Ducci Matrices

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Abstract. For a real square matrix A, its *Ducci map* takes a vector x to $|Ax|$, where the absolute value is meant elementwise. We study the class of matrices with the property that for almost all x their orbit under the Ducci map suddenly "stops" at the zero vector, sometimes after a few iterations, sometimes after thousands of steps, in a seemingly unpredictable way. This generalizes the well known Four Number Game of E. Ducci.

1. DUCCI'S OBSERVATION. In the 1930s, the Italian mathematician Enrico Ducci (1864–1940, [**20**]) observed a peculiar property of the map

$$\delta\colon (x_1, x_2, x_3, x_4) \mapsto (|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_4|, |x_4 - x_1|).$$

For every $x \in \mathbb{N}^4$ the sequence of iterates $x, \delta(x), \delta^2(x), \ldots,$ known as the *Ducci sequence* of x, reaches the null tuple [**10**].

It is striking how fast this happens in most cases. If the components of x are chosen at random in $[0, N]$ for some (not too small) number N, then in about 95% of the cases no more than 7 iterations are needed to reach zero. Moreover, this behavior does not seem to depend very much on the size of N.

On the other hand, there exist 4-tuples with arbitrary long Ducci sequences, among others the tuples $\tau_n = (t_n, t_{n+1}, t_{n+2}, t_{n+3})$, where t_n denotes the nth *Tribonacci number* [**21**]. For τ_n, the number of steps necessary to reach zero is greater than $\frac{3n}{2}$.

What if one allows tuples with real entries? Surprisingly, most Ducci sequences of tuples $x \in \mathbb{R}^4$ also arrive at zero after a few iterations, as in the following case.

$$\begin{pmatrix} \mathrm{e} \\ \pi \\ \sqrt{2} \\ 1 \end{pmatrix} \to \begin{pmatrix} \pi - \mathrm{e} \\ \pi - \sqrt{2} \\ \sqrt{2} - 1 \\ \mathrm{e} - 1 \end{pmatrix} \to \begin{pmatrix} \mathrm{e} - \sqrt{2} \\ \pi - 2\sqrt{2} + 1 \\ \mathrm{e} - \sqrt{2} \\ 2\,\mathrm{e} - \pi - 1 \end{pmatrix} \to \left(\pi - \mathrm{e} - \sqrt{2} + 1\right)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Of course, some tuples may take longer. Starting from $x = (e - 1, \pi, 4\sqrt{2}, 1)^{\mathsf{T}}$, the rather large number of 11 steps is needed. And then there is a tuple where the magic fails completely. The Ducci sequence of $x_0 = (1, r, r^2, r^3)^{\mathsf{T}}$, with $r = 1.839\ldots$ satisfying $1 + r + r^2 = r^3$, never reaches zero as $\delta^k(x_0) = (r - 1)^k x_0 \neq 0$ for all k. Up to trivial variants, x_0 is the only such tuple.

This real-valued version of Ducci's original result has been noted and proved independently at least three times (see [**3**], [**4**], [**14**]).

Over 80 papers have been written on this subject (in [**5**], a fairly comprehensive bibliography up to the year 2007 can be found, some newer articles are listed in the references below). Nevertheless, Ducci's observation is still widely considered to be merely "una interessante curiosità" [**10**], a more or less isolated phenomenon.

It is the purpose of this article to show that this impression is wrong. The map δ is just one representative of a large class of maps with a similar iterative behavior.

2. DUCCI MATRICES. Let A denote a real $n \times n$ matrix. In what follows we are going to study iterates of the *Ducci map* corresponding to A, defined on $\mathbb{R}^n$ by

$$\delta_A(x) = |Ax|,$$

where the absolute value is taken componentwise: $|x| = (|x_1|, \ldots, |x_n|)^\mathsf{T}$. The particular case $A_0 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ yields the original Ducci map $\delta = \delta_{A_0}$.

For $x \in \mathbb{R}^n$, the sequence of iterates $x, \delta_A(x), \delta_A^2(x), \ldots$ will be called the *Ducci sequence* of x with respect to A. If it contains the zero vector, then we say that the Ducci sequence *terminates*.

Our main aim is to prove that besides A_0 there exist many other matrices A whose Ducci sequences terminate for almost all $x \in \mathbb{R}^n$. A matrix having this property will be called a *Ducci matrix*.

Let us begin by looking at a number of examples, with proofs mostly postponed.

Example 1. The map $(x_1, x_2, x_3, x_4) \mapsto (|4x_1 - s|, |4x_2 - s|, |4x_3 - s|, |4x_4 - s|)$, where $s = x_1 + x_2 + x_3 + x_4$, behaves quite similar to Ducci's map. The Ducci sequences of all $x \in \mathbb{R}^4$ terminate, with the exception of those vectors for which three, but not all four, of the components are equal [**11**]. In the above notation, this is δ_{A_1} for the matrix $A_1 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$.
A typical Ducci sequence for A_1 is short:

$$\begin{pmatrix} 1 \\ 10 \\ 17 \\ 3 \end{pmatrix} \to \begin{pmatrix} 27 \\ 9 \\ 37 \\ 19 \end{pmatrix} \to 8 \begin{pmatrix} 2 \\ 7 \\ 7 \\ 2 \end{pmatrix} \to 80 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

About two thirds of all starting vectors x (chosen at random in $[0, N]^4$ for sufficiently large N) terminate after just 4 steps. On average about 4.5 steps are needed, whereas in the classical situation this would be about 5 steps.

If you like to see long Ducci sequences, start with $x = (s, t_1, t_2, t_3)^\mathsf{T}$, where the t_i are similar in size (but not all equal) and s is rather different from them.

If $x_{s,t} = (s, t, t, t)^\mathsf{T}$ with $s \neq t$, then $\delta_{A_1}^n(x_{s,t}) = (|s - t|)^n (3, 1, 1, 1)^\mathsf{T}$ for all $n \geq 1$. Thus the Ducci sequences of such $x_{s,t}$ do not terminate.

Example 2. There are lots of 3×3 Ducci matrices. Some, like $A_2 = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 3 & -4 \\ 0 & 5 & -5 \end{pmatrix}$, produce remarkably short Ducci sequences. For a randomly chosen $x \in [0, N]^3$ (N sufficiently large), zero is reached on average after only 2.5 iterations. A typical Ducci sequence for A_2 looks like this one:

$$\begin{pmatrix} 19360002 \\ 90627986 \\ 96763715 \end{pmatrix} \to 5 \begin{pmatrix} 32188631 \\ 19162180 \\ 6135729 \end{pmatrix} \to 325661275 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is amazingly short. In fact, more than 95% of all Ducci sequences for A_2 reach zero in at most four steps. Trying to find longer-lasting vectors by random search can be really frustrating. However, the following lemma shows that for A_2 arbitrary long Ducci sequences indeed exist.

Lemma 1. *Let the numbers t_n be defined by $t_0 = 2, t_n = \frac{11t_{n-1} - 7}{3t_{n-1} - 1}$ for $n > 0$, with $\lim_{n\to\infty} t_n = 2 + \frac{\sqrt{15}}{3} = t_\infty$. The Ducci sequence of $x_n = (t_n, 1, t_n - 2)^\mathsf{T}$ terminates after $n + 2$ applications of δ_{A_2}. The Ducci sequence of $x_\infty = \lim_{n\to\infty} x_n = (t_\infty, 1, t_\infty - 2)^\mathsf{T}$ does not terminate.* ■

The last statement is confirmed by observing that $\delta_A(x_\infty) = (5 - \sqrt{15})\, x_\infty$, the complete proof is given on page 10.

What exactly is a Ducci matrix? Intuitively, a square matrix A of size n is a Ducci matrix if the Ducci sequence of *almost all* $x \in \mathbb{R}^n$ contains zero. But just how many exceptions do we have to allow for?

There can be many exceptions. For example, if A and B are Ducci matrices of size n and m, then for $C = \left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{smallmatrix}\right)$ a Ducci sequence with the starting point $z = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^{n+m}$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, is composed of two Ducci sequences: $\delta_C^k(z) = \left(\begin{smallmatrix} \delta_A^k(x) \\ \delta_B^k(y) \end{smallmatrix}\right)$. Only if the Ducci sequences for x or y reach zero will the Ducci sequence for z also reach zero.

But still, the Ducci sequence of a randomly chosen $z \in \mathbb{R}^{n+m}$ will terminate, so C should be considered to be a Ducci matrix. Therefore we choose to define "almost all" in the sense of Lebesgue measure:

Definition 1. Let $A \neq 0$ be a real-valued matrix of size $n \times n$.

- The *(Ducci-) length* $\lambda_A(x)$ of $x \in \mathbb{R}^n$ is the smallest k such that $\delta_A^k(x) = 0$, and ∞ if there is no such k.
- The matrix A is called a *Ducci matrix* if the set $\{x \in \mathbb{R}^n : \lambda_A(x) = \infty\}$ has Lebesgue measure 0.
- For a Ducci matrix A, $x \in \mathbb{R}^n$ such that $\lambda_A(x) = \infty$ is called an *exception vector*.

We remark that for $c \in \mathbb{R} \setminus \{0\}$, $\delta_A(cx) = \delta_{cA}(x) = |c|\,\delta_A(x)$ holds, and also $\delta_A(x+y) = \delta_A(x)$ holds for $y \in \operatorname{Ker} A$. This implies $\lambda_A(x) = \lambda_A(cx) = \lambda_{cA}(x) = \lambda_A(x+y)$.

Let us look at a few further examples.

Example 3. The Ducci matrix $A_3 = \left(\begin{smallmatrix} 8 & 1 & -9 \\ 0 & 1 & -1 \\ 8 & -1 & -7 \end{smallmatrix}\right)$, although of the same size as A_2, has far longer Ducci sequences. If we randomly choose $x \in [0, N]^3$ for sufficiently large N, we observe the following distribution of the lengths $\lambda_{A_3}(x)$:

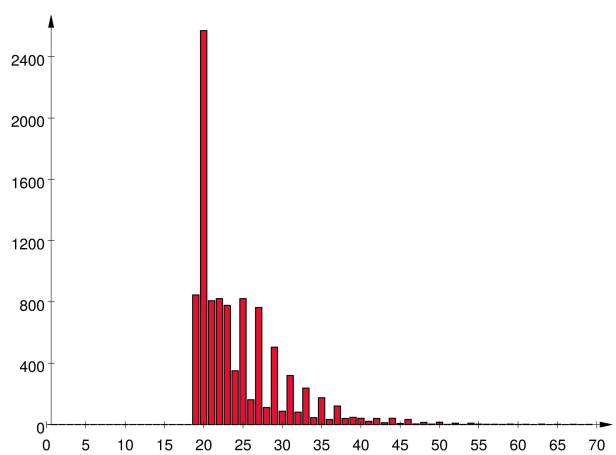

Figure 1. The distribution of lengths $\lambda_{A_3}(x)$ for 10.000 random tuples x.

Several features are worth mentioning here: Not a single vector in the sample had a length under 19. More than a quarter had the *exact* length 20. The largest length was 71, which is enormous when compared with the outcome of the same experiment for, say, A_2 and for many other Ducci matrices of size 3. Also, among the larger Ducci lengths there seems to be a preference for odd numbers.

Example 4. Another outstanding matrix is $A_4 = \begin{pmatrix} 6 & -7 & -8 & 9 \\ 7 & -8 & 9 & 6 \\ 8 & 9 & 6 & -7 \\ 9 & 6 & -7 & -8 \end{pmatrix}$. It produces tremendously long Ducci sequences, although we have to admit that presently it is only a conjecture that A_4 is a Ducci matrix. If a vector is chosen randomly as above, then on average its length is somewhere between 800 and 1000. And, still more surprising, one encounters vectors whose Ducci lengths are still much larger than that. Consider this example:

$$\underbrace{\begin{pmatrix} 5 \\ 33 \\ 11 \\ 47 \end{pmatrix} \to 2\begin{pmatrix} 67 \\ 41 \\ 3 \\ 105 \end{pmatrix} \to 8\begin{pmatrix} 259 \\ 35 \\ 221 \\ 3 \end{pmatrix} \to 16\begin{pmatrix} 216 \\ 43 \\ 226 \\ 485 \end{pmatrix} \to \cdots \to c\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{20430 applications of } \delta_{A_4}}.$$

This Ducci sequence is 20430 steps long. Towards the end, the components of the vectors each have almost 14000 decimal places. The constant in the next to last vector $c\cdot(1,1,1,1)^\mathsf{T}$ is $c = 2^{22982}k$ for some odd integer k still having more than 7000 digits.

Example 5. For our next example we need some further notation.

Definition 2. Let us call a square matrix A a *difference matrix* if it contains the entries 1 and -1 exactly once in every row, and zeros otherwise. Also, we call an integer matrix A of size $n \times n$ a $\mathbb{Z}$*-Ducci matrix* if the length $\lambda_A(x)$ is finite for all $x \in \mathbb{Z}^n$.[1]

The name *difference matrix* is due to the fact that for a matrix of this type the components of Ax all have the form $x_i - x_j$.

Difference matrices have turned out to be good candidates for being Ducci matrices. This is perhaps not too surprising since the archetype A_0 of all Ducci matrices belongs to this class. On the other hand, it is known (see [**10**]) that the difference matrices

$$A_{0,n} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

are $\mathbb{Z}$-Ducci matrices only if n is a power of 2.

We prove an easy lemma about difference matrices.

Lemma 2. *Let A be a difference matrix. If there is some integer k such that A^k contains only even entries, then A is a $\mathbb{Z}$-Ducci matrix.*

Proof. Corresponding entries of the vectors Ax and $\delta_A(x) = |Ax|$ are equal mod 2. Therefore the condition on A^k implies that for $x = (x_1, \ldots, x_n)^\mathsf{T} \in \mathbb{Z}^n$ we have $\delta_A^k(x) = 2y$ for some $y \in \mathbb{N}^n$. Let m be the maximum of $|x_1|, \ldots, |x_n|$. Since A is a difference matrix, the components y_i of y satisfy $0 \le 2y_i \le m$, that is, the maximum of $y_1, \ldots, y_n$ is at most $m/2$. Hence after $\lceil k \log_2 m \rceil$ steps, the Ducci sequence of x must have reached zero. ∎

A computer search produces lots of difference matrices fulfilling the condition of the lemma. Examples of sizes 5 to 9 are

[1] According to this definition, A_1 is not a $\mathbb{Z}$-Ducci matrix. In order to include matrices like A_1 we have to allow for a few $x \in \mathbb{Z}^n$ not to reach zero. It is, however, not entirely clear what "a few" should actually mean. Therefore for our present purposes we prefer the simpler form of the definition.

$$A_5 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix},$$

$$A_8 = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_9 = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

The list could easily be continued. For each of the matrices A_5 to A_9 the condition of the lemma is met since in each case all entries of A_n^n are even. For example, $A_5^5 = 2 \begin{pmatrix} 2 & -4 & 3 & -2 & 1 \\ -1 & 2 & -2 & 2 & -1 \\ 0 & -1 & 1 & -1 & 1 \\ 2 & -3 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 \end{pmatrix}$.

3. MAPS WITH TRAPS. The main question about Ducci matrices is not whether they generate long or short sequences of iterates but rather this: What causes the Ducci sequences of some matrices to suddenly terminate at all after a (large or small) number of iterations? Is there a common reason for this behavior? It is certainly quite different from what one observes with most square matrices.

Before we try to answer this question we would like to remind the reader of one of the lesser known features of dynamical systems, known as *transient chaos* [**17**]. There exist self-maps of, say, the real line where the iterates of most points exhibit seemingly chaotic behavior for a long time and then all of a sudden become quite regular, that is, periodic or even constant. This is not unlike what we observe in Ducci sequences.

As an example, let us consider the *tent map* $f\colon [0,1] \to [0,1]$ which is defined by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

It has the fixed point $x_0 = \frac{2}{3}$. We choose $0 < \varepsilon < \frac{1}{6}$ and modify f to a mapf_ε by installing a *trap* [**16**].

This is done as follows. Outside of the interval $I_\varepsilon = [x_0 - \varepsilon, x_0 + \varepsilon]$ the functions f and f_ε coincide. Within the subinterval $I_{\varepsilon/2} = [x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}]$ we put $f_\varepsilon(x) = x_0$. Finally, in the remaining intervals $(x_0 - \varepsilon, x_0 - \frac{\varepsilon}{2})$ and $(x_0 + \frac{\varepsilon}{2}, x_0 + \varepsilon)$ we define f_ε in such a way that it is linear in both and continuous on $[0,1]$.

It is well known that the tent map is chaotic [**13**]. This implies that in every nonempty subinterval of $[0,1]$ there are points whose f-orbit eventually reaches I_ε. As long as they avoid I_ε, their orbits with respect to f and f_ε coincide. If an f_ε-orbit reaches I_ε, but not $I_{\varepsilon/2}$, it will be repelled: It leaves I_ε quickly and continues its chaotic manner. But once it reaches $I_{\varepsilon/2}$ it stops, since $f_\varepsilon(x) = x_0$ for all $x \in I_{\varepsilon/2}$. These orbits have a fate similar to Ducci sequences: after a certain period of chaotic wandering they suddenly get trapped at a fixed point.

Figure 2 shows the graph of f_ε for $\varepsilon = \frac{1}{30}$ and the f_ε-orbit of $x = \frac{1}{\sqrt{10}}$. It reaches $I_{\varepsilon/2}$ after 126 iterations of f_ε, from then on the orbit is constant:

$$\frac{1}{\sqrt{10}} \to \frac{\sqrt{10}}{5} \to 2 - \frac{2\sqrt{10}}{5} \to \cdots \to n_1 - \frac{n_2\sqrt{10}}{5} \to \frac{2}{3} \to \frac{2}{3} \to \cdots,$$

where n_1 and n_2 are two large integers (each having 38 decimal places).

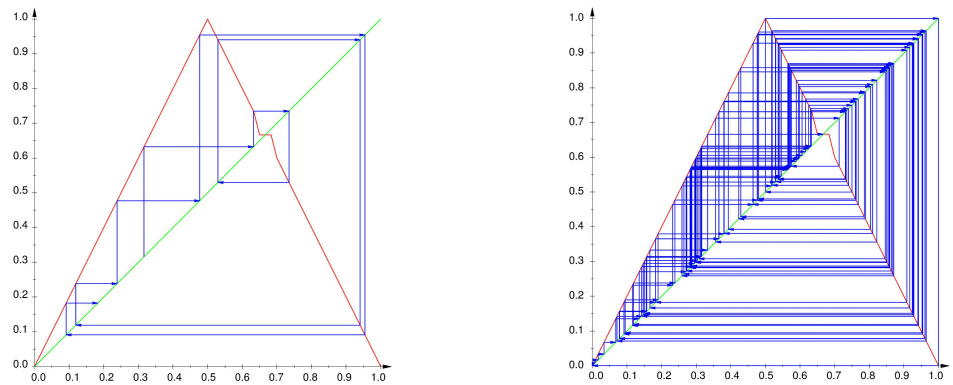

Figure 2. The tent map with a trap. The orbit of $\frac{1}{\sqrt{10}}$ showing 10, respectively 126, iterations.

This is perhaps the most simple case of a map whose orbits show a sudden change in behavior, sometimes after a long time. It also illustrates how by choosing smaller $\varepsilon > 0$ the trap can be made smaller so that on average it will be reached later.

After having seen the concept of a trap it is natural to ask whether a similar device is causing what we have observed in Ducci matrices. This is indeed the case.

4. TRAP MATRICES. But where can we find a trap in connection with a Ducci matrix? The key to the answer is provided by a simple reformulation of the Ducci map.

Definition 3. Let A be a real $n \times n$ matrix. The *signed Ducci map* of A is the map $\sigma_A : u \mapsto A\,|u|$ in $\mathbb{R}^n$. The sequence $x, \sigma_A(x), \sigma_A^2(x), \ldots$ will be called the *signed Ducci sequence* of $x \in \mathbb{R}^n$ with respect to A.

The Ducci sequence of $x \in \mathbb{R}^n$ consists, after the leading element x, of the absolute values of the signed Ducci sequence of $A\,x$, that is, we have

$$\begin{array}{lccccc} \text{Ducci sequence of } x\text{:} & x, & |A\,x|, & |A\,|A\,x||, & |A\,|A\,|A\,x||, & \ldots, \\ \text{Signed Ducci sequence of } Ax\text{:} & & A\,x, & A\,|A\,x|, & A\,|A\,|A\,x||, & \ldots. \end{array}$$

The additional information contained in the signed Ducci sequence will be seen to be quite useful.

We can partition $\mathbb{R}^n$ into regions where σ_A is linear as follows:

Definition 4. For a sign tuple $s = (s_1, \ldots, s_n) \in \{-1, 1\}^n$, let the region $R_s \subset \mathbb{R}^n$ be the closure of $\{(x_1, \ldots, x_n)^\mathsf{T} : s_1 x_1 > 0, \ldots, s_n x_n > 0\}$.

Linearity of σ_A in R_s is obvious: $\sigma_A(x) = A|x| = ASx$ holds for all $x \in R_s$, where $s = (s_1, \ldots, s_n)$ and $S = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}$.

Furthermore, we need not consider σ_A on the whole of $\mathbb{R}^n$. We may restrict it to the image space $\operatorname{Im} A$ of A since the signed Ducci sequence of any $x \in \mathbb{R}^n$ after the first step proceeds completely within $\operatorname{Im} A$. Notice that for a Ducci matrix, $\operatorname{Im} A$ is always a proper subspace of $\mathbb{R}^n$, as the last nonzero element in any nontrivial Ducci sequence clearly belongs to the kernel $\operatorname{Ker} A$ of A.

We can now explain what is meant by a trap in our setting. The definition may appear to be a little unmotivated, but later it will be seen that it is completely analogous to the idea of a trap as described in the preceding section.

Definition 5. Let A be a real $n \times n$ matrix. A *trap* for A is a region R_s such that

for its corresponding sign matrix S the map $u \mapsto ASu$ is not invertible on $\operatorname{Im} A$. The matrix A is called a *trap matrix* if it has at least one trap.

The key point here is the noninvertibility *on the proper subspace* $\operatorname{Im} A$. As a map on the whole vector space $\mathbb{R}^n$, the map $u \mapsto ASu$, where A is a Ducci matrix and S a sign matrix, is never invertible as Ducci matrices are always singular.

The intuition behind this definition of a trap R_s is that σ_A, when acting on the convex cone $C = R_s \cap \operatorname{Im} A$, compresses C in the sense that $\sigma_A(C)$ has a lower dimension than C, quite similar to how the modified tent function f_ε compresses the interval $[x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}]$ to the single point x_0.

The following simple lemma is important.

Lemma 3. *Every Ducci matrix is a trap matrix.*

Proof. The next to last element $x \neq 0$ in a signed Ducci sequence in $\operatorname{Im} A$ satisfies $y = \sigma_A(x) \neq 0$, $\sigma_A(y) = 0$. If S is the sign matrix with the signs of y, then $\sigma_A(y) = ASy = 0$ for $y \in \operatorname{Im} A$. Thus the map $u \mapsto ASu$ is not invertible on $\operatorname{Im} A$. ■

For example, the classical Ducci matrix A_0 has six traps corresponding to the sign tuples $\pm(1,1,-1,-1)$, $\pm(1,-1,1,-1)$, and $\pm(1,-1,-1,1)$. Let us verify this for $s = (1,1,-1,-1)$. Call the corresponding sign matrix S and let $B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$, whose first three column vectors form a basis of $\operatorname{Im} A_0$. The matrix $B^{-1}A_0SB = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ -1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ then represents A_0S with respect to this basis, and its upper left 3×3 submatrix $C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix}$ represents $(u \mapsto A_0Su)|\operatorname{Im} A_0$. Since $\det C = 0$, we see that R_s is a trap.

Calculating the traps of a matrix can be cumbersome. The following lemma from linear algebra is helpful.

Lemma 4. *Let A be an $n \times n$ matrix over the reals. Then the following conditions on A are equivalent.*

1. *The map $u \mapsto Au$ is invertible on* $\operatorname{Im} A$.
2. $\operatorname{Im} A \oplus \operatorname{Ker} A = \mathbb{R}^n$.
3. $\operatorname{rank} A^2 = \operatorname{rank} A$.

If A is singular, the following condition is also equivalent.

4. *If K and C are matrices whose columns constitute a basis of* $\operatorname{Ker} A$ *and of* $\operatorname{Coker} A$ $(= \operatorname{Ker} A^\mathsf{T})$*, respectively, then* $\det(C^\mathsf{T}K) \neq 0$.

Proof. The equivalence of conditions 1–3 is left to the reader. Now assume that $\operatorname{Ker} A \neq \{0\}$. Then $\det(C^\mathsf{T}K) \neq 0$ holds if and only if $C^\mathsf{T}(Kx) \neq 0$ for all $x \neq 0$. Since K is a basis of $\operatorname{Ker} A$, this means $C^\mathsf{T}y \neq 0$ for all $y \in \operatorname{Ker} A$, $y \neq 0$, which can be written as $\operatorname{Ker} A \cap \operatorname{Ker} C^\mathsf{T} = \{0\}$. But $\operatorname{Ker} C^\mathsf{T} = \operatorname{Im} A$, so this amounts to $\operatorname{Ker} A \cap \operatorname{Im} A = \{0\}$. Thus conditions 2 and 4 are equivalent. ■

We call a square matrix fulfilling the conditions of this lemma *range regular*. (There seems to be no well-established name for this elementary matrix property.) For matrices of rank $n-1$, range regularity is particularly easy to check:

Lemma 5. *Let A be a real $n \times n$ matrix of rank $n-1$ and let x, y be vectors generating the kernel and the cokernel, respectively, of A. Then A is range regular if and only if $\langle x, y\rangle \neq 0$. In particular, A is a trap matrix if and only if there exists a sign matrix S such that $\langle Sx, y\rangle = 0$.*

Proof. If $\operatorname{rank} A = n-1$, then in condition 4 of Lemma 4 the matrix K can be taken to be the single column x and C the single column y. Hence $C^{\mathsf{T}}K = (\langle x, y\rangle)$, which proves the first claim of the lemma. The second claim follows immediately from the definition of trap matrices. ∎

The examples in Section 2 were all $n \times n$ matrices of rank $n-1$ with the additional property that their row sums are 0, which is equivalent to the kernel being generated by the constant vector $(1,\dots,1)^{\mathsf{T}}$. In this case, $\langle Sx, y\rangle = \sum_{i=1}^n s_i y_i$, so that A is a trap matrix if and only if for the generating vector y of the cokernel of A this sum is zero for at least one choice of $s_1,\dots,s_n$.

To be still more specific, consider $A = A_{0,n}$ as defined on page 4. Here $y = (1,\dots,1)^{\mathsf{T}}$ generates the cokernel since all column sums are zero. Hence the preceding lemma implies that $A_{0,n}$ is a trap matrix if and only if there is a sign tuple $(s_1,\dots,s_n)$ for which $\sum_{i=1}^n s_i = 0$. This happens if and only if n is an even number. However, we know that $A_{0,n}$ is a $\mathbb{Z}$-Ducci matrix only for $n = 2^k$ [**10**].

Why do the Ducci matrices in Section 2 all have row sum 0? Is this perhaps a necessary condition for being a Ducci matrix? It turns out that this is not so, it is just a convenient "normal form." This follows from the next lemma.

Lemma 6. *Let A and D be real $n \times n$ matrices and assume that D is invertible and $D|x| = |Dx|$ holds for all $x \in \mathbb{R}^n$. Then if A is a Ducci matrix, so is $B = D^{-1}AD$.*

Proof. Writing $f(x) = Dx$ and using $D|x| = |Dx|$, we find that $\sigma_B = f^{-1}\circ\sigma_A\circ f$ and hence $(\sigma_B)^k = f^{-1}\circ(\sigma_A)^k\circ f$ for $k \in \mathbb{N}$. Therefore the signed Ducci sequences according to A and B terminate after the same number of steps. ∎

We emphasize two special cases of the lemma. The first one is $D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$, where $d_i > 0$ for all i. If the kernel of A is generated by $(x_1,\dots,x_n)^{\mathsf{T}}$, then the kernel of B is generated by $(\frac{x_1}{d_1},\dots,\frac{x_n}{d_n})^{\mathsf{T}}$. Thus, if we have a Ducci matrix whose kernel is generated by $(x_1,\dots,x_n)^{\mathsf{T}}$ and $x_i \neq 0$ for all i, then we may simply transform it into a Ducci matrix whose kernel is generated by $(1,\dots,1)^{\mathsf{T}}$, that is, one whose row sums are 0. This works, of course, the other way also: From the matrices in Section 2 we can generate arbitrarily many Ducci matrices with a different kernel. For instance, from $A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 3 & -4 \\ 0 & 5 & -5 \end{pmatrix}$ we obtain the Ducci matrices $B = \begin{pmatrix} 2 & d_2/d_1 & -3\,d_3/d_1 \\ d_1/d_2 & 3 & -4\,d_3/d_2 \\ 0 & 5\,d_2/d_3 & -5 \end{pmatrix}$.

The other special case of the lemma are permutation matrices, that is, matrices whose rows are obtained by a permutation of an identity matrix. If D is of this type, then Dx is a permutation of the entries of x; therefore $D|x| = |Dx|$ obviously holds for all $x \in \mathbb{R}^n$. Thus the two matrices A and $B = D^{-1}AD$ generate the same Ducci sequences up to a renaming of the variables.

5. THE REDUCED DUCCI MAP. By switching from δ_A to σ_A we have lowered the dimension of the relevant domain of our maps from n to $\operatorname{rank} A = r < n$. We are now going to cut this down to $r-1$ by taking into account the fact that σ_A is homogeneous.

Recall that $\delta_A(c\,x) = |c|\,x$ and $\sigma_A(c\,x) = |c|\,\sigma_A(x)$, for $c \neq 0$, so that the Ducci sequences of x and $c\,x$ are the same up to a constant factor. Hence we may consider vectors $x, y \in \mathbb{R}^n$ such that $y = c\,x$ for some $c \neq 0$ as equivalent, $x \sim y$, with $[x]$ denoting the equivalence class of x. Also, we write $\mathbf{0}$ for the equivalence class $[0]$ of

the zero vector. The iterative behavior of σ_A on equivalent vectors will be identical, so we may as well factor out by this equivalence.

The quotient space $\mathbb{R}^n/_\sim$ by this relation, with the quotient topology, is the *real projective space* $\mathbb{R}\mathrm{P}^{n-1}$, augmented by the isolated point $\mathbf{0}$. It is best thought of as the unit sphere in $\mathbb{R}^n$ with antipodal points identified, plus the point $\mathbf{0}$ at its center.

Definition 6. Let A be a real-valued $n \times n$ matrix, and $\mathbb{P}_A = \operatorname{Im} A/_\sim$. The map $\rho_A : \mathbb{P}_A \to \mathbb{P}_A$ defined by $\rho_A([x]) = [\sigma_A(x)]$ is called the *reduced Ducci map* of A.

Clearly, this map is well-defined. Since $\operatorname{Im} A$ is an r-dimensional real vector space, where $r = \operatorname{rank} A$, its domain $\mathbb{P}_A$ is homeomorphic to $\mathbb{R}\mathrm{P}^{r-1} \cup \{\mathbf{0}\}$. The map ρ_A is continuous on $(\operatorname{Im} A \setminus \operatorname{Ker} A)/_\sim$.

It is obvious that the Ducci sequence and the Ducci length of any $x \in \mathbb{P}_A$ are well-defined notions. The sets $[R_s] = \{[x] \in \mathbb{P}_A : x \in R_s \cap \operatorname{Im} A\}$ define regions within $\mathbb{P}_A$ in a natural way. Clearly, $[R_s] = [R_{-s}]$ since $R_{-s} = -R_s$. With this in mind, we may omit the brackets and speak of the regions R_s of $\mathbb{P}_A$ restricted to which ρ_A is the quotient of a linear map.

To calculate ρ_A for a specific matrix A, we have to do two things:

1. Find matrix representations of the maps $(u \mapsto ASu)|_{\operatorname{Im} A}$.
2. Choose a representation for $\mathbb{R}\mathrm{P}^{r-1}$ and then calculate an explicit form of ρ_A.

For step 1 we choose a matrix B whose column vectors include a basis for $\operatorname{Im} A$ and calculate $B^{-1}ASB$ for the sign matrices S. In step 2, we can represent the lines of $\operatorname{Im} A$ by their intersection vectors with a hyperplane $H \subset \operatorname{Im} A$ not containing zero. Lines parallel to H will be considered as points at infinity.

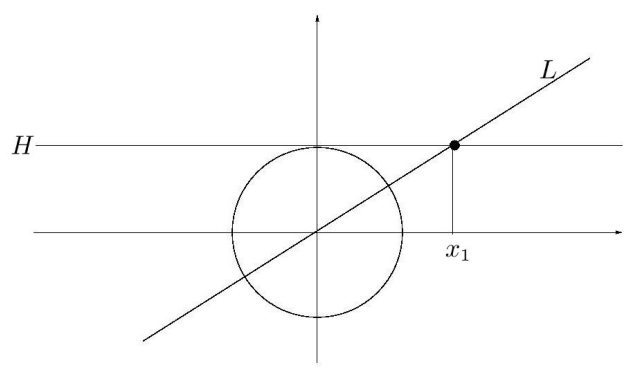

Figure 3. The real projective space $\mathbb{R}\mathrm{P}^1$.

Figure 3 shows the idea for $\mathbb{R}\mathrm{P}^1$. We choose $H = \{(x_1, x_2)^\mathsf{T} \in \operatorname{Im} A : x_2 = 1\}$. For $x = (x_1, x_2)^\mathsf{T} \in \operatorname{Im} A$ with $x_2 \neq 0$, $[x]$ can be identified with $[(t, 1)^\mathsf{T}]$, where $t = \frac{x_1}{x_2}$, and hence with $t \in \mathbb{R}$. The equivalence class $[(x_1, 0)^\mathsf{T}]$ can be identified with a point of $\mathbb{R}\mathrm{P}^1$ at infinity. Note that $[(x_1, 0)^\mathsf{T}] = [(-x_1, 0)^\mathsf{T}]$. The real projective space $\mathbb{R}\mathrm{P}^1$ has the topology of a circle (Figure 3 illustrates this). The nonzero points on the line L are represented by the point at $(x_1, 1)^\mathsf{T}$ on the line parallel to the x_1-axis through $(0, 1)^\mathsf{T}$. The x_1-axis itself "is" the point at infinity.

Example 2 (continued). Let us demonstrate this approach for $A = A_2 = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 3 & -4 \\ 0 & 5 & -5 \end{pmatrix}$ of Section 2. A basis for $\operatorname{Im} A$ is given by $\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\}$, and $\operatorname{Coker} A$ is spanned by

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Since $\operatorname{Im} A \oplus \operatorname{Coker} A = \mathbb{R}^3$, the column vectors of $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$. From $B^{-1}ASB = \begin{pmatrix} 2s_1+3 & s_2-6 & 2s_1-2s_2-3 \\ s_1+4 & 3s_2-8 & s_1-6s_2-4 \\ 0 & 0 & 0 \end{pmatrix}$, where $S = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we obtain that $A^{[s]} = \begin{pmatrix} 2s_1+3 & s_2-6 \\ s_1+4 & 3s_2-8 \end{pmatrix}$ is the matrix representation of $(u \mapsto ASu)|_{\operatorname{Im} A}$ with respect to the above basis of $\operatorname{Im} A$.

We choose $H = \{h_t : t \in \mathbb{R}\}$, where $h_t = t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} t \\ 1 \\ 2-t \end{pmatrix}$ as our hyperplane in $\operatorname{Im} A$. Then by definition, $t \in R_s$ holds if and only if either $Sh_t \geq 0$ or $Sh_t \leq 0$ (componentwise).

In the Table 1, the regions $R_s \subset \mathbb{R}\mathrm{P}^1$ and the matrices $A^{[s]} = AS_s$ are given explicitly. The second line serves to show the calculation. Line 3 contains simplifications of the conditions in line 2.

s	$(1,1,1)$	$(-1,1,1)$	$(1,1,-1)$	$(1,1,-1)$
R_s	$t \geq 0 \wedge 2-t \geq 0$	$t \leq 0 \wedge 2-t \geq 0$	$t \leq 0 \wedge 2-t \leq 0$	$t \geq 0 \wedge 2-t \leq 0$
	$0 \leq t \leq 2$	$t \leq 0$	$-$	$2 \leq t$
$A^{[s]}$	$\begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix}$	$\begin{pmatrix} 1 & -5 \\ 3 & -5 \end{pmatrix}$	$\begin{pmatrix} 5 & -7 \\ 5 & -11 \end{pmatrix}$	$\begin{pmatrix} 1 & -7 \\ 3 & -11 \end{pmatrix}$

Table 1. Computation of the regions and the corresponding matrices $A^{[s]}$ for the Ducci matrix of Example 2.

We observe that the region according to $s = (1,1,-1)$ is empty, as $R_s \cup R_{-s}$ has empty intersection with $\operatorname{Im} A \setminus \{0\}$. If $s = (1,1,1)$, then R_s is a trap since $A^{[s]}$ is singular.

Using that $A^{[s]}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_2 \neq 0$ means $\rho_A(\begin{bmatrix} t \\ 1 \end{bmatrix}) = \begin{bmatrix} x_1/x_2 \\ 1 \end{bmatrix}$, we now can give the explicit form of $\rho_A : \mathbb{P}_A \to \mathbb{P}_A$, where we identify $\begin{bmatrix} t \\ 1 \end{bmatrix}$ with t and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with ∞:

$$\rho_A(t) = \begin{cases} \frac{t-5}{3t-5} & \text{if } t \leq 0, \\ 1 & \text{if } 0 \leq t \leq 2,\ t \neq 1, \\ \frac{t-7}{3t-11} & \text{if } 2 \leq t \text{ and } t \neq \frac{11}{3}, \\ \infty & \text{if } t = \frac{11}{3}, \\ \frac{1}{3} & \text{if } t = \infty, \\ \mathbf{0} & \text{if } t = 1 \text{ or } t = \mathbf{0}. \end{cases}$$

Figure 4 shows its graph with the trap $[0,2]$ and a sample ρ_A-orbit (of $t = \frac{17}{5}$, corresponding to the vector $(17,5,-7)^\mathsf{T}$). Note that, contrary to what the figure suggests, $\rho_A(1) = \mathbf{0}$. The pole at $t = \frac{11}{3}$ has no deeper significance, it reflects our choice of H. Topologically, $\mathbb{P}_A$ is a circle (plus the isolated point $\mathbf{0}$), on which ρ_A is continuous except at $t = 1$.

It is now easy to give the proof of Lemma 1.

Proof. If $0 \leq t \leq 2$, $t \neq 0$, then $\rho_A(t) = 1$. If $t \leq 0$, then $0 \leq \rho_A(t) \leq 1$ and $\mathbf{0}$ is reached in two steps. In $(2,\infty)$, ρ_A has a unique fixed point at $t_\star = 2 + \frac{\sqrt{15}}{3}$. Any other value $t > 2$, $t \neq t_\star$, is repelled from $t_\star$ and its ρ_A-iterates eventually reach one of the two other regions, and from there, finally $\mathbf{0}$. Thus the Ducci sequence of every $t \neq t_\star$ reaches $\mathbf{0}$. This proves that A is a Ducci matrix.

The sequence t_n of Lemma 1 was defined as $t_0 = 2$, $t_n = \rho_A^{-1}(t_{n-1})$. (The function $t \to \rho_A(t)$ is invertible for $2 < t < \frac{11}{3}$.) Also, $x_n = |y_n|$ holds, where

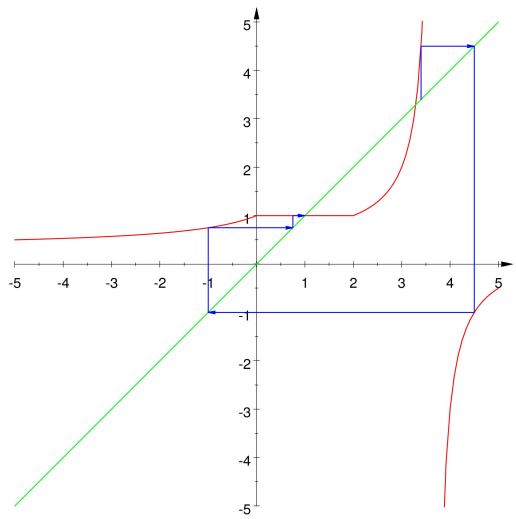

Figure 4. The reduced Ducci function ρ_{A_2} with the trap $[0, 2]$.

$y_n = (t_n, 1, 2 - t_n)^\mathsf{T} \in \operatorname{Im} A$. Thus $[\sigma_A(y_n)] = [y_{n-1}]$, and from $\lambda_A(x_0) = 2$ we obtain $\lambda_A(x_n) = n + 2$. ■

The reduced form of the Ducci map also makes it obvious why the Ducci sequences of A_2 tend to be so remarkably short: For $t \leq 2$ the length is bounded by 3, and for $t > 2$ it is large only as long as t is close to $t_\star$. But the slope of ρ_{A_2} at $t_\star$ is $4 + \sqrt{15} = 7.87\ldots$, a rather large value, so that the ρ_{A_2}-orbits of all points near $t_\star$ are repelled very quickly from this fixed point. The point $t_\star$ corresponds to $x_\star = (6 + \sqrt{15}, 3, -\sqrt{15})^\mathsf{T}$, which is an eigenvector of $A_2 S$ for $S = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. Its absolute value $|x_\star|$ is (up to multiples and the addition of constant vectors) the only exception vector of A_2.

6. FAMILIES OF DUCCI MATRICES. The preceding proof can be summarized by saying that points $t \neq t_\star$ are driven away from $t_\star$ until they reach the trap. Next we prove a result for a family of Ducci matrices of size 4 whose dynamics are similar to this.

For $p \in \mathbb{R}$, define $A_c(p) = \left(\begin{smallmatrix} p{+}1 & -1 & 1{-}p & -1 \\ -1 & p{+}1 & -1 & 1{-}p \\ 1{-}p & -1 & p{+}1 & -1 \\ -1 & 1{-}p & -1 & p{+}1 \end{smallmatrix}\right)$. The subscript indicates that these are *circulant* matrices [**12**]. These matrices satisfy the nice functional equation $A_c(p)A_c(q) = 4A_c(\frac{pq}{2})$. The value $p = 2$ produces the matrix $A_1 = \left(\begin{smallmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{smallmatrix}\right)$ of Example 1.

To analyze $A_c(p)$, we need some further notation. As the number of regions increases exponentially with the size of the matrices, it will be helpful to assign indices to them. Which indexing we choose is not really relevant; we shall use $\iota(s_1, \ldots, s_n) = \sum_{k=1}^{n} \frac{1-s_k}{2} 2^{k-1}$. Henceforth we shall write R_i for the regions and S_i, with $i = \iota(s_1, \ldots, s_n)$, for the sign matrices, instead of $R_{(s_1,\ldots,s_n)}$ and $S_{(s_1,\ldots,s_n)}$. Again, R_i can denote a subset of either $\mathbb{R}^n$ or $\mathbb{P}_A$. Also, for brevity, we shall write A instead of $A_c(p)$ in this section.

We want to determine the reduced Ducci function ρ_A. As a basis for $\mathbb{R}^4 = \operatorname{Im} A \oplus \operatorname{Coker} A$, we choose the columns of $B = \left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{smallmatrix}\right)$. Explicitly, $\operatorname{Im} A$ consists

of the vectors $x = (x_1, x_2, x_3, -x_1 - x_2 - x_3)^\mathsf{T}$, from which the regions R_i, defined by the inequalities $S_i x \geq 0$, can be calculated as in the preceding section. Instead of a table, we display the regions graphically in Figure 5. As in the preceding example, we have one empty region. In Figure 5 it is R_0.

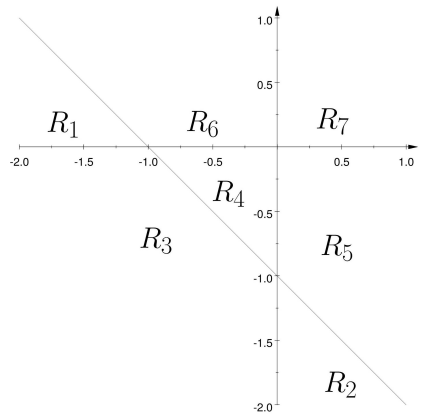

Figure 5. The seven regions of $\mathbb{P}_{A_c(s)}$.

Next we calculate the matrices $A^{[i]} = (B^{-1}AS_iB)|\operatorname{Im} A$:

$$A^{[1]} = \begin{pmatrix} -p & 0 & 2-p \\ p & 2p & p-2 \\ p & 0 & p+2 \end{pmatrix}, A^{[2]} = \begin{pmatrix} p+2 & 2 & 2-p \\ p-2 & -2 & p-2 \\ 2-p & 2 & p+2 \end{pmatrix}, A^{[3]} = \begin{pmatrix} -p & 2 & 2-p \\ p & -2 & p-2 \\ p & 2 & p+2 \end{pmatrix}, A^{[4]} = \begin{pmatrix} p+2 & 0 & p \\ p-2 & 2p & p \\ 2-p & 0 & -p \end{pmatrix},$$

$$A^{[5]} = \begin{pmatrix} -p & 0 & p \\ p & 2p & p \\ p & 0 & -p \end{pmatrix},\ A^{[6]} = \begin{pmatrix} p+2 & 2 & p \\ p-2 & -2 & p \\ 2-p & 2 & -p \end{pmatrix},\ A^{[7]} = \begin{pmatrix} -p & 2 & p \\ p & -2 & p \\ p & 2 & -p \end{pmatrix}.$$

In region R_i, the reduced function ρ_A is given by $[x] \mapsto \left[A^{[i]}x\right]$, and if we represent $[(x_1, x_2, x_3)^\mathsf{T}]$ by $(\frac{x_1}{x_3}, \frac{x_2}{x_3})^\mathsf{T} \in \mathbb{R}^2$ we obtain the explicit form of ρ_A. In order to avoid indices, we write x, y instead of x_1, x_2. We also omit the cases where the denominator becomes zero.

$$\rho_A(x,y) = \begin{cases} (\frac{-p-px+2}{p+px+2}, \frac{p+px+2py-2}{p+px+2}) & \text{if } x+y \leq -1 \wedge 0 \leq y \quad (R_1), \\ (\frac{2x-p+2y+px+2}{p+2x+2y-px+2}, \frac{-2x+p-2y+px-2}{p+2x+2y-px+2}) & \text{if } x+y \leq -1 \wedge 0 \leq x \quad (R_2), \\ \frac{p-2y+px-2}{p+2y+px+2}(-1,1) & \text{if } x+y \leq -1 \wedge x \leq 0 \wedge y \leq 0 \\ & \quad \wedge (x,y) \neq (-1,-1) \quad (R_3), \\ (\frac{-p-2x-px}{p-2x+px}, \frac{-p+2x-px-2py}{p-2x+px}), & \text{if } -1 \leq x+y \wedge x \leq 0 \wedge y \leq 0 \ (R_4), \\ (-1, \frac{x+2y+1}{x-1}) & \text{if } -1 \leq x+y \wedge 0 \leq x \wedge y \leq 0 \\ & \quad \wedge (x,y) \neq (1,-1) \quad (R_5), \\ (\frac{p+2x+2y+px}{p-2x-2y+px}, -1) & \text{if } -1 \leq x+y \wedge x \leq 0 \wedge 0 \leq y \\ & \quad \wedge (x,y) \neq (-1,1) \quad (R_6), \\ (\frac{p+2y-px}{2y-p+px}, \frac{p-2y+px}{2y-p+px}), & \text{if } 0 \leq x \wedge 0 \leq y \quad (R_7). \end{cases}$$

The formula may look somewhat forbidding, but one just has to compute $(\frac{x_1}{x_3}, \frac{x_2}{x_3})$ from $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{[i]} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.

Theorem 1. *For every $p \in \mathbb{R}$, the matrix $A = A_c(p)$ is a Ducci matrix. If $p \neq 0$, the reduced Ducci function ρ_A has four exception points in $\mathbb{P}_A$. For $p > 0$ these are fixed points, and for $p < 0$ they are 2-periodic points of ρ_A. There exist points with arbitrarily long Ducci lengths.*

Proof. From the explicit representation of ρ_A it is obvious that regions 3, 5 and 6 are traps. If $v \in \mathbb{P}_A$ is in one of these regions, then v has a length of at most 3: R_3 is mapped onto the line $x + y = 0$, $\rho_A(x, -x) = (-1, -1)$, and $\rho_A(-1, -1) = \mathbf{0}$. Regions 5 and 6 are similarly mapped onto the lines $x = -1$ and $y = -1$, respectively, and then in two steps to $\mathbf{0}$. Thus all points in the three traps have a finite length.

The other four regions each contain a fixed or 2-periodic point of ρ_A. We discuss R_4 in detail. Regions 1, 2 and 7 can be dealt with in a completely analogous way.

Region R_4 is the triangle with corners $(0, 0), (0, -1)$ and $(-1, 0)$. Let us denote $\rho_A|\, R_4$ by f, that is, $f(x, y) = (\frac{-p-2x-px}{p-2x+px}, \frac{-p+2x-px-2py}{p-2x+px})$. The image $f(R_4)$ is the triangle, call it T_0, with the vertices $v_1 = (-1, 1), v_2 = (1, -1)$ and $v_3 = (-1, -1)$. Since R_4 is not a trap region, $A^{[4]}$ is invertible and, therefore, f is a bijection between R_4 and $T_0 = f(R_4)$. Its inverse is $g(x, y) = f^{-1}(x, y) = (-\frac{p+px}{p-2x+px+2}, -\frac{2y+2}{p-2x+px+2})$.

Now let us assume that $p > 0$. Then the images $T_n = g^n(T_0)$ form a series of triangles such that the vertices of T_n lie on the sides of T_{n-1} $(n > 0)$. One side of each triangle is parallel to the x-axis (since the first component of f does not depend on y). For each n, triangle T_n consists of four smaller triangles: T_n^i $(i = 0, \ldots, 3)$, where $T_n^0 = T_{n+1}$ and T_n^i contains the corner $g^{(n)}(v_i)$ for $i = 1, 2, 3$. Triangle T_n^i is mapped by f bijectively to T_{n-1}^i $(n > 0,\ i = 0, \ldots, 3)$.

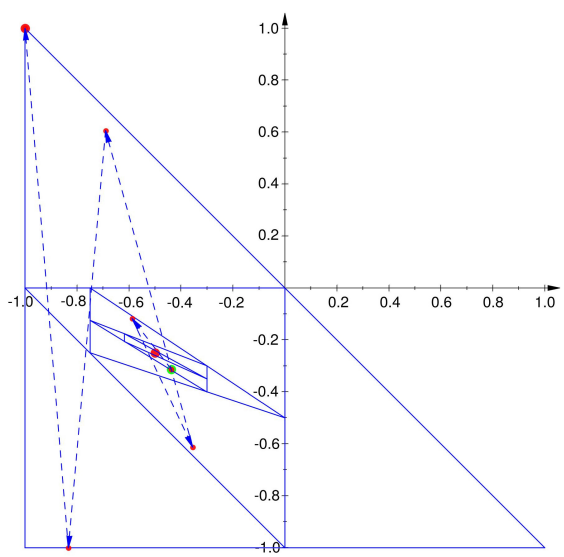

Figure 6. The $\rho_{A(6)}$-orbit of $(-\frac{7}{16}, -\frac{5}{16})$

The intersection of the triangles T_n is the sole fixed point $t_\star(p) = (x_\star, y_\star)$ of f. It depends on p: $t_\star(2) = -(\frac{1}{3}, \frac{1}{3})$ and $t_\star(p) = (\frac{r-p-1}{p-2}, \frac{3-r}{2\,p-4})$ if $0 < p \neq 2$, where $r = \sqrt{4\,p+1}$.

Note that $\begin{pmatrix} x_\star \\ y_\star \\ 1 \end{pmatrix}$ is an eigenvector of $A^{[4]}$ in region 4 for all $p > 0$. Note further that

$\lim_{p\to\infty} t_\star(p) = (-1,0)$ and $\lim_{p\downarrow 0} t_\star(p) = (0,-\frac{1}{2})$ and that the fixed points $t_\star(p)$ all lie on the line segment connecting these two limit points.

The dynamics of f in $T_1 = R_4$ are now easily described: For every $v \in T_1 \setminus \{t_\star(p)\}$ there is some n such that $v \in T_n \setminus T_{n+1}$. Then $v \in T_n^i$ for some $i \in \{1,2,3\}$, and $f^k(x) \in T_{n-k}^i$ for $k = 0,\dots,n$. Now $T_0^1 \subset R_6$, $T_0^2 \subset R_5$ and $T_0^3 \subset R_3$, that is, the orbit of x reaches a trap region after n steps, and from there, $\mathbf{0}$ in at most three further steps.

The proofs for regions 1, 2 and 7 are analogous. The fixed points of ρ_A in these regions are $(-\frac{r+p+1}{p}, \frac{r+1}{2p})$, $(1, -\frac{r+3}{2})$ and $(1, \frac{r-1}{2})$ (with r as above).

If $p < 0$, the situation differs in one detail: The preimages of the points in R_4, that is, the points $g(v)$ with $v \in R_4$ all lie in R_1, not in R_4. But their preimages again lie in R_4. It is not difficult to see that points $v \in R_4 \setminus \{t_\star\}$ are driven away from $t_\star$ by repeated application of ρ_A^2 until they reach one of the three traps. The point $t_\star \in R_4$ (and its counterpart in R_1) is a 2-periodic point of ρ_A in this case. But as for $p > 0$, its Ducci sequence does not terminate. Figure 7 shows the fixed points and the 2-periodic points, respectively, of $\rho_{A_c(p)}$ for $p = 6$ and $p = -6$.

Finally, the case $p = 0$ is trivial. All vectors reach zero in at most two steps. ■

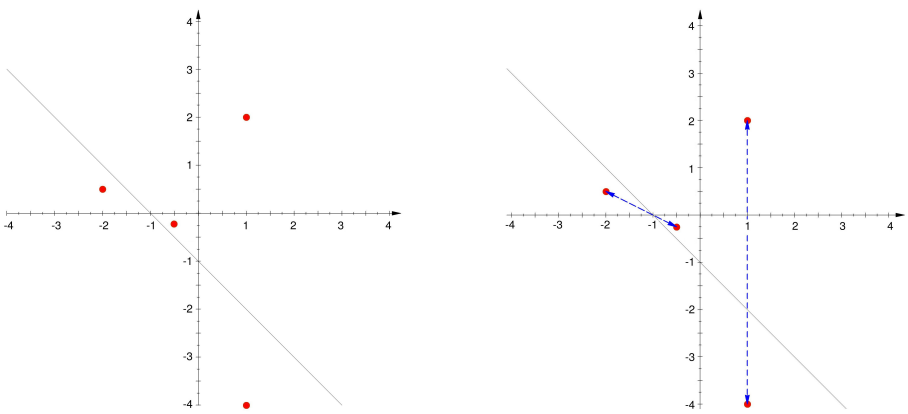

Figure 7. The exception sets of $A_c(6)$ (left, four fixed points) and of $A_c(-6)$ (right, four 2-periodic points).

There are other families of Ducci matrices which are similar. For instance, the matrices $A_a(p) = \begin{pmatrix} p+1 & -1 & 1-p & -1 \\ -1 & 1-p & -1 & p+1 \\ 1-p & -1 & p+1 & -1 \\ -1 & p+1 & -1 & 1-p \end{pmatrix}$ show almost the same behavior. The subscript a indicates that these are *anticirculant* matrices (see [**12**]). The exception set of $A_a(p)$ for $p \neq 0$ again consists of four points, but now two of these are fixed points of $\rho_{A_a(p)}$, whereas the other two are 2-periodic.

There is a counterpart family $B_c(p) = \begin{pmatrix} 1 & -1 & p & -p \\ -p & 1 & -1 & p \\ p & -p & 1 & -1 \\ -1 & p & -p & 1 \end{pmatrix}$, $p \in \mathbb{R}$, to $A_c(p)$ which, for $p = 0$, contains Ducci's original matrix. The matrices $B_c(p)$ as well as their anticirculant companions $B_a(p) = \begin{pmatrix} 1 & -1 & p & -p \\ -1 & p & -p & 1 \\ p & -p & 1 & -1 \\ -p & 1 & -1 & p \end{pmatrix}$ are Ducci matrices for all $p \in \mathbb{R}$. The exception set each consists of eight points. The exception set of $B_c(p)$ consists, for $|p| > 1$, of two 2-periodic pairs and one 4-cycle, for $|p| < 1$, of four fixed points and one 4-cycle. The exception set of $B_a(p)$ consists, for $|p| \neq 1$, of one 8-cycle.

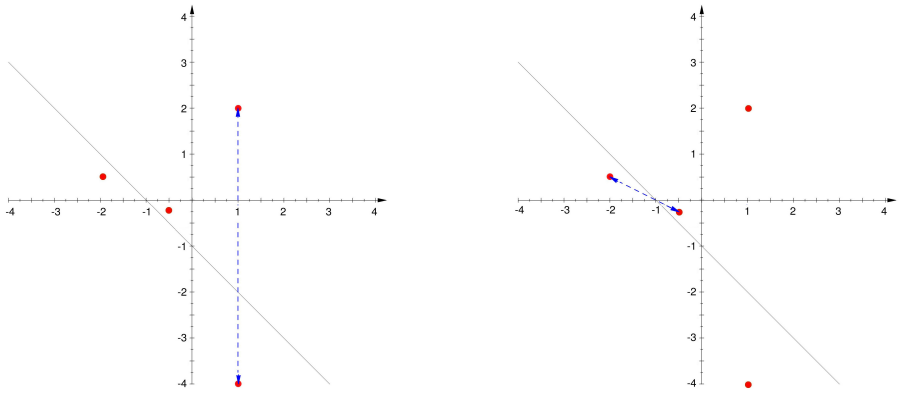

Figure 8. The exception sets of $A_a(6)$ (left) and $A_a(-6)$ (right).

The proof that these matrices are Ducci and have the indicated exception sets is lengthy but elementary. We omit it but note that all matrices have their kernel and cokernel spanned by $(1,1,1,1)^\mathsf{T}$, so that they all have the same regions (see Figure 5). In all cases, regions 3, 5 and 6 are traps and the exception points are situated in the non-trap regions 1, 2, 4 and 7. For the calculation of $\mathbb{P}_A$ we can use the same basis as in the proof of Theorem 1.

As a partial substitute for the missing proofs, let us look at a few details for the matrix $A = B_a(0) = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$. Since $A^4 = 2\begin{pmatrix} 3 & -2 & 1 & -2 \\ -2 & 3 & -2 & 1 \\ 1 & -2 & 3 & -2 \\ -2 & 1 & 2 & -3 \end{pmatrix}$, Lemma 2 tells us that A is a $\mathbb{Z}$-Ducci matrix. The exception set of A is shown in Figure 9.

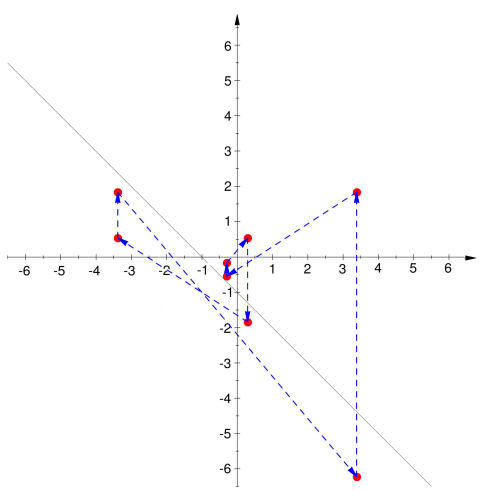

Figure 9. The exception set of $B_a(0)$.

The upper left exception point is $v_0 = \left(\begin{smallmatrix} -3.382\ldots \\ 1.839\ldots \end{smallmatrix}\right) = -\frac{1}{14}\left(\begin{smallmatrix} 2x^2+22x+46 \\ x^2+4x+26 \end{smallmatrix}\right)$, where $x = 0.061\ldots$ is the only real root of $f(x) = x^3 + 16x^2 + 64x - 4$. An explicit form of x is $x = \frac{(3a-8)^2}{9a}$ with $a = \frac{1}{3}(42\sqrt{33} + 566)^{\frac{1}{3}}$. The point v_0 is a fixed point of ρ_A^8. The ρ_A-iterates of any point $v \neq v_0$ close enough to v_0 approximately follow the same cycle as those of v_0, with increasing distance to the fixed points until eventually one of the trap regions is reached. After that, zero is reached in at most five additional steps.

From Figure 9 one can see that the exception points cycle through the non-trap regions in the order 7-4-4-7-2-1-1-2. Let us, for example, start with the approximation $v = \frac{1}{1000}\left(\begin{smallmatrix}-3382\\1839\end{smallmatrix}\right)$ of v_0. Since a point $v = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) \in \mathbb{P}_A$ corresponds with the vector $\tilde{v} = (a, b, 1, -(1+a+b))^\mathsf{T} \in \mathbb{R}^4$, we have to calculate the σ_A-Ducci sequence of $\tilde{v} = (-3383, 1839, 1000, 544)^\mathsf{T}$:

$$\begin{pmatrix}-3383\\1839\\1000\\544\end{pmatrix} \to \begin{pmatrix}1544\\-2839\\456\\839\end{pmatrix} \to \begin{pmatrix}-1295\\-705\\-383\\2383\end{pmatrix} \to 2\begin{pmatrix}295\\544\\-1000\\161\end{pmatrix} \to 2\begin{pmatrix}-249\\-134\\839\\-456\end{pmatrix} \to 2\begin{pmatrix}115\\207\\383\\-705\end{pmatrix} \to$$

$$4\begin{pmatrix}-46\\295\\-161\\-88\end{pmatrix} \to 4\begin{pmatrix}-249\\42\\73\\134\end{pmatrix} \to 4\begin{pmatrix}207\\-115\\-61\\-31\end{pmatrix} \to \cdots \to 32\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \to 32\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \to \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

The whole σ_A-Ducci sequence is 18 steps long, and the numbers of the regions visited are 1, 2, 7, 4, 4, 7, 2, 1, 1, 2, 7, 4, 4, 6, 2, 3, 4, 5. The first trap this sequence reaches is R_6 after 13 steps. We notice that the vector $(207, -115, -61, -31)^\mathsf{T}$, reached after the first 8 steps, corresponds to $\begin{pmatrix}-207/61\\115/61\end{pmatrix} = \left(\begin{smallmatrix}-3.393..\\1.885..\end{smallmatrix}\right)$, which is still rather close to $v = \left(\begin{smallmatrix}-3.382\\1.839\end{smallmatrix}\right)$.

7. LARGE EXCEPTION SETS. The families of Ducci matrices in the preceding section all have exception sets of a fixed size. Our next topic is a family of 3×3 Ducci matrices whose exception sets can become arbitrarily large. It will be seen that, as a consequence, these matrices produce Ducci sequences whose *average* length tends to be large. This is in stark contrast to the behavior exhibited by most other Ducci matrices.

For $p > 0$ let $A_l(p) = \begin{pmatrix}p & 1 & -p-1\\0 & 1 & -1\\p & -1 & -p+1\end{pmatrix}$. Obviously, this family is equivalent to the family of matrices $\begin{pmatrix}a & b & -a-b\\0 & b & -b\\a & -b & -a+b\end{pmatrix}$ where $a, b > 0$. In $A_3 = A_l(8)$ of Example 3, we have already encountered a typical representative of this family.

Theorem 2. *For every $p \in \mathbb{N}$ there exists a 3×3 Ducci matrix such that the Ducci sequence of a random vector needs at least p steps until it terminates.*

If $p \in \mathbb{N}$, then $A = A_l(p)$ is such a Ducci matrix. Its exception set $\mathscr{X}_A \subset \mathbb{P}_A$ consists of $2p+1$ points forming a cycle with respect to the reduced Ducci map.

Keep in mind that, although $A_l(p)$ is defined for real parameters $p > 0$, this theorem is only concerned with integer values of p. To prove it, let us calculate the reduced Ducci functions of the matrices $A = A_l(p)$. We do this for all $p > 0$. First, it is easy to check that $\operatorname{Im} A$ is spanned by the vectors $(1, 0, 1)^\mathsf{T}$ and $(0, 1, -2)^\mathsf{T}$ and that the cokernel of A is spanned by $(-1, 2, 1)^\mathsf{T}$. If we choose $H = \{h_t : t \in \mathbb{R}\}$ with $h_t = t\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}t\\1\\t-2\end{pmatrix}$ as normalizing hyperplane, then $t \in \mathbb{P}_A$ corresponds to $h_t \in \operatorname{Im} A$.

The columns of $B = \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\\1 & -2 & 1\end{pmatrix}$ form a basis of $\mathbb{R}^3$. From $B^{-1}ASB = \begin{pmatrix}ps_1-p-1 & 2p+s_2+2 & -ps_1+p+2s_2-1\\-1 & s_2+2 & 2s_2-1\\0 & 0 & 0\end{pmatrix}$, with $S = \begin{pmatrix}s_1 & 0 & 0\\0 & s_2 & 0\\0 & 0 & 1\end{pmatrix}$ (where $s_1, s_2 \in \{-1, 1\}$), we find that the maps $(u \to ASu)_{|\operatorname{Im} A}$ have the matrix representations $A^{[s]} = \begin{pmatrix}ps_1-p-1 & 2p+s_2+2\\-1 & s_2+2\end{pmatrix}$ with respect to B. Using these matrices, we calculate the reduced Ducci map of A in exactly the same way as in Section 5. We obtain

$$\rho_A(t) = \begin{cases} 1 + \frac{2p}{1-t} & \text{if } t \le 0, & (R_2) \\ 2p+1 & \text{if } 0 \le t \le 2,\ t \ne 1, & (R_3) \\ 1 + \frac{2p}{3-t} & \text{if } 2 \le t \text{ and } t \ne 3, & (R_0) \\ \infty & \text{if } t = 3, & \\ 1 & \text{if } t = \infty, & \\ \mathbf{0} & \text{if } t = 1 \text{ or } t = \mathbf{0}. & \end{cases}$$

Region $R_3 = [0, 2]$ is the only trap of A. It does not contain a fixed point of ρ_A. The exception points of A stem from fixed points of ρ_A^{2p+1} as shown in the next lemma.

Lemma 7. *For $p \in \mathbb{N}$, the ρ_A-iterates $t_i = \rho_A^i(t_0)$ of $t_0 = \sqrt{1+p^2} + p + 2$ form a cycle of length $2p+1$. The values in this cycle are $t_{2i} = t_0 - 2i$ $(i = 1, \ldots, p)$ and $t_{2i+1} = 1 - \frac{2p}{t_{2i}-3}$ $(i = 0, \ldots, p-1)$.*

Proof. Clearly, the explicit values of the points t_i satisfy $t_0 > t_2 > \cdots > t_{2p} = 2 + \sqrt{1+p^2} - p > 2$. Hence $t_{2i} \in R_0$ and from the explicit form of ρ_A we immediately see that $\rho_A(t_{2i}) = t_{2i+1}$ for $i < p$. Likewise, it is clear that $t_{2p-1} < t_{2p-3} < \cdots < t_1 < 0$ for $i < p$, that is, $t_{2i+1} \in R_2$. Using $\rho_A(t) = 1 + \frac{2p}{1-t}$ for $t \in R_2$ we obtain $\rho_A(t_{2i+1}) = t_{2i+2}$ for $i < p$. Finally, as $t_{2p} > 2$, $\rho_A(t_{2p}) = 1 + \frac{2p}{3-t_{2p}} = t_0$. ■

Let us call $t_0, \ldots, t_{2p}$ the *exception cycle* of A. A better insight into this cycle is gained if one looks at the square of ρ_A. For any (not necessarily integer-valued) $p > 0$ we find $\rho_A^2(t) = t - 2$ if $t \in [3, 2p+3]$. If $p \in \mathbb{N}$, the points $t_0, \ldots, t_{2p}$ also constitute a cycle with respect to the map ρ_A^2, but now in the strictly decreasing order $t_0 \to t_2 \to \cdots \to t_{2p} \to t_1 \to t_3 \to \cdots \to t_{2p-1}$. Figure 10 shows the trajectory of t_0 under ρ_A^2. Notice how closely the transition from $t_{2p} = \sqrt{1+p^2} - p + 2$ with $t_{2p} > 2$ to $t_1 = \rho_A^2(t_{2p}) = p - \sqrt{1+p^2}$ with $t_1 < 0$ avoids the trap $R_3 = [0, 2]$ which if reached would break the cycle.

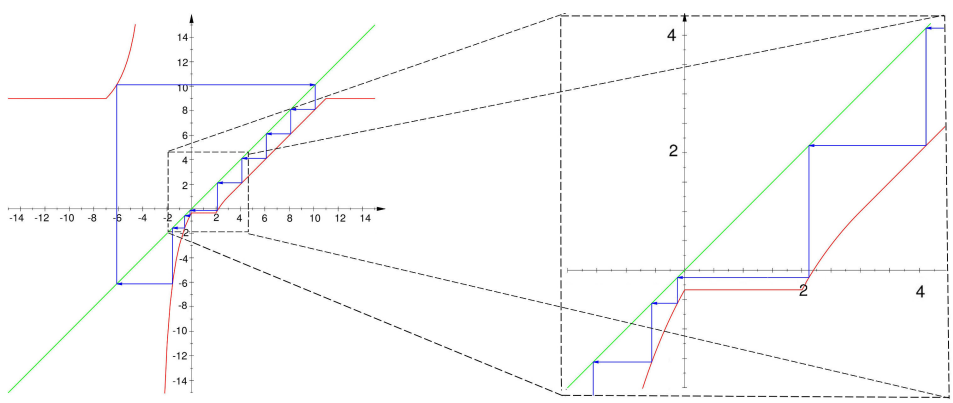

Figure 10. The graph of $\rho_{A_l(4)}^2$ with its cycle of exception points.

For all points $t \in \mathbb{R}$ not in the exception cycle, the sequence of iterates $(\rho_A^i(t))_{i=0,1,\ldots}$ eventually reaches 1 (and then $\mathbf{0}$). To see this, we take a closer look at ρ_A^{2p+1}. It is defined on $\mathbb{P}_A$, but as with ρ_A we shall work on the circle $\mathbb{R} \cup \{\infty\}$.

Define the $2p+1$ intervals $I_0, \ldots, I_{2p}$ by $I_i = (t_{i+2}, t_i)$ for $i = 0, \ldots, 2p-2$, $I_{2p-1} = (t_0, t_{2p-1})$, and $I_{2p} = (t_1, t_{2p})$. Interval I_{2p-1} is to be understood in the circular sense as $(t_0, t_{2p-1}) = (\mathbb{R} \cup \{\infty\}) \setminus [t_{2p-1}, t_0]$.

Also, let $u_0 = 2p + \frac{1}{p+1}, r_0 = 2p+1, v_0 = 2p+2$, and $u_i = \rho_A^i(u_0)$, $r_i = \rho_A^i(r_0)$, $v_i = \rho_A^i(v_0)$ for $i = 1, \ldots, 2p$.

Lemma 8. *Let $p \in \mathbb{N}$ and $i \in \{1, \ldots, 2p\}$. Then $r_i \in (u_i, v_i) \subset I_i$ and $\rho_A^{2p+1}|\,[u_i, v_i] \equiv r_i$ holds. For $i = 2p-1$, where $r_i = \infty$, the statement is meant in the circular sense.*[2]

Proof. The first statement, for the particular case $i = 0$, amounts to $t_2 = \sqrt{1+p^2} + p < u_0 = 2p + \frac{1}{p+1} < r_0 = 2p+1 < v_0 = 2p+2 < t_0 = \sqrt{1+p^2} + p + 2$, which is true for all $p > 0$. For $i > 0$ the statement then follows from the (circular) monotonicity of ρ_A. To obtain the second statement, one first verifies $\rho_A^{2p}(u_0) = 0$, $\rho_A^{2p+1}(u_0) = 2p+1$, and hence $\rho_A^{2p+1}(u_i) = r_i$. The same conclusion for $\rho_A^{2p+1}(v_i)$ follows from $\rho_A^{2p}(v_0) = 2$, and likewise we get $\rho_A^{2p+1}(t) = r_i$ for $t \in [u_i, v_i]$. ■

The sequence $r_0, \ldots, r_{2p}$ is a kind of counterpart to the exception cycle of A. Explicitly it looks as follows: $r_{2i} = r_0 - 2i$ for $i = 1, \ldots, p$, $r_{2i+1} = 1 - \frac{p}{p-i-1}$ for $i = 0, \ldots, p-2$, and $r_{2p-1} = \infty$. If we alter ρ_A by replacing $\rho_A(1) = \mathbf{0}$ with $\rho_A(1) = 2p+1$ (thereby making ρ_A continuous on the circle), then $r_0, \ldots, r_{2p}$ is a period of ρ_A into which all Ducci sequences not starting in the exception cycle would eventually run:

Lemma 9. *For every $t \in \mathbb{R}$ not in $\{r_0, \ldots, r_{2p}, t_0, \ldots, t_{2p}\}$, the reduced Ducci sequence $(\rho_A^i(t))_{i=1,2,\ldots}$ ends with $r_0, \ldots, r_{2p}, \mathbf{0}$.*

Proof. In the closure of I_i, ρ_A^{2p+1} has the fixed points t_{i+2}, r_i, and t_i by Lemmas 7 and 8. Also, ρ_A^{2p+1} is strictly increasing in $[t_{i+2}, u_i]$, constant in $[u_i, v_i]$, and strictly increasing in $[v_i, t_i]$. To prove this, we note that as long as the sequence $(\rho_A^k(t))_{k=1,2,\ldots}$ for some $t \in [v_i, t_i]$ does not reach region 3, that is, the trap $[0, 2]$, it runs through the intervals $I_i, I_{(i+1) \bmod 2p+1}, \ldots$. Thus, after $2p+1$ steps it returns to I_i. The region numbers in which $t, \rho_A(t), \ldots, \rho_A^{2p}(t)$ lie therefore are given by some circular permutation of $\underbrace{0, 2, 0, 2, \ldots, 0, 2}_{p \text{ times}}, 0$.

For example, if $t \in (v_0, t_0)$, then this is the sequence $0, 2, \ldots, 2, 0$. Let $f_0(t) = 1 + \frac{2p}{3-p}$ and $f_2(t) = 1 + \frac{2p}{1-p}$ be the explicit form of ρ_A in regions 0 and 2, respectively. Then, for $t \in (v_0, t_0)$, we know that $\rho_A^{2p+1} = f_0 \circ (f_2 \circ f_0)^p$, and as $f_2(f_0(t)) = t - 2$, we obtain $\rho_A^{2p+1}(t) = f_0(t - 2p) = \frac{4p+3-t}{2p+3-t}$. If we start with $t \in (t_2, u_0)$, then the sequence of region numbers in which $t, \rho_A(t), \ldots, \rho_A^{2p}(t)$ lie is $\underbrace{0, 2, 0, 2, \ldots, 0, 2}_{p-1 \text{ times}}, 0, 0, 2$, and we arrive at $\rho_A^{2p+1}(t) = f_2(f_0(f_0(t - 2(p-1)))) = \frac{t-1}{2p-t+1}$. Figure 11 shows the situation in the interval I_0 for $p = 4$.

The same calculation as for I_0 can be done for the intervals I_i $(i = 1, \ldots, 2p)$. It follows that the orbit of every $t \in I_i$ arrives in $[u_i, v_i]$ after finitely many iterations of ρ_A^{2p+1}, and from there in $\{r_0, \ldots, r_{2p}, \mathbf{0}\}$. Since $I_0 \cup \cdots \cup I_{2p} \cup \{t_0, \ldots, t_{2p}\} = \mathbb{R} \cup \{\infty\}$, this proves the lemma. ■

[2] That is, $v_{2p-1} < t_{2p-1}, t_{2p+1} = t_0 < u_{2p-1}$, and $\rho_A^{2p+1}|\,\mathbb{R} \cup \{\infty\} \setminus (v_{2p-1}, u_{2p-1}) \equiv \infty$.

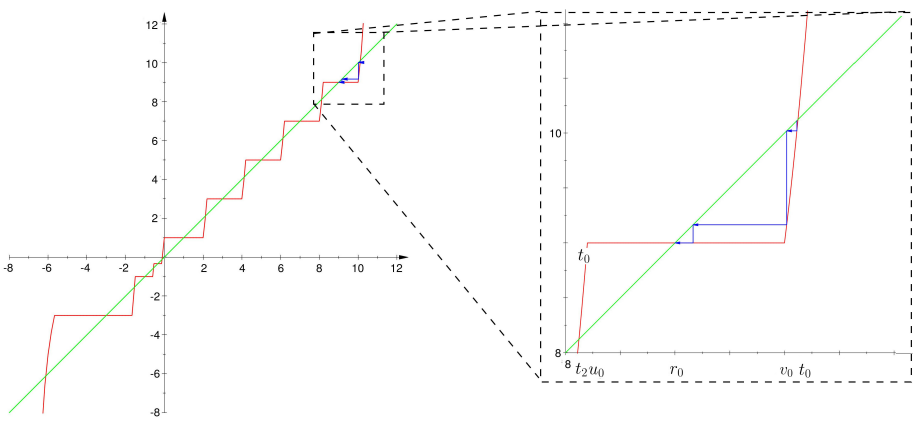

Figure 11. The graph of $\rho_{A_l(p)}^{2p+1}$ for $p = 4$, with a sample trajectory in I_0.

Lemma 9 concludes the proof of Theorem 2. Although the dynamics of the Ducci matrices $A_l(p)$ with $p \in \mathbb{N}$ are more intricate than those of most other examples of size $n = 3$, the basic mechanism is similar: Repelling fixed points and a trap are cooperating in sending all but finitely many starting points to zero.

What can be said about the Ducci sequences for $A = A_l(p)$ if p is not an integer? Numerical evidence indicates that for the continuous version[3] $\overline{\rho}_A$ of ρ_A the iterates $r_i = \rho_A^i(r_0)$ of $r_0 = 2p + 1$ still form a finite cycle $r_0, \ldots, r_L$ into which the ρ_A-orbits of all but finitely many $t \in \mathbb{R}$ eventually run. The finitely many exceptions apparently also form a ρ_A-cycle, $t_0, \ldots, t_L$, of the same length. The length L, however, can vary wildly, in particular for values $p = n + \frac{1}{2} + \varepsilon$ ($n \in \mathbb{N}$, $\varepsilon > 0$ small) it seems to get huge.

Two particular values of p deserve special mention:

Proposition 1. *For $p = \frac{2}{3}$, $A = A_l(p) = \frac{1}{3}\begin{pmatrix} 2 & 3 & -5 \\ 0 & 3 & -3 \\ 2 & -3 & 1 \end{pmatrix}$ is a Ducci matrix. The Ducci sequence of $t \in \{\frac{2\sqrt{3}}{3} + 3, -\frac{2\sqrt{3}}{3} + 1, \frac{2\sqrt{3}}{3} + 1, \frac{\sqrt{3}}{3} + 2\}$ circulates through these four points. The ρ_A-orbit of $r_0 = \frac{7}{3}$ is $\frac{7}{3}, 3, \infty, 1, \mathbf{0}$. The orbit of all other $t \in \mathbb{R}$ contains $\frac{7}{3}$, and hence $\mathbf{0}$.*

Proof. The continuous version $\overline{\rho}_A$ of the reduced function is

$$\overline{\rho}_A(t) = \begin{cases} \frac{3\,t-7}{3\,t-3} & \text{if} \quad t \le 0, \\ \frac{7}{3} & \text{if} \quad t \in [0, 2], \\ \frac{3\,t-13}{3\,t-9} & \text{if} \quad 2 \le t \wedge t \ne 3, \\ \infty & \text{if} \quad t = 3, \text{ and } 1 \text{ if } t = \infty, \end{cases}$$

from which the proof follows. We omit the details.[4] ■

Proposition 2. *For $p = \frac{1}{2}$, $A = A_l(p) = \frac{1}{2}\begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -2 \\ 1 & -2 & 1 \end{pmatrix}$ is* not *a Ducci matrix. The ρ_A-orbits of $t \in \{1, 3, \infty\} \cup \{2 + \frac{1}{n} : n \in \mathbb{N}\}$ contain $\mathbf{0}$. The orbits of all other points contain the fixed point $t_0 = 2$ of ρ_A.*

Proof. Again, the statement follows easily once $\overline{\rho}_A$ is calculated. It is given by

[3] defined by $\overline{\rho}_A(x) = \rho_A(x)$ for $x \ne 1$ and $\overline{\rho}_A(1) = 2p + 1$

[4] Recall that $t \in \mathbb{R}$ corresponds to $(t, 1, t-2)^\mathsf{T} \in \operatorname{Im} A$ and $t = \infty$ to $(1, 0, 1)^\mathsf{T} \in \operatorname{Im} A$.

$$\overline{\rho}_A(t) = \begin{cases} \frac{t-2}{t-1} & \text{if} \quad t \le 0, \\ 2 & \text{if} \quad t \in [0,2], \\ \frac{t-4}{t-3} & \text{if} \quad 2 \le t \wedge t \ne 3, \\ \infty & \text{if} \quad t = 3, \text{ and } 1 \text{ if } t = \infty. \end{cases}$$

At $t = 2$, this function is tangential to the diagonal. We leave the detailed proof as an exercise. ■

The value $t = 2$ corresponds to the eigenvector $(2,1,0)^\mathsf{T}$ of A. A typical (unsigned) Ducci sequence for A looks like this one:

$$\begin{pmatrix}343\\141\\81\end{pmatrix} \to 2\begin{pmatrix}181\\50\\81\end{pmatrix} \to 2^2\begin{pmatrix}19\\31\\81\end{pmatrix} \to 2^3\begin{pmatrix}81\\50\\19\end{pmatrix} \to 2^4\,31\begin{pmatrix}2\\1\\0\end{pmatrix} \to 2^5\,31\begin{pmatrix}2\\1\\0\end{pmatrix} \to \cdots,$$

which can be viewed as a kind of termination at $(2,1,0)^\mathsf{T}$ instead of $(0,0,0)^\mathsf{T}$.

8. OPEN QUESTIONS. We have only scratched the surface of the subject of Ducci matrices. This is a safe claim because it is by far easier to find matrices that with high plausibility are suspected to be Ducci than to verify them as such. There are lots of candidates awaiting confirmation, and it is easy to produce further ones by diverse heuristics (Lemma 4 helps to curtail the search space).

In fact, not even the matrices $A_{0,2^k}$ $(k > 2)$, proved to be $\mathbb{Z}$-Ducci matrices in [**10**] (see also Lemma 2), have been verified to be Ducci matrices.

Or, consider $A_q(p) = \begin{pmatrix} p & -p-1 & -p-2 & p+3 \\ -p-1 & -p-2 & p+3 & p \\ -p-2 & p+3 & p & -p-1 \\ p+3 & p & -p-1 & -p-2 \end{pmatrix}$ where $p \in \mathbb{R}$. These matrices, among them $A_4 = A_q(6)$ of Example 4, are remarkable for generating Ducci sequences of astronomical length when $|p|$ is not too small. Are they Ducci matrices?

Worse still, for $n \ge 5$ there are no $n \times n$ matrices confirmed to be Ducci apart from those composed of smaller Ducci matrices.

For sizes $n = 3$ and $n = 4$ a number of problems are open; concerning the exception sets, which can become infinite; concerning their classification, not by some parametrizations but by deeper properties; and many more.

We would like to conclude with a question which may have a simple answer. All integer Ducci matrices A of size n considered in this paper (and all others we have found and mostly not verified yet) have the property that A^n is even in the sense that $A^n = 2B$ for some integer matrix B. Are there Ducci matrices without this property?

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