*. If A is a C^*-algebra then $K_0(A)$ is the abelian group generated by all equivalence classes (with respect to Murray-von Neumann equivalence) $\langle p\rangle_0$ of projections p in $\mathscr{F} \otimes A$ with relations $\langle p\rangle_0 + \langle q\rangle_0 = \langle p_1 + q_1\rangle_0$ $(p_1 \in \langle p\rangle_0,\ q_1 \in \langle q\rangle_0,\ p_1 \perp q_1)$. Of course, if two projections p and q are equivalent in the sense of Murray-von Neumann (via a partial isometry) in $\mathscr{F} \otimes A$, then also $p \equiv q$ in $\mathscr{F} \otimes A$. Therefore there is a homomorphism $\Phi: K_0(A) \to K_0^*(A)$ such that $\Phi([p]_0) = [p]$ where $[p]_0$ denotes the image of the projection $p \in \mathscr{F} \otimes A$ in K_0.

D. Handelman (private communication) has shown that Φ is an isomorphism if A is a von Neumann-algebra.

For (infinite-dimensional) AF-algebras the situation is totally different. Consider for instance a UHF-algebra with dimension function $\bar{R}_\tau$ induced by the trace τ. If $x \in A$ is such that $\bar{R}_\tau(x)$ is irrational, then $[x] \neq [p]$ in $K_0^*(A)$ for every projection p in A, since for a projection the value $\bar{R}_\tau(p)$ must be rational. While Φ is injective for UHF-algebras this is not true for the algebra constructed by Elliott in [4, 6.5]. This algebra is factorial in the sense of [2] and contains two non-zero projections having the same trace which are not comparable in K_0 (i.e. in the sense of Murray and von Neumann). In particular, these projections have distinct

images in K_0. On the other hand it can be shown that their images in K_0^* coincide. In fact, we have the following general result. If A is a factorial AF-algebra in the sense of [2] with canonical dimension function λ then the mapping $[x] \rightarrow \lambda(x)$ $(x \in \mathscr{F} \otimes A)$ extends to an isomorphism of ordered groups from $K_0^*(A)$ onto R with the natural order. This follows almost immediately from [2, 3.7].

For AF-algebras there is a bijective relation between traces, dimension functions, order preserving homomorphisms from K_0^* to R and order preserving homomorphisms from K_0 to R. Since there are simple AF-algebras with many different order preserving homomorphisms from K_0 to R, this class also provides examples of algebras with many different dimension functions.

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