MATH 2A03 Fall 2014 Final Exam SAMPLE

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Final Exam SAMPLE
Duration of test: 180 minutes
McMaster University
2014

Name:____

Student No.:____________________________________

There are 5 questions each worth 5 marks. The test is marked out of 25. You should answer all the questions.
Please write your answers on the paper provided, not directly on the question paper.
You should show your working. Partial credit may be given for correct working despite an incorrect answer.
Please be sure to head each sheet of paper you write on with your name and student number.

You may consult your textbook and course notes during the test.
Only the McMaster standard student calculator may be used for this test.
The maximal number of marks on this test is 25.

Score

Question	1	2	3	4	5	Total
Points	5	5	5	5	5	25
Score						

1. [5] Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable map. Let $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ be the following parametrisation of the plane $x+y+z=0$:

$$\mathbf{r}(u,v) = (u,\ v-u,\ -v).$$

   (i) [2] Compute, in terms of the partial derivatives of f, the partial derivatives g_u and g_v of the composition

$$g = f \circ \mathbf{r}.$$

   Now let $f(x,y,z) = x + \frac{y^2}{2} + z^2$.

   (ii) [1] Compute ∇f.

   (iii) [2] Using your answers to parts (i) and (ii), or otherwise, find the minimal value that $f(x,y,z)$ takes on the plane $x+y+z=0$. You may make use of the fact that such a minimal value exists.

2. [5] Let $\mathbf{c}$ be the path $\mathbf{c} : [0,1] \to \mathbb{R}^3$

$$\mathbf{c}(t) = (t^2 - t,\ t^3 - t^2,\ t^3).$$

   Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, defined on the whole of $\mathbb{R}^3$, and suppose $\operatorname{curl} \mathbf{F} = \mathbf{0}$ everywhere on $\mathbb{R}^3$.

   However, you only know the value of $\mathbf{F}(x,y,z)$ when $x=y=0$: for all $t \in \mathbb{R}$, the value at $(0,0,t)$ is

$$\mathbf{F}(0,0,t) = (t^2,\ t^2,\ t^2).$$

   (i) [3] Find

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

   (ii) [2] Suppose now that $\mathbf{F}$ has all the properties described above, except that now it is defined only everywhere but the line $x=y=2$. In notation: $\operatorname{dom} \mathbf{F} = \mathbb{R}^3 \setminus \{(2,2,z) \mid z \in \mathbb{R}\}$.

   Can you still conclude that your answer in part (i) is correct? Briefly explain your answer.

3. [5] Using the spherical change of co-ordinates map

$$T(\rho,\phi,\theta) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi),$$

find the integral $\iiint_V f dV$ of

$$f(x,y,z) = z^2$$

over the unit ball

$$V = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}.$$

4. [5] A flying saucer hovers overhead at noon, casting a sinister shadow. We model the situation as follows: a horizontal disc of radius 1 centred above $(x,y) = (0,0)$ casts a shadow directly downwards onto a surface which is the graph of the continuously differentiable function $f(x,y)$.

   (a) [3] Express, as an integral over the unit disc, the area of the surface which is in shadow - i.e. the area of that part of the graph of f which lies directly below the disc.

   (b) [2] What is the minimum possible area of the shadow? Give an example of a function f for which the shadow has this minimal area.

5. [5] Let

$$\mathbf{F}(x,y,z) = (x,\ y,\ (z^2-1)e^{x^2}+z^2).$$

By integrating over its boundary, find the average divergence of $\mathbf{F}$ in the cylinder centred at $\mathbf{0}$ with unit radius and height 2,

$$V = \{(x,y,z) \mid x^2+y^2 = 1,\ -1 \leq z \leq 1\}.$$

(Do not attempt to directly integrate $\operatorname{div}\mathbf{F}$ over V; you will fail.)