Theorem (Ben Yaacov - Tsankov)

Let $G = \mathrm{Aut}(M)$ for M an $\aleph_0$-categorical structure TFAE

1) $Th(M)$ is stable
2) $R(G) = W(G)$, that is, every Roelcke uniformly continuous function on G is weakly almost periodic

Polish G, fix a compatible left-invariant metric d_L. We also get d_R right-invariant $d_R(f,g) = d_L(f^{-1}, g^{-1})$. We define the Roelcke metric by

$$d_{L\wedge R}(g,h) = \inf_{f \in G} \max\{d_R(g,f), d_L(f,h)\}$$

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left uniformity is generated by

$\mathcal{U}_L = \{(h,f) \in G^2 : h^{-1}f \in U\}$ as U varies / nbhds of 1 in G

Similarly for the right uniformity

$\mathcal{U}_R = \{(h,f) \in G^2 : hf^{-1} \in U\}$ — // —

and the Roelcke uniformity

$\mathcal{U}_{L\wedge R} = \{(h,f) \in G^2 : h \in UfU\}$ — // —

- A collection $\mathcal{U} \subseteq P(X^2)$ is called the <u>uniformity</u> if:

1) $\Delta_X \subseteq U \quad \forall U \in \mathcal{U}$
   diagonal
2) $\forall U \in \mathcal{U} \ \exists V \in \mathcal{U}$ s.t. $V^2 \subseteq U$
3) $\forall U \in \mathcal{U}$ if $V \supseteq U$, $V \in \mathcal{U}$
4) $U, V \in \mathcal{U} \Rightarrow U \cap V \in \mathcal{U}$

→ In a metric space (X, d) one can take $U_\varepsilon = \{(x,y) : d(x,y) < \varepsilon\}$, $\varepsilon > 0$

Roelcke completion:

$(\hat{G}, \widehat{d_{L \wedge R}})$

G is called <u>Roelcke precompact</u> if its Roelcke compl. is compact.
Equivalently, $\forall U$ open nbhd of 1 $\exists$ finite set $F \subseteq G$ s.t. $G = UFU$.

- Let $(\hat{G}, \hat{d}_L)$ be the left-completion of G.
Whenever $G \curvearrowright (X, d)$ by isometries,

the map $\alpha: G \times X \to X$ extends to a map $\hat{\alpha}: \hat{G} \times \hat{X} \to \hat{X}$ (for every $g \in \hat{G}_L$ the $x \to g \cdot x$ extends to an isometric embedding).

In particular, when $G \curvearrowright (G, d_L)$ by left-translations we get a map $\hat{G}_L \times \hat{G}_L \to \hat{G}_L$, which determines a semigroup structure on G_L

Def Whenever $G \curvearrowright (X, d)$ acts by isometries, for $x \in X$ we let $[x]$ denote the closure of the orbit of x.

We equip $X /\!/ G = \{ [x] : x \in X \}$ with the metric

$$d([x],[y]) = \inf \{ d(u,v) : u \in [x], v \in [y] \}$$

Def We say that $G \curvearrowright (X, d)$ is approximately oligomorphic if for every n, $X^n /\!/ G$ is compact.

Fact If (X, d) is complete then $G \curvearrowright (X, d)$ is approximately oligomorphic iff $X^{\mathbb{N}} /\!/ G$ is compact (idea: $X^{\mathbb{N}} /\!\!/ G = \varprojlim X^{n} /\!\!/ G$)

Lemma Let X be a separable complete metric space and $\xi \in X^{\mathbb{N}}$ enumerating a dense subset. Let $G \leq \mathrm{Iso}(X)$ be a closed subgroup and let $\Xi = [\xi]$ (orbit in $X^{\mathbb{N}}$).

Then $d_L(g,h) = d(g\xi, h\xi)$ is a compatible left-invariant metric on G and the map $(\hat{G}_L, \hat{d}_L) \to \Xi$ is an isometric bijection.

The diagonal action $\hat{G}_L \curvearrowright X^{\mathbb{N}}$ (restricted to Ξ) coincides with multiplication in $\hat{G}_L$.

- Therefore we have $R(G) =: \hat{G}_L^2 /\!\!/ G$ (we have $G \curvearrowright \hat{G}_L$ and act diagonally)

$(G, d_{\wedge})\to R(G)$ given by

$g \mapsto [1_G, g] = [g^{-1}, 1_G]$ is an isometric embedding, so $R(G)$ is the Roelcke completion.

Continuous logic

- A structure is a complete bounded metric space (X, d) with predicates $P_i : M^k \to [0,1]$ that are uniformly continuous.

- Formulas :

→ Atomic formulas

→ If $u : [0,1]^m \to [0,1]$ is continuous and $\varphi_1, \dots, \varphi_n$ are formulas, so is $u(\varphi_1, \dots, \varphi_n)$

→ if φ is a formula, $\inf_x \varphi$, $\sup_x \varphi$ are formulas

- We say that M is $\aleph_0$-categorical if M is separable and $Th(M)$ has unique model up to isomorphism.

- $Aut(M)$ is the subgroup of $Iso(M)$ consisting of isometries that preserve the predicates (i.e. $P_i(g \cdot \bar{a}) = P_i(\bar{a})$ $\forall g \in Aut(M)$ $\bar{a} \in M^k$)

- There is a continuous version of Ryll-Nardzewski : TFAE
  - $\to$ $Th(M)$ is $\aleph_0$-categorical
  - $\to$ $Aut(M) \curvearrowright M$ is approximately oligomorphic

<u>Theorem</u>

For a Polish G TFAE:

1) G is Roelcke precompact
2) Whenever $G \curvearrowright X$ and $G \curvearrowright Y$ (X, Y - complete)
   $X /\!/ G$, $Y /\!/ G$ compact $\Rightarrow$ $X \times Y /\!/ G$ compact
3) Whenever $G \curvearrowright X$ s.t. $X /\!/ G$ compact, then $G \curvearrowright X$ approximately oligomorphic.
4) There exists a (separable, complete) X, a homeomorphic embedding $G \mapsto Iso(X)$ s.t. $G \curvearrowright X$ is approx. oligomorphic.

$G \curvearrowright M \quad \rightsquigarrow \quad \hat{G}_L \curvearrowright M$

Every $x \in \hat{G}_L$ induces an elementary embedding $x : M \preccurlyeq M$.

Fix M $\aleph_0$-categorical

Identify $\widehat{G}_L$ with $\Xi = [\,\{\}\,] \subseteq M^{\mathbb{N}}$ enumerating a dense subset of M

Now $R(G) \cong \Xi^2 /\!/ G$ we can identify $[x,y] \in R(G)$ with $tp(x,y)$ (or $tp(x(M), y(M))$)

Note: If $Im(x) = Im(y)$ then $x^{-1}y \in Aut(M)$

$R(G) = \{ tp(x(M), y(M)) \mid x,y : M \overset{\sim}{\to} M \}$

Examples

1) $\mathbb{N}$ $\quad Aut(\mathbb{N}) = S_\infty$ $\quad$ What is $R(S_\infty)$?

$x, y : \mathbb{N} \to \mathbb{N}$ injections

$x(n) = y(m)$ or $x(n) \neq y(m)$ $\quad m, n \in \mathbb{N}$

So we can identify $tp(x,y)$ with the partial bijection $x^{-1}y$

We get

$R(S_\infty) = \{ \text{partial bijections} \}$

2) $(\mathbb{Q}, \leq)$ rationals $\mathrm{Aut}(\mathbb{Q}, \leq) \ni x, y$

$tp(x,y)$ is giving linear order on $x(\mathbb{Q}) \cup y(\mathbb{Q})$

$$R(\mathrm{Aut}(\mathbb{Q})) \subseteq 3^{\mathbb{Q} \times \mathbb{Q}}$$

$$\alpha(q,p) = \begin{cases} -1 & \text{if } x(q) < y(p) \\ 0 & \text{if } x(q) = y(p) \\ 1 & \text{if } x(q) > y(p) \end{cases}$$

3) H separable Hilbert space

$U(H) = \mathrm{Aut}(H)$

But $H /\!/_{U(H)} \cong [0, \infty)$

Let's consider instead $U(H) \curvearrowright S(H)$

Note $S(H) /\!/_{U(H)} \cong \{*\}$ the unit sphere

$R(U(H)) =$ "pairs of embeddings"

For $x, y : H \to H$ we look at $\langle x(\xi), y(\eta) \rangle \ \forall_{\xi, \eta \in H}$

So $p \in R(U(H))$ determines a bilinear form $\langle -, - \rangle_p$ satisfying $|\langle \xi, \eta \rangle_p| \leq 1$ on $S(H)$. But this is the same as determining a linear contraction T_p by $\langle T_p \xi, \eta \rangle = \langle \xi, \eta \rangle_p \quad \forall \xi, \eta \in H$

$R(U(H)) = B(H)_1$