ECMA and its imaginaries — Paris Feb '06 — Joint w/ M. Hils & R. Moosa

TA

- T complete thy with QE and EI
- $T_\sigma :=$ ~~[scribble]~~ Theory of the elementary class $\{(M,\sigma) \mid M \models T,\ \sigma \in Aut(M)\}$
- $TA :=$ model companion $=$ theory of ~~[scribble]~~ $\{$existentially closed models of $T_\sigma\}$
  ~~[scribble]~~ if they form an elementary class
  ("TA exists")
- Suppose TA exists.
- Facts: ~~[scribble]~~
  - completions determined by ~~[scribble]~~ choice of substructure $(acl^T(\emptyset), \sigma_0)$ ~~[scribble]~~ $\sigma_0 \in Aut^T(acl(\emptyset))$
  - acl^{TA}-closed sets are exactly of form (A, σ_A) ~~[scribble]~~ $A = acl^T(A)$, $\sigma_A \in Aut^T(A)$; i.e. $A = acl^{TA}(A) \Leftrightarrow (A = acl^T(A)$ and $\sigma(A) = A)$
    and every such is a substructure of a model of TA
  - QE: if $A = acl^{TA}(A)$, $qftp^{TA}(A) \models tp^{TA}(A)$
    ~~[scribble]~~
  - T (super)stable $\Rightarrow$ TA (super)simple $\qquad A \mathop{\downarrow}^{TA}_{C} B \Leftrightarrow acl^{TA}(AC) \mathop{\downarrow}^{T}_{acl^{TA}(C)} acl^{TA}(BC)$

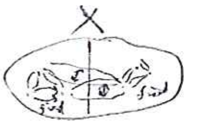
Imaginaries in TA

Example: $T :=$ theory of an infinite connected groupoid with $\pi_1 := \mathbb{Z}/2\mathbb{Z}$
$=$ cat^y with infinitely many objects and precisely two morphisms $x \to y$ for each x, y, each invertible
language: $(Mor, \circ)$ ~~[scribble]~~
(totally categorical)

~~[scribble]~~

TA exists
~~[scribble]~~
T has EI
But TA hasn't:
$X :=$ fixed objects (identifying objects with id^y morphisms)
$E(x,y) :\equiv \sigma$ fixes both morphisms $x \to y$

Then $|X/E| = 2$

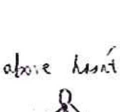


Not eliminable: $X/E \subseteq acl^{eq}(\emptyset)$, but $acl^{TA}(\emptyset) = \emptyset$. But is eliminated if add any $x \in X$ as parameter.

~~Say $X/E = \{x_1/E, x_2/E\}$~~
~~Sps $x_1/E \in dcl^{eq}(dcl^{eq}(x_1/E) \cap$ reals$)$~~
~~$dcl^{eq}(x_1/E) \not\subseteq acl^{eq}(\emptyset)$ since $|X/E| = 2$~~
~~and $acl^{eq}(\emptyset) \cap$ reals $= acl^T(\emptyset) = \emptyset$~~
~~but $x_1 \equiv^{TA} x_2$~~

Def^n: T stable with EI has 3-uniqueness $/ A = acl^T(A)$

e.g. T above hasn't 3-uniqueness $/\emptyset$



if for any triple (b_0, b_1, b_2) ind^t $/A$,
$$dcl(acl(Ab_1b_0) \cup acl(Ab_2b_0)) \cap acl(Ab_1b_2) = dcl(acl(Ab_1) \cup acl(Ab_2))$$

T has 3-uniqueness if it has 3-uniqueness over any $A = acl^T(A)$

Th^m [Hrushovski]: ~~[scribble]~~ Suppose T superstable with EI and TA exists.

(a) TFAE
(i) T has 3-uniqueness
(ii) ~~[scribble]~~ All completions of TA satisfy the Ind^nce Th^m over acl^{TA}-closed sets
(iii) All completions of TA have EI

i.e. if $b_1 \mathop{\downarrow}^{TA}_{A} b_2$, $A := acl^{TA}(A)$, $b_0^1 \mathop{\downarrow}^{TA}_{A} b_1$, $b_0^2 \mathop{\downarrow}_{A} b_2$, $b_0^1 \equiv^{TA}_{A} b_0^2$ then exists $b_0^1 \equiv^{TA}_{Ab_1} b_0 \equiv^{TA}_{Ab_2} b_0^2$, $b_0 \mathop{\downarrow}_{A} b_1 b_2$

(b) T_A has gEI i.e. $\forall e \in \mathbb{M}^{eq},\ e \in \mathrm{acl}^{eq}(\mathrm{acl}^{eq}(e) \cap \mathbb{M})$

pf sketch:

(a) (i)$\Leftrightarrow$(ii) : Imaginary Galois theory

(ii)$\Rightarrow$(iii): $e = b/_E$, $A := \mathrm{acl}^{eq}(e) \cap \mathbb{M}$

STS $e \in \mathrm{dcl}^{eq}(A)$

Else,

<u>Lemma</u>: exists b' s.t. $b' \equiv^{+A}_A b$ and $b' \downarrow^{+A}_A b$ and $b' E b$

Note: (b) follows: $e \in \mathrm{acl}^{eq}(b) \cap \mathrm{acl}^{eq}(b') = \mathrm{acl}^{eq}(A)$

Also exists " " " " " " " $b' \not E b$

IT/A $\Rightarrow$  amalgamable $\cancel{X}$ Transitivity

(iii)$\Rightarrow$(ii): supersimple $\Rightarrow$ elimination of hyperimaginaries

so EI for T_A $\Rightarrow$ types over acl^{TA}-closed are Lascar strong types

$\Rightarrow$ IT / " "

$\square$

<u>Remark</u>: $M \models T \Rightarrow T$ has 3-uniqueness/M (coheir)

So e.g. ACFA has EI, since acl^T-closed sets are models.

<u>CCM</u>

Structure $\mathcal{C}$: Sort for each compact complex manifold

relation for each closed analytic subset of a product of sorts

$\nwarrow$ locally zero set of holomorphic f^ns

CCM $:= \mathrm{Th}(\mathcal{C})$

Note $\mathcal{C} = \mathrm{dcl}(\emptyset)$

Chow $\Rightarrow$ structure on a proj^{ve} complex alg^c var^y

is just algebraic Zariski structure;

$\mathbb{F} := \mathbb{P}^1 \setminus \{\infty\}$ is pure field

<u>Facts</u>: CCM has QE & EI, each sort has finite RM,

and CCMA exists.

By above, CCM has gEI. But

<u>Theorem</u>: CCM does not have 3-uniqueness

so CCMA does not have EI.

<u>pf Sketch</u>: Consider holomorphic principal $\mathbb{C}^*$-bundle $X \downarrow Y$.

so $\mathbb{C}^*$ acts principally on fibres X_a, uniformly definably in $\mathcal{C}^{M}$,

(equiv: holomorphic line bundle minus 0-section)

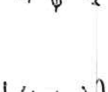
Work in monster $\mathcal{C}' \succcurlyeq \mathcal{C}$. Let $K := \mathbb{F}(\mathcal{C}') \succcurlyeq \mathbb{C}$.

Let $a \in Y$ gen^c/$\emptyset$ (i.e. / $\mathcal{C}$);

Let $(b_0, b_1, b_2) \in (X_a)^3$ gen^c/a.

Let $n \geq 2$.

Let $\phi \in (b_2/b_1)^{1/n} \subseteq K^\times$.



So $\phi \in (b_2/b_1)^{1/n} = (b_2/b_0)^{1/n} \cdot (b_0/b_1)^{1/n}$

$\subseteq \mathrm{dcl}(\mathrm{acl}(b_2 b_0) \cup \mathrm{acl}(b_1 b_0))$.

So contradict 3-uniqueness unless $\phi \in \mathrm{dcl}(\mathrm{acl}(b_1) \cup \mathrm{acl}(b_2))$.

Now $\phi \notin \mathrm{dcl}(b_1, b_2) = \mathrm{dcl}(b_1, \phi^n)$ since $\phi \notin \mathrm{dcl}(\phi^n)$ (by purity)

and $b_1 \downarrow \phi^n$ [so $\mathrm{tp}(b_1/\phi^n)$ stationary

so $\mathrm{dcl}(b_1 \phi^n) \cap \mathrm{acl}(\phi^n) = \mathrm{dcl}(\phi^n)$]

So STS $\mathrm{acl}(b_i) = \mathrm{dcl}(b_i)$

i.e. any generically finite dom^{n} $X' \to X$ has a generic section

(CCMA Imag' Paris cont^d)

Finite covers of principal $\mathbb{C}^*$-bundles $\begin{matrix} \mathbb{C}^* \\ \downarrow \\ X \\ \downarrow \\ Y \end{matrix}$:

(I) Base change :
$$\begin{tikzcd} \mathbb{C}^* \arrow[d] & \mathbb{C}^* \arrow[l, phantom, "="] \arrow[d] \\ X' \arrow[r, two heads, dashed, "\exists \text{fin}"'] \arrow[d, dashed] & X \arrow[d] \\ Y' \arrow[r, two heads, "\text{fin}"'] & Y \end{tikzcd}$$

(II) quotient by action of μ_n
$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\mu_n"] \arrow[d] & \mathbb{C}^* \arrow[d] \\ X' \arrow[r, two heads, dashed, "\text{fin}"'] \arrow[d] & X \arrow[d] \\ Y \arrow[r, phantom, "="] & Y \end{tikzcd}$$

<u>Fact</u>: Holomorphic $\mathbb{C}^*$-bundles over Y are classified by $H^1(\mathcal{O}_Y^*)$ ← sheaf of holomorphic $\mathbb{C}^*$-valued fⁿˢ

(II) corresponds to mult^n by n on $H^1(\mathcal{O}_Y^*)$

<u>Fact</u>: Exists Y simply connected strongly minimal smooth compact surface with $H^1(\mathcal{O}_Y^*) \cong \mathbb{Z}$

Let $\begin{matrix} X \\ \downarrow \\ Y \end{matrix} \sim 1 \in \mathbb{Z} \cong H^1(\mathcal{O}_Y^*)$

~~Y s.c. s.m. $\Rightarrow$ no $Y' \xrightarrow[\text{fin}]{} Y$~~
~~Y s.m. $\Rightarrow X$ has no ramified finite covers~~
~~$\Rightarrow$ any finite cover is as in (II)~~
~~but not divisible in $H^1(\mathcal{O}_Y^*)$~~
~~so no non-trivial finite cover.~~ ~~$\square$~~

Suppose $X' \xrightarrow[\text{dom}]{} X$ generically finite ; WMA all fibres finite (Stein)

Y s.m. $\Rightarrow X' \to X$ unramified finite

Y s.c. $\Rightarrow$ no non-trivial unramified $Y' \xrightarrow[\text{fin}]{} Y$

so get $X' \to X$ is of form (II)

(after $Y \to Y$ (fin))

so trivial since $\begin{matrix} X \\ \downarrow \\ Y \end{matrix}$ not divisible in $H^1(\mathcal{O}_Y^*)$.

(nor in $H^1(\mathcal{O}^*_{Y \text{fin}})$) $\square$

<u>More on CCMA</u>

- $(\mathbb{F}, \sigma) \models ACFA$ purely stably embedded
- CBP via jet spaces
  for finite-dim^l types
  (gen^c types of $(X, F)^\#$ $\begin{tikzcd} & X \times X^\sigma & \\ & F \arrow[u, phantom, "\cup"] \arrow[dl] \arrow[dr] & \\ X & & X^\sigma \end{tikzcd}$)
  $\quad = \{x \in X \mid (x, \sigma x) \in F\}$ dom^nt

$\Rightarrow$ trichotomy for f.d. min^l types

p not 1-based $\Rightarrow$ non-$\perp$ to $(\mathbb{P}^1, id)^\#$ = fixed field

p 1-based non-trivial $\Rightarrow$ non-$\perp$ to some $(G, H)^\#$ with G and H def^ble gr^ps

$\langle C_0, \beta \rangle$

$C_0 \underset{fg}{\triangleleft} A'' \triangleleft \Gamma$ V/A $\dim V = n$

(I) $\alpha \in V \cap \Gamma^n$

$\eta\alpha \in A^m$ $m < n$ max'l

$0 \leq \delta(\alpha/A) = td(\alpha/A) - (n-m)$

$\dim(V \cap L) \geq n-m > 0$

$L = LH \times H$ $\dim L = 2m$

$V \cap LG \times H$ typical $\Rightarrow$

$V \cap LG \times H = V \cap LH \times H$

i.e. "$H \underset{V}{\Rightarrow} LH$"

e.g. $\Delta^2 \cap LG \times \Delta = \Delta^2 = \Delta \cap \Delta \cdot \Delta$

WTS $|\{\text{such } H\}| < \infty$

(II) $\alpha \in V \cap A^n$

$\theta\beta + \gamma$ $\in C_0^n$