Outline

1 Introduction
   - Naive approach to non-Archimedean analytic functions
   - Motivation for a more sophisticated theory

2 Classical rigid spaces à la Tate, Grauert, Remmert, Kiehl, ...

3 Rigid spaces via Raynaud’s formal schemes
Non-Archimedean Fields

Definition

An *absolute value* on a field $K$ is a map $|·|: K \rightarrow \mathbb{R}_{\geq 0}$ such that

- $|\alpha| = 0 \iff \alpha = 0$
- $|\alpha \beta| = |\alpha||\beta|$
- $|\alpha + \beta| \leq |\alpha| + |\beta|$

$|·|$ is called *non-Archimedean* if

- $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$

Fundamental examples: $\mathbb{Q}_p$, $\mathbb{C}_p$ for $p \in \mathbb{N}$ prime

$|·|_p: \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}$, $|\alpha|_p := \begin{cases} 0 & \text{if } \alpha = 0 \\ \frac{1}{p^n} & \text{if } \alpha = \frac{a}{b}p^n, \ a, b \in \mathbb{Z}, \ p \nmid ab \end{cases}$

$\mathbb{Q}_p = \text{completion of } \mathbb{Q} \text{ under } |·|_p$ ($p$-adic analogue of $\mathbb{R}$)

$\mathbb{C}_p = \text{completed algebraic closure of } \mathbb{Q}_p$ ($p$-adic analogue of $\mathbb{C}$)
### Non-Archimedean Calculus

**$K$ field with complete non-Archimedean absolute value**

- $\sum_{\nu=0}^{\infty} \alpha_{\nu}$ converges in $K \iff \lim_{\nu \to \infty} |\alpha_{\nu}| = 0$
- $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$ if $|\alpha| \neq |\beta|$.

### Proof of 2nd assertion

\[
|\beta| < |\alpha| \implies |\alpha + \beta| \leq \max\{|\alpha|, |\beta|\} = |\alpha| = |\alpha + \beta - \beta| \\
\leq \max\{|\alpha + \beta|, |\beta|\} = |\alpha + \beta|
\]

### Geometric implications

- Triangles in $K$ are isosceles
- $D_1, D_2 \subset K$ disks, $D_1 \cap D_2 \neq \emptyset \implies D_1 \subset D_2$ or $D_2 \subset D_1$
- Disk $D = \{x ; |x - a| < r\}$ is open and closed
- $K$ is totally disconnected
Locally Analytic Functions

Problems
- have no non-trivial continuous paths
- have no line integrals in the usual sense
- have no identity theorems

Example
\[ f : K \rightarrow K, \quad f(x) = \begin{cases} 
0 & \text{if } |x| < 1 \\
1 & \text{if } |x| \geq 1 
\end{cases} \]

is locally analytic on \( K \), identically zero in neighborhood of 0, but is not identically zero globally

Conclusion
Notion of local analyticity is rather weak, cannot derive meaningful global properties
Motivating Example (J. Tate, 1959)

\( K, \ | \cdot | \) algebraically closed field, complete, non-Archimedean

\[ \mathcal{O}(K^*) := \{ \sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu ; \lim_{|\nu| \to \infty} |c_\nu| r^\nu = 0 \text{ for all } r > 0 \} \]

globally convergent Laurent series on \( K^* \)

\[ \mathcal{M}(K^*) := \text{Frac}(\mathcal{O}(K^*)) \]

field of “meromorphic” functions on \( K^* \)

let \( q \in K, 0 < |q| < 1 \)

\[ \mathcal{M}^q(K^*) := \{ f \in \mathcal{M}(K^*) ; f(q\zeta) = f(\zeta) \} \]

field of \( q \)-periodic meromorphic functions on \( K^* \)

Observation

\( \mathcal{M}^q(K^*) \) is an elliptic function field with non-integral \( j \)-invariant; i. e. \( |j| > 1 \). Points of associated Tate elliptic curve \( E_K \):

\[ K^*/q^\mathbb{Z} \cong E_K(K) \hookrightarrow \mathbb{P}^2(K) \]

But: Quotient \( K^*/q^\mathbb{Z} \) does not make sense in algebraic geometry
Comparison with complex elliptic curves

\[ \Gamma = \mathbb{Z} \oplus \mathbb{Z} \omega \text{ a lattice in } \mathbb{C} \text{ with } \omega \notin \mathbb{R} \]

Weierstrass \( \wp \)-function defines an isomorphism \( \mathbb{C}/\Gamma \sim \rightarrow E \subset \mathbb{P}^2(\mathbb{C}) \)
on to an elliptic curve \( E \) in \( \mathbb{P}^2 \)

use exponential map \( \mathbb{C} \longrightarrow \mathbb{C}^*, \ z \longmapsto e^{2\pi i z} \), get isomorphism \( \mathbb{C}/\mathbb{Z} \sim \rightarrow \mathbb{C}^* \), thus

\[ \mathbb{C}^*/q^{\mathbb{Z}} \sim \mathbb{C}/\Gamma \sim E \subset \mathbb{P}^2(\mathbb{C}), \quad q = e^{2\pi i \omega} \]

uniformization of \( E \) from multiplicative point of view

Program of Tate

Develop theory of analytic functions over non-Archimedean fields such that \( K^*/q^{\mathbb{Z}} \) makes sense as a Riemann surface, which can be algebrained to yield an elliptic curve over \( K \)
Outline

1 Introduction

2 Classical rigid spaces à la Tate, Grauert, Remmert, Kiehl, . . .
   - Tate algebras
   - Affinoid algebras
   - Localization of affinoid algebras
   - Affinoid spaces and their subdomains
   - Tate’s Acyclicity Theorem
   - Grothendieck topologies
   - Sheaves
   - Construction of global rigid spaces
   - Some advanced results

3 Rigid spaces via Raynaud’s formal schemes
Basic Literature

- J. Tate: Rigid analytic spaces. Private notes, distributed with(out) his permission by IHES (1962). Reprinted in Inv. math. 12, 257-289 (1971)
- S. Bosch: Lectures on formal and rigid geometry. SFB-Preprint Münster (2005)
  wwwmath.uni-muenster.de/sfb/about/publ/heft378.ps
**Restricted power series**

**K** complete, with non-Archimedean \(|\cdot|\)

\(\overline{K}\) alg. closure, with extension of \(|\cdot|\), not necessarily complete

\(B^n(\overline{K}) = \{(x_1, \ldots, x_n) \in \overline{K}^n ; |x_i| \leq 1 \text{ for all } i\}\)

A power series \(\sum_{\nu} c_\nu \zeta^\nu\) with coefficients in \(K\) and variables \(\zeta = (\zeta_1, \ldots, \zeta_n)\) converges globally on \(B^n(\overline{K})\) iff \(\lim_{\nu} c_\nu = 0\)

**Definition**

\(T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle = \{\sum_{\nu} c_\nu \zeta^n ; \lim_{\nu} c_\nu = 0\}\) is called the Tate algebra of restricted power series in the variables \(\zeta_1, \ldots, \zeta_n\)

**Properties**

\(T_n\) is noetherian, factorial, jacobson.

*Noether normalization*: for any ideal \(\mathfrak{a} \subsetneq T_n\) there is a finite monomorphism \(T_d \hookrightarrow T_n/\mathfrak{a}\)
Corollary to Noether normalization

\[ m \subseteq T_n \text{ maximal ideal} \implies [T_n/m: K] < \infty \]

Proof

Choose \( T_d \hookrightarrow T_n/m \) finite. Then:

\[ T_n/m \text{ a field} \implies T_d \text{ a field,} \]

hence, \( d = 0, \ T_d = K, \) and \( [T_n/m: K] < \infty \)

\[ \square \]

Definition

\( \text{Sp } T_n = \{ \text{maximal ideals } \subseteq T_n \} \) is called the max spectrum of \( T_n \)
For $x = (x_1, \ldots, x_n) \in B^n(K)$, have evaluation homomorphism

$$\text{eval}_x : T_n \longrightarrow K(x_1, \ldots x_n) \hookrightarrow \bar{K},$$

$$\sum_{\nu} c_{\nu} \zeta_{\nu} \longmapsto \sum_{\nu} c_{\nu} x_{\nu}$$

Note:

$K(x_1, \ldots, x_n)$ is finite over $K$, hence complete

$\Longrightarrow$ eval$_x$ factors surjectively through $K(x_1, \ldots, x_n)$,

$\Longrightarrow$ $m_X := \ker(\text{eval}_x)$ is a maximal ideal in $T_n$

Proposition

The map $B^n(K) \longrightarrow \text{Sp } T_n, x \longmapsto m_X$, induces a bijection

$$B^n(K)/\text{Aut}_K(K) \sim \text{Sp } T_n$$
$T_n$ as a Banach algebra

**View elements $f \in T_n$ as functions on $\text{Sp } T_n$**
- for $x \in \text{Sp } T_n$ let $f(x)$ be residue class of $f$ in $T_n/m_x$
- $f(x) = 0 \iff f \in m_x$
- $|f(x)|$ is well-defined

**Norms on $T_n$**
- **Gauß norm**: $|\sum_{\nu} c_{\nu} \zeta^\nu|_{\text{Gauß}} \coloneqq \max |c_{\nu}|$
- **Supremum norm**: $|f|_{\text{sup}} = \sup_{x \in \text{Sp } T_n} |f(x)|$

**Proposition**
- $|f|_{\text{Gauß}} = |f|_{\text{sup}}$ for all $f \in T_n$;
- $T_n$ is a Banach $K$-algebra under $| \cdot | = \text{Gauß or sup norm}$

**Note**: $|f|_{\text{Gauß}} \geq |f|_{\text{sup}}$ is obvious; for full proof need reduction
Method: Reduction of coefficients

\[ R := \{ \alpha \in K \mid |\alpha| \leq 1 \} \]  
valuation ring of \( K \)

\[ m := \{ \alpha \in K \mid |\alpha| < 1 \} \]  
valuation ideal, maximal ideal in \( R \)

\[ k := R/m \]  
residue field

Canonical projection \( R \longrightarrow k, \alpha \longmapsto \tilde{\alpha} \), induces a reduction map

\[ R\langle \zeta \rangle \longrightarrow k[\zeta], \quad f = \sum_{\nu} c_{\nu} \zeta^{\nu} \longmapsto \tilde{f} = \sum_{\nu} \tilde{c}_{\nu} \zeta^{\nu}, \]

compatible with evaluation at points \( x \in B^n(K) \)

Maximum Principle

Given \( f \in T_n \), there is \( x \in \text{Sp } T_n \) such that \( |f|_{\text{Gauß}} = |f(x)| \)

Classical rigid spaces à la Tate, Grauert, Remmert, Kiehl, . . .  
Tate algebras
Gauß norm on $T_n$

Proof of Maximum Principle

let $f \in T_n$, may assume $|f|_{\text{Gauß}} = 1$; then $f \in R\langle \zeta \rangle$ and $\tilde{f} \neq 0$
choose $\tilde{x} \in \overline{k}^n$ such that $\tilde{f}(\tilde{x}) \neq 0$, lift to $x \in B^n(\overline{K})$
get $f(x) = \tilde{f}(\tilde{x}) \neq 0$, hence $|f(x)| = 1$
in particular, get $|f|_{\text{Gauß}} \leq |f|_{\sup}$, hence, $|f|_{\text{Gauß}} = |f|_{\sup}$

Properties of Gauß resp. sup norm $| \cdot |$

- $|f| = 0 \iff f = 0$
- $|fg| = |f||g|
- $|f + g| \leq \max\{|f|, |g|\}$

Proposition (non-trivial)

$\alpha \subset T_n$ ideal $\implies \alpha$ is closed in $T_n$
Affinoid $K$-algebras

Definition

Call a $K$-algebra $A$ **affinoid** if there is an epimorphism $\pi: T_n \to A$; call $\text{Sp } A = \{\text{maximal ideals } \subset A\}$ an **affinoid $K$-space**

Properties

- $\text{Sp } A \xrightarrow{\sim} \text{V}(a) = \{x \in \text{Sp } T_n ; a \subset m_x\}$ for $a = \ker \pi$
- Set $|f| = \inf\{|g| ; g \in \pi^{-1}(f)\}$ for $f \in A$
- $|\cdot|$ is a so-called **residue norm** on $A$, satisfies
  - $|f| = 0 \iff f = 0$ (since $a$ is closed in $T_n$)
  - $|fg| \leq |f||g|
  - $|f + g| \leq \max\{|f|, |g|\}$
- $A$ is complete with respect to residue norm $|\cdot|$

  - also have sup norm $|\cdot|_{\text{sup}}$ on $A$, is **semi-norm**,
    - $|f|_{\text{sup}} = 0 \iff f$ is nilpotent
    - $|f|_{\text{sup}} \leq |f|$
Fact: Let \( A \rightarrow B \) finite homomorphism of affinoid \( K \)-algebras, \( f \in B \). Then exist \( a_1, \ldots, a_r \in A \) such that 
\[
f^r + a_1 f^{r-1} + \ldots + a_r = 0, \quad |f|_{\text{sup}} = \max_i |a_i|_\text{sup}^{1/i}
\]

Proposition (non-trivial)

Let \( A, |\cdot| \) be an affinoid \( K \)-algebra with a residue norm, \( f \in A \). Then:
- \( |f|_{\text{sup}} \leq 1 \iff |f^n| \) for \( n \in \mathbb{N} \) is bounded (\( f \) is power bounded)
- \( |f|_{\text{sup}} < 1 \iff \lim_n |f^n| = 0 \) (\( f \) is topologically nilpotent)
- \( |\cdot|_{\text{sup}} \) equivalent to \( |\cdot| \iff A \) reduced

Corollary

Let \( \varphi : A \rightarrow B \) be a homomorphism of affinoid \( K \)-algebras; fix any residue norms on \( A \) and \( B \). Then \( \varphi \) is continuous.
Hence, all residue norms on an affinoid \( K \)-algebra are equivalent.
Let $A$ affinoid $K$-algebra, $f_0, \ldots, f_r \in A$, assume $V(f_0, \ldots, f_r) = \emptyset$; i.e., $f_0, \ldots, f_r$ generate unit ideal in $A$

set $A\langle \frac{f}{f_0} \rangle = A\langle \frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0} \rangle = A\langle \zeta_1, \ldots, \zeta_r \rangle / (f_i - f_0 \zeta_i; i = 1, \ldots, r)$

**Lemma**

Canonical homomorphism $\varphi : A \longrightarrow A\langle \frac{f}{f_0} \rangle$ satisfies:

- $\varphi(f_0)$ is a unit
- $\frac{\varphi(f_i)}{\varphi(f_0)}$, $i = 1, \ldots, r$, are power bounded

Furthermore, $\varphi$ is universal: any $\varphi' : A \longrightarrow B$ satisfying above conditions factors uniquely through $\varphi : A \longrightarrow A\langle \frac{f}{f_0} \rangle$

**Remark**

$A\langle \frac{f}{f_0} \rangle$ is (non-canonical) completion of localization $A[f_0^{-1}]$
Proof of localization Lemma

Let $x \in \text{Sp } \mathcal{A}(\frac{f}{f_0})$, assume $f_0(x) = 0$ (drop $\varphi$ in notation), have $f_i = f_0 \zeta_i$ in $\mathcal{A}(\frac{f}{f_0})$, get $f_i(x) = 0$ for all $i$, hence, $(f_0, \ldots, f_r) \neq (1)$, contradiction!

Thus: $f_0$ has no zero on $\text{Sp } \mathcal{A}(\frac{f}{f_0})$, hence, $f_0$ is unit in $\mathcal{A}(\frac{f}{f_0})$.

have $\zeta_i = \frac{f_i}{f_0}$ in $\mathcal{A}(\frac{f}{f_0})$; hence, by continuity of $\mathcal{A}(\zeta) \longrightarrow \mathcal{A}(\frac{f}{f_0})$: all $\frac{f_i}{f_0}$ are power bounded in $\mathcal{A}(\frac{f}{f_0})$.

Universal property:

$$
\begin{array}{ccc}
A & \longrightarrow & \mathcal{A}(\zeta_1, \ldots, \zeta_r) & \longrightarrow & \mathcal{A}(\frac{f}{f_0}) \\
\varphi' & \downarrow & & \downarrow \varphi'(f_i) \\
& & \mathcal{A}(\frac{f}{f_0}) & \longrightarrow & B
\end{array}
$$
Examples of localization

\[ A = T_1 = K \langle \zeta \rangle \] restricted power series, \( \pi \in K^* \)

\[ A\langle \frac{\zeta}{\pi} \rangle = K \langle \zeta, \eta \rangle / (\zeta - \pi \eta) \]
\[ = \begin{cases} \{ \sum_{\nu \in \mathbb{N}} c_\nu \zeta^\nu \} \text{ converging for } |\zeta| \leq |\pi| \text{ if } |\pi| \leq 1 \\ K \langle \zeta \rangle \text{ if } |\pi| \geq 1 \end{cases} \]

\[ A\langle \frac{\pi}{\zeta} \rangle = K \langle \zeta, \eta \rangle / (\pi - \zeta \eta) \]
\[ = \begin{cases} \{ \sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu \} \text{ conv. for } |\pi| \leq |\zeta| \leq 1 \text{ if } |\pi| \leq 1 \\ 0 \text{ if } |\pi| > 1 \end{cases} \]
Lemma

Any $\varphi: A \longrightarrow B$ homomorphism of affinoid $K$-algebras induces a map

$$a \varphi: \text{Sp } B \longrightarrow \text{Sp } A, \quad m \longmapsto \varphi^{-1}(m)$$

Proof

$A \xrightarrow{\varphi} B$

$K \xhookrightarrow{A/\varphi^{-1}(m)} \xhookrightarrow{B/m}$

$B/m$ is finite over $K$, hence $B/m$ is integral over $A/\varphi^{-1}(m)$; $B/m$ field $\implies A/\varphi^{-1}(m)$ field

Definition

$a \varphi: \text{Sp } B \longrightarrow \text{Sp } A$ as in Lemma is called the morphism of affinoid $K$-spaces associated to $\varphi: A \longrightarrow B$
Let $A$ affinoid $K$-algebra, $f_0, \ldots, f_r \in A$ without common zeros

**Lemma**

The canonical map $\varphi : A \rightarrow A\langle \frac{f}{f_0} \rangle$ induces a bijection

\[ a\varphi : \text{Sp } A\langle \frac{f}{f_0} \rangle \sim \rightarrow U = \{ x \in \text{Sp } A ; |f_1(x)|, \ldots, |f_r(x)| \leq |f_0(x)| \} \]

**Definition**

A subset $U \subset \text{Sp } A$ as in Lemma is called a **rational subdomain** of the affinoid $K$-space $\text{Sp } A$

Consider $X = \text{Sp } A$ together with its rational subdomains as a category, inclusions as morphisms, define a **presheaf** $\mathcal{O}_X$ on it by

\[ X \supset U \simeq \text{Sp } A\langle \frac{f}{f_0} \rangle \sim A\langle \frac{f}{f_0} \rangle \]

restriction morphisms by universal property of localizations $A\langle \frac{f}{f_0} \rangle$
Tate’s Acyclicity Theorem

Let $X = \text{Sp} A$ an affinoid $K$-space with presheaf $\mathcal{O}_X$
$\mathcal{U} = (U_i)_{i \in I}$ a finite covering of $X$ by rational subdomains $U_i \subset X$

**Theorem**

$\mathcal{U}$ is $\mathcal{O}_X$-acyclic; i.e., the augmented Čech complex

$$
0 \to \mathcal{O}_X(X) \to \prod_{i_0 \in I} \mathcal{O}_X(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{O}_X(U_{i_0} \cap U_{i_1}) \to \ldots
$$

is exact

**Corollary**

$\mathcal{O}_X$ is a sheaf in the following sense:
given $f_i \in \mathcal{O}_X(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j$,

there is a unique $f \in \mathcal{O}_X(X)$ such that $f|_{U_i} = f_i$ for all $i$.
Proof of Acyclicity Theorem, basic case

\( X = \mathbb{B}_K^1 = \text{Sp} \ K \langle \zeta \rangle \) unit disk, let \( \pi \in K^*, |\pi| < 1 \)

\( U_1 = \{ x \in \mathbb{B}_K^1 ; |x| \geq |\pi| \} \), \( |x| := |\zeta(x)| \)

\( U_2 = \{ x \in \mathbb{B}_K^1 ; |x| \leq |\pi| \} \)

\( \mathcal{O}_X(U_1) = \{ \sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu ; \text{converging on } |\pi| \leq |\zeta| \leq 1 \} \)

\( \mathcal{O}_X(U_2) = \{ \sum_{\nu \in \mathbb{N}} c_\nu \zeta^\nu ; \text{converging on } |\zeta| \leq |\pi| \} \)

\( \mathcal{O}_X(U_1 \cap U_2) = \{ \sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu ; \text{converging on } |\zeta| = |\pi| \} \)

\[
\sum_{\nu \in \mathbb{Z}} a_\nu \zeta^\nu \quad \leftrightarrow \quad \bullet \quad \leftrightarrow \quad \sum_{\nu \in \mathbb{N}} b_\nu \zeta^\nu
\]

coincidence on \( U_1 \cap U_2 \) means \( a_\nu = b_\nu, \nu \geq 0 \) and \( a_\nu = 0, \nu < 0 \),
\( \sum_{\nu \in \mathbb{Z}} a_\nu \zeta^\nu \) converges on \( U_1 \), e. g. at \( x = 1 \) (max. ideal is \( (\zeta - 1) \))

\[
\lim_{n \to \infty} |a_\nu| = 0 \implies \sum_{\nu \in \mathbb{N}} b_\nu \zeta^\nu \in K \langle \zeta \rangle = \mathcal{O}_X(\mathbb{B}_K^1)\]
Implications from Tate’s theorem

Strategy for constructing global analytic spaces

- glue affinoid $K$-spaces as local parts
- do this under the constraints of Tate’s Acyclicity Theorem: glue with respect to admissible open overlaps ($=\text{rational subdomains in our case}$), respect finite coverings on affinoid pieces
- get so-called rigid analytic spaces

Technical tool: Grothendieck topology on a topological space $X$

- specify certain open sets $U \subset X$ as admissible open sets
- specify certain open coverings $U = \bigcup_{i \in I} U_i$, where $U$ and all $U_i$ are admissible open in $X$, as admissible open coverings
- require a minimal amount of compatibility conditions such that Čech cohomology can be done in the admissible setting
A Grothendieck topology (short: G-topology) consists of

- a category $\text{Cat } T$
- for each object $U \in \text{Cat } T$ a set $\text{Cov } U$ of families $(U_i \xrightarrow{\Phi_i} U)_{i \in I}$ of morphisms in $\text{Cat } T$, called coverings

such that:

- $\Phi: U' \to U$ isomorphism $\implies (\Phi) \in \text{Cov } U$
- composition: $(U_i \to U)_i \in \text{Cov } U$ and $(U_{ij} \to U_i)_j \in \text{Cov } U_i$
  for all $i \implies (U_{ij} \to U_i \to U)_{ij} \in \text{Cov } U$
- restriction: $(U_i \to U)_i \in \text{Cov } U$ and $V \to U$ a morphism in $\text{Cat } T$ $\implies (U_i \times_U V \to V)_i \in \text{Cov } V$; in particular, the fiber products $U_i \times_U V$ must exist in $\text{Cat } T$

**Note**, for a $G$-topology on a topological space $X$:

$\text{Cat } T =$ category of admissible open sets in $X$, with inclusions as morphisms; fiber products exist: $U_i \times_U V = U_i \cap V$
Topologies on an affinoid $K$-space $X = \text{Sp} A$

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<th><strong>Canonical topology</strong></th>
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<td>Topology generated by all <strong>rational subdomains</strong> $U \subset X$</td>
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<th><strong>Weak $G$-topology</strong></th>
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<td><strong>admissible open sets</strong>: all rational subdomains in $X$</td>
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<th><strong>Strong $G$-topology</strong></th>
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| $U \subset X$ **admissible open** $\iff$ exists covering $U = \bigcup_{i \in I} U_i$ by rational subdomains $U_i \subset X$ such that:  
for any morphism of affinoid $K$-spaces $\varphi: Y \longrightarrow X$ with $\varphi(Y) \subset U$ the covering $(\varphi^{-1}(U_i))_i$ of $Y$ admits a finite refinement by rational subdomains of $Y$  
$U = \bigcup_{i \in I} U_i$ (with $U, U_i$ admissible open) is **admissible** $\iff U = \bigcup_{i \in I} U_i$ satisfies same condition as before |  |
**Definition**

Let $X$ be a set with a $G$-topology. A **presheaf** (of rings) on $X$ is a contravariant functor

$$\mathcal{F}: (\text{admissible opens } \subset X) \rightarrow (\text{rings})$$

It is called a **sheaf** if for any admissible covering $U = \bigcup_{i \in I} U_i$ the sequence

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is exact

**Remark**

The strong $G$-topology on affinoid $K$-spaces $X$ is a best possible refinement of the weak $G$-topology such that:

- morphisms of affinoid $K$-spaces remain **continuous**, 
- any **sheaf** with respect to the weak $G$-topology on $X$ extends uniquely to a **sheaf** with respect to the strong $G$-topology.
Let $X = \text{Sp} A$ be an affinoid $K$-space, with weak $G$-topology, have \textit{presheaf of affinoid functions} on $X$:

$$\mathcal{O}_X : \text{Sp} A\langle \frac{f}{f_0} \rangle \mapsto A\langle \frac{f}{f_0} \rangle$$

\textbf{Tate’s Theorem}

- The presheaf of affinoid functions $\mathcal{O}_X$ is a sheaf.
- $\mathcal{O}_X$ extends uniquely to a sheaf with respect to the strong $G$-topology, again denoted by $\mathcal{O}_X$.
- $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$

\textbf{Definition of affinoid $K$-spaces, revisited}

An \textit{affinoid $K$-space} is a pair $(X, \mathcal{O}_X)$, where $X = \text{Sp} A$, for an affinoid $K$-algebra $A$, is equipped with the \textit{strong} $G$-topology, and where $\mathcal{O}_X$ is (the extension of) the sheaf of affinoid functions on $X$.
Completeness conditions for strong $G$-topology

The **strong $G$-topology** on $X = \text{Sp} \ A$ satisfies:

(G₀) $\emptyset, X$ are admissible open

(G₁) let $U = \bigcup_{i \in I} U_i$ be an admissible covering, let $V \subset U$ be a subset such that $V \cap U_i$ is admissible open for all $i$. Then: $V$ is admissible open in $X$

(G₂) let $U = \bigcup_{i \in I} U_i$ be a covering with $U, U_i$ admissible open for all $i$, and assume that $(U_i)_i$ has an admissible refinement. Then: $(U_i)_i$ is an admissible covering of $U$

**Fact**

Let $X$ be a set with a $G$-topology which satisfies (G₀), (G₁), (G₂). Let $X = \bigcup_{i \in I} X_i$ be an admissible covering. Then the $G$-topology on $X$ can be **uniquely recovered** from

- the restriction of the $G$-topology to each $X_i$,
- knowing the overlaps $X_i \cap X_j$ as admissible opens in $X_i, X_j$
Global rigid spaces

**Definition**

A **rigid** $K$-space is a pair $X = (X, \mathcal{O}_X)$ where

- $X$ is a set with a $G$-topology on it, satisfying $(G_0)$, $(G_1)$, $(G_2)$,
- $\mathcal{O}_X$ is a sheaf of $K$-algebras on $X$ such that there exists an admissible covering $X = \bigcup_{i \in I} X_i$ where each $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid $K$-space.

A **morphism of rigid** $K$-spaces is a morphism in the sense of locally ringed spaces over $K$.

**Proposition**

Let $A, B$ be affinoid $K$-algebras, and let $X, Y$ be the associated affinoid $K$-spaces (in the sense of rigid $K$-spaces as above). Then the canonical map

$$\text{Hom}_K(A, B) \longrightarrow \text{Hom}_K(Y, X)$$

is bijective.
Construction of rigid spaces

Remark

Rigid $K$-spaces can be constructed by the usual glueing techniques. Need:

- a set $X$ with a covering $X = \bigcup_{i \in I} X_i$,
- each $X_i$ with structure of affinoid (or rigid) $K$-space, compatible with overlaps $X_i \cap X_j$ (admissible open in $X_i, X_j$).

Then, thanks to $(G_0), (G_1), (G_2)$, there is a unique structure as a rigid $K$-space on $X$ such that $X = \bigcup_{i \in I} X_i$ is an admissible covering (structural admissible covering of $X$)

Example: Affine $n$-space $\mathbb{A}^n_K$

Choose $\pi \in K$, $0 < |\pi| < 1$. Let $\zeta = (\zeta_1, \ldots, \zeta_n)$. Then

$$\text{Sp } K\langle \zeta \rangle \hookrightarrow \text{Sp } K\langle \pi \zeta \rangle \hookrightarrow \text{Sp } K\langle \pi^2 \zeta \rangle \hookrightarrow \ldots$$

is increasing sequence of balls with radii $1, |\pi|^{-1}, |\pi|^{-2}, \ldots$ gives structural admissible covering, get the rigid version of $\mathbb{A}^n_K$
Exotic structures on unit disk

- $\mathcal{B}_K^1 = \text{Sp } T_1$ (affinoid $K$-space, as usual, connected)
- $\tilde{\mathcal{B}}_K^1 = \{x ; |x| < 1\} \cup \{x ; |x| = 1\}$ as structural covering (not connected)

- $K$ algebraically closed, not sperically complete (e.g., $K = \mathbb{C}_p$)
  $\implies$ exists decreasing sequence of disks $\mathcal{B}_K^1 \supset D_1 \supset D_2 \supset \ldots$
  with empty intersection,
  take $\tilde{\mathcal{B}}_K^1 = \bigcup_i (\mathcal{B}_K^1 - D_i)$ as structural covering, get connected rigid $K$-space
  have canonical morphism $\tilde{\mathcal{B}}_K^1 \to \mathcal{B}_K^1$, becoming a (non-bijective) open immersion after enlarging $K$ suitably
More examples

Serre’s GAGA functor

The construction of the rigid affine $n$-space works more generally for Zariski closed subschemes of $\mathbb{A}^n_K$. Thereby get functor

\[(K\text{-schemes of locally finite type}) \longrightarrow (\text{rigid } K\text{-spaces})
\]

$X \mapsto X_{\text{rig}}$

In particular, the rigid version of projective $n$-space $\mathbb{P}^n_K$ is defined (can be covered by $n + 1$ unit balls)

Tate elliptic curves

Let $q \in K$, $0 < |q| < 1$. Then the quotient

\[E = \mathbb{G}_{m,K} / q^\mathbb{Z}\]

makes sense as a rigid $K$-space; is obtained by glueing the annuli $\{x \in \mathbb{B}^1_K ; |q|\leq |x| \leq |q|^{1/2}\}$ and $\{x \in \mathbb{B}^1_K ; |q|^{1/2} \leq |x| \leq 1\}$ via multiplication by $q$. As in complex analytic geometry, $E$ can be algebraized to yield an algebraic curve in $\mathbb{P}^2_K$.
Notions to translate from algebraic geometry

- open, resp. closed immersions
- separated, resp. proper morphisms
- coherent modules

Grothendieck’s Proper Mapping Theorem (Kiehl)

Let $\varphi : X \to Y$ be a proper morphism of rigid $K$-spaces and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then the higher direct images $R^i \varphi_*(\mathcal{F})$, $i \geq 0$, are coherent $\mathcal{O}_Y$-modules

Applications

- proper $\implies$ closed
- Stein factorization
- Serre’s GAGA theorems (Köpf)
- Chow’s Theorem
Outline

1. Introduction

2. Classical rigid spaces à la Tate, Grauert, Remmert, Kiehl, ...

3. Rigid spaces via Raynaud’s formal schemes
   - Motivating example
   - Admissible formal schemes
   - Generic fiber of a formal scheme
   - Admissible formal blowing-up
   - An equivalence of categories
   - Relative rigid spaces
   - Zariski-Riemann space
   - Some advanced results
Motivating example

Tate elliptic curve $\mathbb{G}_m, K / q^\mathbb{Z}$, Mumford style

$R$ valuation ring of $K$, field with complete non-Archimedean val.
$\text{Spec } R = \{\eta, s\}$
$\eta$ generic point ($\simeq$ zero ideal $\subset R$)
$s$ special point ($\simeq$ maximal ideal $\subset R$)
Tate elliptic curve, continued

Tate elliptic curve $G_m, \mathbb{K} / q^\mathbb{Z}$ comes equipped with $R$-model

Program

Make generic fibre of a formal scheme visible as rigid space!
Base ring $R$ replacing field $K$

**$R$ ring, assume:**

- $R$ **complete** and **separated** with respect to $I$-adic topology, for some finitely generated ideal $I = (g_1, \ldots, g_s) \subset R$
- $R$ has **no $I$-torsion**; i.e.:
  
  $$(I\text{-torsion})_R = \{ r \in R ; \ I^n r = 0 \text{ for some } n \in \mathbb{N} \} = 0$$
- no $I$-torsion is equivalent to: $R \hookrightarrow \prod_{i=1}^{s} R_{g_i}$
- consider only following cases:
  
  (N) $R$ is noetherian, or
  
  (V) $R$ is an adic valuation ring with a finitely generated ideal of definition $I$

- **Main example:** $R$ valuation ring of a field $K$ with a complete non-Archimedean absolute value $| \cdot |$, with $I = (t)$ for some $t \in R$, $0 < |t| < 1$
Definition

An $R$-algebra $A$ is called
- of tf (topologically finite) type if $A = R\langle \zeta_1, \ldots, \zeta_n \rangle / \mathfrak{a}$
- of tf presentation if $A = R\langle \zeta_1, \ldots, \zeta_n \rangle / (a_1, \ldots, a_s)$
- admissible if $A$ is of tf presentation and has no $I$-torsion

Theorem (based on flattening techniques of Raynaud-Gruson)

Let $A$ an $R$-algebra of tf presentation. Then $A$ is a coherent ring. In particular, any $A$-module $M$ of finite presentation is coherent; i.e., $M$ is finitely generated, and each finite submodule of $M$ is of finite presentation

Corollary

Let $A$ an $R$-algebra of tf type. If $A$ has no $I$-torsion, $A$ is of tf presentation and, hence, admissible
Admissible formal schemes

Corollary

Any $R$-algebra $A$ of tf type is $I$-adically complete and separated

For $\lambda \in \mathbb{N}$ set: $R_\lambda = R/I^{\lambda+1}$, $A_\lambda = A \otimes_R R_\lambda$; then $A = \lim_\leftarrow A_\lambda$

Construction of formal scheme $\text{Spf} \ A$

$$\text{Spf} \ A = \lim_\rightarrow \text{Spec} \ A_\lambda$$

more precisely: $\text{Spf} \ A$ is a locally ringed space $(X, \mathcal{O}_X)$ where

- $X = \text{Spec} \ A_0 \subset \text{Spec} \ A$ with Zariski topology
- $\mathcal{O}_X : D(f) \rightsquigarrow A\langle f^{-1} \rangle := \lim_\leftarrow A_\lambda [f^{-1}]$, for $f \in A$

Definition

A formal $R$-scheme of locally tf type is a locally ringed space which is locally of type $\text{Spf} \ A$ with $A$ an $R$-algebra of tf type.

Same for tf presentation, admissible
Properties of formal schemes

Proposition

Let $X = \text{Spf} \ A$ be a formal $R$-scheme, where $A$ is $I$-adically complete and separated. Equivalent:

- $X$ is locally of tf type
- $A$ is of tf type

Same assertion for tf presentation, admissible

Lemma

Let $A$ be an $R$-algebra of tf type; let $f_0, \ldots, f_r \in A$ generate the unit ideal. Then

$$A \longrightarrow \prod_{i=0}^{r} A\langle f_i^{-1} \rangle$$

is faithfully flat
Assume for a moment: $R$ valuation ring of field $K$ with complete non-Archimedean absolute value $|\cdot|$

**Proposition**

There is a canonical functor

\[
\text{rig} : \left(\text{formal } R\text{-schemes of locally tf type} \right) \longrightarrow (\text{rigid } K\text{-spaces})
\]

which on affine formal $R$-schemes is given by

\[
\text{Spf } A \leadsto \text{Sp } A_{\text{rig}}, \quad A_{\text{rig}} := A \otimes_R K
\]

**Proof**

\[
A = R\langle \zeta_1, \ldots , \zeta_n \rangle / \alpha \implies A_{\text{rig}} = A \otimes_R K = K\langle \zeta_1, \ldots , \zeta_n \rangle / (\alpha)
\]

is an affinoid $K$-algebra; globalization:

\[
A\langle f^{-1} \rangle_{\text{rig}} = A\langle \zeta \rangle / (1 - f \zeta) \otimes_R K = A_{\text{rig}} \langle \zeta \rangle / (1 - f \zeta) = A_{\text{rig}} \langle f^{-1} \rangle
\]
Question

How far is rig from being an equivalence of categories?

Observation

- Let $A$ an $R$-alg. of tf type, let $f_0, \ldots, f_r \in A$, $(f_0, \ldots, f_r) = A$

- Zariski covering $\text{Spf } A = \bigcup_i \text{Spf } A\langle f_i^{-1} \rangle$ yields via rig a formal covering $\text{Sp } A_{\text{rig}} = \bigcup_i \text{Sp } A_{\text{rig}}\langle f_i^{-1} \rangle$ of the associated affinoid $K$-space; recall

$$\text{Sp } A_{\text{rig}}\langle f_i^{-1} \rangle = \{x \in \text{Sp } A_{\text{rig}} ; |f_i(x)| \geq 1 = |f_i|_{\text{sup}}\}$$

- Exist more general coverings on $\text{Sp } A_{\text{rig}}$: let $f_0, \ldots, f_r \in A_{\text{rig}}$ such that $(f_0, \ldots, f_r) = A_{\text{rig}}$. Then $\text{Sp } A_{\text{rig}} = \bigcup_i \text{Sp } A_{\text{rig}}\langle f f_i \rangle$ is admissible (rational) covering, generally not induced via rig

- Note: $A_{\text{rig}}\langle f \rangle = A_{\text{rig}}\langle \zeta_0, \ldots, \hat{\zeta}_i, \ldots, \zeta_r \rangle/(f_j - f_i \zeta_j ; j \neq i)$ alludes to blow-up; recall

$$\text{Sp } A_{\text{rig}}\langle f f_i \rangle = \{x \in \text{Sp } A_{\text{rig}} ; |f_j(x)| \leq |f_i(x)| \text{ for all } j\}$$
Let $R$ base ring, general, of type (N) or (V), $I \subset R$ ideal of definition; let $X$ a formal $R$-scheme, locally of tf presentation.

An ideal $\mathcal{A} \subset \mathcal{O}_X$ is **coherent** if, locally on affine open pieces $\text{Spf} \mathcal{A} \subset X$, it is associated to an ideal of finite type $a \subset A$.

An ideal $\mathcal{A} \subset \mathcal{O}_X$ is called **open** if, locally on $X$, it contains powers of type $I^n \mathcal{O}_X$.

**Definition**

Let $\mathcal{A} \subset \mathcal{O}_X$ be a coherent open ideal. Then

$$X_\mathcal{A} = \lim_{\lambda \in \mathbb{N}} \text{Proj} \left( \bigoplus_{d=0}^{\infty} \mathcal{A}^d \otimes (\mathcal{O}_X/I^\lambda \mathcal{O}_X) \right)$$

together with canonical projection $X_\mathcal{A} \longrightarrow X$ is called the **formal blowing-up** of $\mathcal{A}$ on $X$. 

Properties of admissible formal blowing-up

\[ X_A = \lim_{\lambda \in \mathbb{N}} \text{Proj} \left( \bigoplus_{d=0}^{\infty} A^d \otimes (\mathcal{O}_X/I^{\lambda} \mathcal{O}_X) \right) \]

**Proposition**

- \( X = \text{Spf } A \) affine, \( A \subset \mathcal{O}_X \) associated to the coherent open ideal \( \mathfrak{a} \subset A \). Then \( X_A \) equals the \( I \)-adic completion of the scheme theoretic blowing-up of \( \mathfrak{a} \) on \( \text{Spec } A \)
- \( X \) admissible \( \implies \)
  - \( X_A \) admissible
  - \( A\mathcal{O}_{X_A} \) is invertible on \( X_A \)
  - \( X_A \rightarrow X \) satisfies universal property:
    - Any morphism of formal schemes \( \varphi : Y \rightarrow X \) such that \( A\mathcal{O}_Y \) is invertible on \( Y \) factors uniquely through \( X_A \rightarrow X \)
Explicit description of admissible formal blowing-up

**Basic Lemma**

Let $X = \text{Spf } A$ be admissible, assume that the coherent open ideal $A \subset \mathcal{O}_X$ is associated to the ideal $a = (f_0, \ldots, f_r) \subset A$. Then:

- Let $U_i \subset X_A$ be the locus where $A\mathcal{O}_{X_A}$ is generated by $f_i$, $i = 0, \ldots, r$. Then the $U_i$ define an affine open covering of $X_A$.

- Let
  
  $$C_i = A\langle \frac{f_i}{f_i} \rangle = A\langle \zeta_0, \ldots, \hat{\zeta}_i, \ldots, \zeta_r \rangle/(f_i\zeta_j - f_j; j \neq i)$$

  $$A_i = C_i/(I\text{-torsion})_{C_i}$$

  The $I$-torsion of $C_i$ coincides with its $f_i$-torsion, and

  $$U_i = \text{Spf } A_i$$

**Method of Proof**

Start with scheme theoretic blowing-up of $a$ on $\text{Spec } A$, and use a flatness lemma of Gabber for $I$-adic completion.
The classical case, again

Observation

Let the situation be as in Basic Lemma, but assume $R$ to be the valuation ring of a complete non-Archimedean field $K$. Then the functor $\text{rig}$ (which “is” tensoring over $R$ with $K$) transforms the open affine covering

$$X_A = \bigcup_{i=0}^{r} U_i = \bigcup_{i=0}^{r} \text{Spf} \left( A\left\langle \frac{f}{f_i} \right\rangle / (I\text{-torsion}) \right)$$

into the admissible rational covering

$$X_{\text{rig}} = \bigcup_{i=0}^{r} U_{i,\text{rig}} = \bigcup_{i=0}^{r} \text{Sp} A_{\text{rig}}\left\langle \frac{f}{f_i} \right\rangle$$

of the rigid $K$-space $X_{\text{rig}}$

In particular, $\text{rig}$ transforms admissible formal blow-ups into isomorphisms.
An equivalence of categories

Raynaud’s Theorem

Let $R$ be the valuation ring of a field $K$ with a complete non-Archimedean absolute value. Then the functor

$$\text{rig} : \left( \text{formal } R\text{-schemes} \right) \rightarrow (\text{rigid } K\text{-spaces})$$

induces an equivalence between

1. the category of all quasi-paracompact admissible formal $R$-schemes, localized by the class of admissible formal blowing-ups, and
2. the category of all quasi-separated quasi-paracompact rigid $K$-spaces

$X$ quasi-paracompact means: $X$ admits an (admissible) covering of finite type by quasi-compact (admissible) open subspaces $U_i \subset X$; i. e., such that each $U_i$ is disjoint from almost all other $U_j$
Relative rigid spaces

Base schemes $S$

(V') $S$ an admissible formal $R$-scheme, where $R$ is an adic valuation ring of type (V), as considered above

(N') $S$ is a noetherian formal scheme (of quite general type) such that the topology of $\mathcal{O}_S$ is generated by a coherent ideal $\mathcal{I} \subset \mathcal{O}_S$, where $\mathcal{O}_S$ does not admit $\mathcal{I}$-torsion

Definition

Let $S$ be a formal scheme of type (V') or (N'), let $(\text{FSch}/S)$ be the category of admissible formal $S$-schemes. Then the category $(\text{Rig}/S)$ of rigid $S$-spaces is obtained from $(\text{FSch}/S)$ by localization via admissible formal blowing-ups

Remark

Essentially, we take the category (1) in Raynaud’s Theorem as definition for rigid spaces over non-classical bases
Families

Idea

Relative rigid space $X$ is a family of classical rigid spaces $X_s$ over suitable rigid points $s \in S$

Example: Universal Tate elliptic curve

$R = \mathbb{Z}[[Q]]$, $Q$ a variable, with $Q$-adic topology

$S = \text{Spf} \mathbb{Z}[[Q]]$ formal base scheme

$\mathbb{B}^1_S = \text{Spf} R\langle \zeta \rangle$ unit disk over $S$

blow up ideal $\mathcal{A} = (\zeta^3, Q\zeta, Q^2) \subset \mathcal{O}_{\mathbb{B}^1_S}$ on $\mathbb{B}^1_S$, get $X_\mathcal{A} \rightarrow \mathbb{B}^1_S$

$X_1 = \text{Spf} R\langle \zeta, Q\zeta^{-2} \rangle$ part in $X_\mathcal{A}$ where $\zeta^3$ generates $\mathcal{A}$

$X_2 = \text{Spf} R\langle \zeta, Q\zeta^{-1}, Q^{-1}\zeta^2 \rangle$ part in $X_\mathcal{A}$ where $Q\zeta$ generates $\mathcal{A}$

identify $X_1(\zeta^{-1})$ with $X_2(Q^{-1}\zeta)$ via multiplication with $Q$

associated rigid $S$-space $E$ is universal Tate elliptic curve:

Any Tate elliptic curve $E_q = \mathbb{G}_{m,K}/q^Z$ over complete non-Archimedean field $K$ with valuation ring $R_K$ is pull-back of $E$ with respect to point given by $\mathbb{Z}[[Q]] \rightarrow R_K$, $Q \mapsto q$
Zariski-Riemann space

Definition

Let $S$ be a base scheme of type $(V')$ or $(N')$. Let $X$ be an admissible formal $S$-scheme, which is quasi-separated and quasi-paracompact. Then

$$[X] = \lim \leftarrow \bigcup_{A \subseteq \mathcal{O}_X} X_A$$

where $A$ is a coherent open ideal $A \subseteq \mathcal{O}_X$.

is called the **Zariski-Riemann space** associated to $X$.

Proposition

- $[X]$ is non-empty if $X$ is non-empty.
- $[X]$ is a $T_0$-space, but not necessarily Hausdorff.
- $[X]$ is quasi-compact if $X$ is quasi-compact.
Proposition

Let $X$ be a formal model of a classical rigid $K$-space $X_K$. There is a canonical specialization map

$$sp : X_K \longrightarrow [X]$$

where

- $sp$ is injective
- the image of $sp$ is dense in $[X]$ with respect to the constructible topology
- $sp$ induces an equivalence between the category of abelian sheaves on $X_K$ and the one on $[X]$
Notion

Let $X$ be an admissible formal $S$-scheme with ideal of definition $\mathcal{I} \subset \mathcal{O}_S$ (assume $X$ quasi-paracompact and quasi-separated)

say that associated rigid space $X_{\text{rig}}$ satisfies certain property (P) if there exists an open affine covering $(\text{Spf} \, A_i)_i$ of $X$ such that (P) is satisfied on all schemes $\text{Spec} \, A_i - V(\mathcal{I}A_i)$

in above situation, say $\text{rig-}(P)$ satisfied on $X$

Of course, this makes sense only if $\text{rig-}(P)$ is independent from the chosen covering $(\text{Spf} \, A_i)_i$ and invariant under admissible formal blowing-up of $X$

General problem

Start with rigid $S$-space $X_{\text{rig}}$ enjoying certain property (P)

Find formal $S$-model $X$ such that $\text{rig-}(P)$ extends to (P) on $X$
Theorem (Raynaud-Gruson, Bosch-Lütkebohmert)

Let $\varphi : X \rightarrow Y$ be quasi-compact morphism of admissible formal $S$-schemes, which is rig-flat. Then there exists commutative diagram of admissible formal $S$-schemes

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow \varphi & & \downarrow \varphi' \\
Y & \leftarrow & Y'
\end{array}
\]

where $\varphi'$ is flat, $Y' \rightarrow Y$ is formal blowing-up of some coherent open ideal $\mathcal{A} \subset \mathcal{O}_Y$, and where $X' \rightarrow X$ is formal blowing-up of $\mathcal{A}\mathcal{O}_X \subset \mathcal{O}_X$ on $X$. 
Theorem (Bosch-Lütkebohmert-Raynaud)

$X/S$ quasi-compact admissible, flat, fibers $\neq \emptyset$ equidimensional. $f \in \Gamma(X, \mathcal{O}_X)$ not nilpotent on fibers of $X_{rig}/S_{rig}$. Then:

$Y' \quad \downarrow \quad X'$
$\quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$
$X \quad \leftarrow \quad X'$
$\quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$
$S \quad \leftarrow \quad S'$

- $X' = X \times_S S'$,
- $S' \to S$ rig-flat, rig-quasi-finite, rig-surjective
- $Y'_{rig} \to X'_{rig}$ an isomorphism,
- exists $\beta \in \Gamma(S', \mathcal{O}_{S'})$ rig-invertible, $g \in \Gamma(Y', \mathcal{O}_{Y'})$ invertible on an open part of $Y'$ covering $S'$ such that $f = \beta g$ on $Y'$.
Theorem (Bosch-Lütkebohmert-Raynaud)

$X/S$ quasi-compact admissible formal scheme, flat, $X_{\text{rig}}/S_{\text{rig}}$ with reduced geometric fibers, equidimensional. Then:

- $Y' \rightarrow X' \leftarrow X$
- $S' \rightarrow S$ surjective and $S'_{\text{rig}} \rightarrow S_{\text{rig}}$ étale,
- $Y' \rightarrow X'$ finite and $Y'_{\text{rig}} \rightarrow X'_{\text{rig}}$ an isomorphism,
- $Y' \rightarrow S'$ flat and has reduced geometric fibres.