<u>Theorem</u> M an ω-categorical str. in the cts sense. Then $Th(M)$ is stable iff every Roelcke uniformly ctns function $Aut(M) \to \mathbb{C}$ is weakly almost periodic (wap)

<u>Recall</u> M ω-categ. str. M is Polish

$G = Aut(M) \leq Iso(M)$ is also Polish

$\uparrow$ pointwise topology

- On G we have uniformities:
  - → left uniformity $\mathcal{U}_L : \{(x,y) : x^{-1}y \in V\}$, V nbhd of 1
  - → right uniformity $\mathcal{U}_R : \{(x,y) : xy^{-1} \in V\}$
  - → Roelcke unif. $\mathcal{U}_{L \wedge R}$ generated by $\mathcal{U}_L \cap \mathcal{U}_R$

- $f : G \to \mathbb{C}$ is right u.c. if

  $\forall \varepsilon > 0$ $\exists$ nbhd V of 1_G s.t. $gh^{-1} \in V \Rightarrow |f(g) - f(h)| < \varepsilon$

- $f : G \to \mathbb{C}$ is Roelcke u.c. if it is unif. cont wrt $\mathcal{U}_{L \wedge R}$

  $\Leftrightarrow$ it is right u.c. and left u.c.

- G metrisable : d_L - left inv. metric
  d_R - right inv. metric

$$d_R(g,h) = d_L(g^{-1}, h^{-1})$$

We have
$$d_{L\wedge R}(g,h) = \inf_{f\in G} \max\{d_L(f,g), d_R(f,h)\}$$

$\hat{G}_L \times \hat{G}_L \to \hat{G}_L$ is a semigroup

$\hat{G}_L \times M \to M$ is an action by isometries (isometric embeddings)

<u>Lemma</u> M ω-categ. str. $G = Aut(M)$
$\xi \in M^{\mathbb{N}}$ enumerating a dense set of M
$\Xi := \overline{G\xi} \subseteq M^{\mathbb{N}}$

Then $d_\xi(g,h) = d(g\xi, h\xi)$ is a compatible left invariant metric on G.

$\hat{G}_L \to \Xi$ $a \longmapsto a\xi$ is an isometric bijection,

and $\hat{G}_L \times \Xi \to \Xi$ corresponds to semigroup multiplication on $\hat{G}_L$

Proof: d_ξ is a metric $g\xi = h\xi \iff g = h$

d_ξ left-inv.: G acts by isometries

To see d_ξ is compatible with $\mathcal{U}_L$ we'll do one direction. Take $\varepsilon > 0$.

Let's show $U_\varepsilon = \{(g,h) : d_\xi(g,h) < \varepsilon\}$ is in $\mathcal{U}_L$

$$d_\xi(g,h) < \varepsilon \iff d(g\xi, h\xi) < \varepsilon \iff d(\xi, g^{-1}h\xi) < \varepsilon$$

Write $\xi = (x_0, x_1, x_2, \ldots)$ Then there are n, η s.t. if $d(x_i, g^{-1}h x_i) < \eta$ for all $i < n$, then $d(\xi, g^{-1}h\xi) < \varepsilon$

So we need to show: for $x \in M$, $\eta > 0$ that $\{(g,h) : d(x, g^{-1}h x) < \eta\} \in \mathcal{U}_L$

$$d(x, g^{-1}h x) < \eta \iff g^{-1}hx \in B_\eta(x)$$

$\mathrm{Isom}(M)$ has ptwise conv. topology.

Subbase given by

nbhd of id $\iff x \in U$

$$O_{(x,U)} = \{g : gx \in U\}$$

$\blacksquare$

M ω-cat. $G = Aut(M)$

$RUCB(G) = \{ f : G \to \mathbb{C} \ \ f \text{ r.u.c. } \& \text{ b.d.} \}$

This is a Banach space with sup norm

- The <u>weak top.</u> on $RUCB(G)$ is the coarsest topology making

$$\hat{g} : f \mapsto f(g) \qquad g \in G$$

continuous

- $f \in RUCB(G)$ is <u>weakly almost periodic</u> if $\overline{Gf}$ in the weak top. is compact.

→ <u>Grothendieck</u>

f is w.a.p. $\Leftrightarrow$ for $(g_n) \subseteq G$, $(h_m) \subseteq G$ if

$\lim_m \lim_n f(g_n h_m)$ and $\lim_n \lim_m f(g_n h_m)$ exist

then they are equal

→ <u>Fact</u> (see Ruppert "semit semig")

Every w.a.p. f is Roelcke unif. cts.

→ so we get $R(G) \to W(G)$ (see below for def of $W(G)$)

WAP(G) - weakly almost periodic functions on G
it is a C^*-alg.
W(G) = WAP compactification of G;
it is a compact semi-t. semigroup

Def 1 of W(G) (the universal property)

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & W(G) \arrow[d, dashed, "\exists!"] \\ & H \end{tikzcd}$$

cont. semigroup homom.

some other compact semit. semigr.

Def 2 (using Gelfand-Naimark)

$W(G) = \{$ non-zero algebra homs $W(G) \to \mathbb{C} \}$

$WAP(G) \cong C(W(G))$

Stability

A formula $f(x_1, \dots, x_n)$ has interpretation in M as a function $f: M^n \to \mathbb{C}$, which is uniformly cts, bd G-invariant.

We also consider unif. limits of formulas:
u.c. G-inv. funs $M^{\mathbb{N}} \to \mathbb{C}$

- For $X \subseteq M^{\mathbb{N}}$ G-inv. and closed, and consider restriction of $f: M^{\mathbb{N}} \to \mathbb{C}$ to $X \to \mathbb{C}$, or simply take continuous G-inv. $X \to \mathbb{C}$

(notions coincide by Tietze ext. thm and Ryll-Nardzewski)

- $X, Y \subseteq M^{\mathbb{N}}$ G-inv. & closed

  $f : X \times Y \to \mathbb{R}$ a formula

The f has <u>OP</u> (order property) if there are sequences $(x_n) \subseteq X$, $(y_m) \subseteq Y$, $r < s \in \mathbb{R}$ s.t.

$$n < m \Rightarrow \varphi(x_n, y_m) \leq r$$
$$n > m \Rightarrow \varphi(x_n, y_m) \geq s$$

- A formula f is <u>stable</u> if it does not have OP

- $Th(M)$ is <u>stable</u> if every formula is stable

- We say that φ satisfies Grothendieck's condition if

  $\forall (x_n) \subseteq X \quad \forall (y_m) \subseteq Y$

$$\lim_m \lim_n \varphi(x_n, y_m) = \lim_n \lim_m \varphi(x_n, y_m),$$

whenever the two limits exist.

<u>Lemma</u> φ - stable wrt X, Y $\Leftrightarrow$ φ has Gro wrt X, Y.

<u>Proof</u> (sketch) We show OP $\Leftrightarrow \neg$ Gro

Say φ has OP witnessed by $(x_n), (y_m)$, $r<s$

suppose the two limits exist.

Fix $m_0 \in \mathbb{N}$. For $n > m_0$, $\varphi(x_n, y_{m_0}) \geq s$

$\Rightarrow \lim_n \varphi(x_n, y_{m_0}) \geq s$

m_0 was arbitrary, so $\lim_m \lim_n \varphi(x_n, y_m) \geq s$

Similarly $\lim_n \lim_m \varphi(x_n, y_m) \leq r$. So get $\neg$Gro

(And in case limits don't exist we can always pass to subsequences so that limits exist.)

<u>Lemma</u>

φ a formula on $X \times Y$

Let for $x \in X$, $y \in Y$, $\tilde{\varphi}_{x,y} : G \to \mathbb{C}$

$$\tilde{\varphi}_{x,y}(g) = \varphi(x, gy)$$

i) φ stable on $X \times Y$ iff $\tilde{\varphi}_{x,y}$ w.a.p $\forall x \in X$ $\forall y \in Y$

ii) In the case $X = [x_0]$, $Y = [y_0]$,
$\overline{Gx_0} \subseteq M^{\mathbb{N}}$

φ stable $\Leftrightarrow$ $\tilde{\varphi}_{x_0,y_0}$ w.a.p

<u>Proof of ii)</u> on X, Y :

φ stable $\Leftrightarrow$ for $x_n = g_n x_0$, $y_m = h_m y_0$,

$$\lim_m \lim_n \varphi(g_n x_0, h_m y_0) = \lim_n \lim_m \varphi(g_n x_0, h_m y_0)$$

(whenever both limits exist)

$\Leftrightarrow$ Gro for $\tilde{\varphi}_{x_0, y_0}$ $\Leftrightarrow$ w.a.p.

(We have

$$\varphi(g_n x_0, h_m y_0) = \varphi(x_0, g_n^{-1} h_m y_0) = \tilde{\varphi}_{x_0, y_0}(g_n^{-1} h_m)\ )$$

For the proof of i) see Lemma 5.1 in Ben Yaacov - Tsankov

<u>Theorem</u> $Th(M)$ - stable (1)

$\Leftrightarrow$ all formulas on Ξ^2 are stable (2)

$\Leftrightarrow$ If $f: G \to \mathbb{C}$ is Roelcke unif. cont., then it is w.a.p. (3)

$\Leftrightarrow$ $W(G) = R(G)$ (4)

<u>Proof</u> (sketch)

(2) $\Leftrightarrow$ (3)

$\Xi \cong \hat{G}_L$ $\qquad f: \Xi^2 \to \mathbb{C}$ $\leftarrow$ So by G-inv. these are formulas from $R(G)$ and these are same as Roelcke unif. cont. functions

$R(G) = \hat{G}_L^2 /\!/ G$

(orbit closures)

$g \mapsto [1_G, g]$

(1) $\iff$ (2)

φ formule on $M^{\mathbb{N}} \times M^{\mathbb{N}}$

$x, y \in M^{\mathbb{N}}$

$\bar{\varphi}_{x,y}(x', y') = \varphi(x'x, y'y)$ a formula on Ξ^2

φ stable

$\iff \bar{\varphi}_{x,y}$ stable $\forall x, y$