Multisets

Example:
A bag of Scrabble tiles contains 100 tiles: 10 A’s, 2 B’s, 2 C’s, 5 D’s and so on.

When you start a game, you take 7 letters from the bag, and put them on a rack. How many possible hands can you get, if we say that the order of the tiles on the rack matters? How about if it doesn’t?

A multiset is a "set with multiplicity".

Notation: \{2 \ast a, 3 \ast b, 1 \ast c\}
(Think of a "bag" with 2 a’s, 3 b’s and a c in it.)

We also allow "infinite multiplicity", denoted \{\infty \ast a\}.

Multiplicities are also called "repetition numbers".

The size of a multiset is the sum of the multiplicities (may be \infty).

An \(r\)-permutation of a multiset is an ordered list of \(r\) elements from the multiset;
e.g. the 2-permutations of \{2 \ast a, 1 \ast b\} are
\[ aa, ab, ba; \]
the 3-permutations of \{\infty \ast a, 2 \ast b\} are
\[ aaa, aab, aba, baa, abb, bab, bba. \]

A permutation of a multiset of size \(n\) is an \(n\)-permutation.

Example: how many permutations are there of the unfortunate scrabble hand \{4 \ast U, 1 \ast J, 2 \ast K\}?

Theorem:
Let \(S\) be a multiset with \(k\) types with finite multiplicities \(n_1, \ldots, n_k\).

Let \(n = \sum n_i\) be the size of \(S\).

Then the number of permutations of \(S\) is
\[ n! / n_1! \ast n_2! \ast \ldots \ast n_k!. \]

Proof:
Label the elements 1, ..., \(n\). Each permutation of \{1, ..., \(n\}\} yields a permutation of \(S\), and two yield the same permutation precisely when we can get one from another by permuting the labels on elements of the same type.

So there are \(n_1! \ast n_2! \ast \ldots \ast n_k!\) permutations of \{1, ..., \(n\}\) per permutation of \(S\). We conclude by the division principle.

Example:
We have 4 black rooks and 4 white rooks. How many ways are there of putting them on a chess board such that no two are attacking (/defending) each other? e.g.
Solution:
First, just choose the 8 squares for them to occupy.

By listing off the filled columns row-by-row, a choice corresponds to a permutation of the columns, so there are 8!.

For a given such choice, a choice of colours corresponds to a permutation of the multiset \( \{4 \ast r, 4 \ast R\} \).

So the answer is 
\[
8! \ast (8!/4! \ast 4!) = 2822400
\]

An \( r \)-submultiset, or \( r \)-combination, of a multiset \( S \) is a multiset \( S' \) of size \( r \) such that for all \( x \), the multiplicity of \( x \) in \( S' \) is at most the multiplicity of \( x \) in \( S \).

E.g. the 2-submultisets of \( \{3 \ast a, b\} \) are \( \{2 \ast a\}, \{1 \ast a, 1 \ast b\} \).
Theorem:
The number of $r$-combinations of a multiset with $k$ types each with multiplicity at least $r$ is
\[ \binom{r+k-1}{r} \]

Example:
If we have a bag containing red, green and blue marbles, with many of each, and we draw 5 marbles from the bag, how many possible results (numbers of each colour drawn) are there?
Answer: \( \binom{5+3-1}{5} = \binom{7}{5} = \frac{7!}{2!5!} = 21 \)

Proof:
We can identify an $r$-submultiset with an arrangement of $k-1$ partitions interspersed among $r$ identical objects, by counting the numbers of objects between the partitions; e.g. with $k = 6$ and $r = 8$
\[ oo|ooo||o|oo| \]

corresponds to
\[ \{2 \ast a_1, 3 \ast a_2, 0 \ast a_3, 1 \ast a_4, 2 \ast a_5, 0 \ast a_6\}. \]

These arrangements correspond to choosing $r$ of the $r+k-1$ characters to be 'o's, so the number of such arrangements is \( \binom{r+k-1}{r} \).