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Sharply n-transitive groups in o-minimal structures

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Abstract. We classify infinite sharply n-transitive groups definable in o-minimal structures, for $n = 2, 3$. For $n = 2$ we show that these are definably isomorphic to the general affine linear groups $G \cong K^+ \rtimes K^*$ where K is either a real closed field $\mathscr{R}$, its algebraic closure $\mathscr{R}(i)$ or the skew field $\mathbb{H}(\mathscr{R})$ of quaternions over $\mathscr{R}$. For $n = 3$, these groups are definably isomorphic to groups of the form $\mathrm{PGL}_2(K)$ for $K = \mathscr{R}$ or $K = \mathscr{R}(i)$. There are no 4-transitive groups definable in o-minimal structures.

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1 Preliminaries: Group actions and near-domains

We classify infinite sharply 2-transitive and sharply 3-transitive groups definable in o-minimal structures. Tits [Ti4] showed that there are no infinite sharply n-transitive groups for $n \geq 4$, and in fact we show here that in o-minimal structures there are no infinite 4-transitive groups at all. Note that we do not assume the groups to be definably connected. For background about groups definable in o-minimal structures refer to [Pi] and [PPS1].

Let us first collect some general facts about 2-transitive groups. Note first that 2-transitivity easily implies that the action is primitive. Assume that G acts effectively (i.e. only the identity fixes all elements of X) and 2-transitively on a set X with $|X| > 2$.

The following facts are well-known:

1.1 Facts. (i) For any $x \in X$, the stabilizer G_x is a maximal subgroup of G (see eg. [Ja], Thm 1.12) and clearly, the action of G on X is equivalent to the action of G by left multiplication on the coset space G/G_x.

(ii) Any nontrivial normal subgroup N of G acts transitively. If N is an abelian, the action of N is regular and G is a semidirect product $G = N \rtimes G_x$. This clearly implies that G is centerless and does not contain finite normal subgroups if X is infinite.

(iii) In particular, we will make use of the fact due to Tits that for a *sharply* 2-transitive group G the following are equivalent (see [Ti2], p. 208):

(a) G has a non-trivial normal subgroup.

(b) G has a non-trivial *abelian* normal subgroup.

(c) The set $I^2 = \{i \cdot j : i^2 = j^2 = 1, i, j \in G, i, j \neq 1\}$ is a subgroup of G in which case it is abelian and normal.

In this case we say that G *splits*.

The study of sharply 2-transitive groups is equivalent to the study of so called *near-domains*. A near-domain is a structure $(D, +, \cdot, 0, 1)$, where $(D^*, \cdot, 1)$ is a group, $(D, +, 0)$ is a loop (i.e. $x + 0 = 0 + x = x$ holds for all $x \in D$, and the equations $a + x = b$ and $x + a = b$ have unique solutions in x for all $a, b \in D$) with $0 \cdot a = a \cdot 0 = 0$ and $(a + b) \cdot c = ac + bc$. Furthermore, for any $a, b \in D$ there exists a unique $d_{a,b} \in D^*$ with $a + (b + x) = (a + b) + d_{a,b}x$. A near-domain D where $(D, +, 0)$ is a (necessarily commutative) group is called *nearfield*, hence a nearfield satisfies all axioms of a field except possibly the left distributive law. Note that if $(D^*, \cdot, 1)$ is an abelian group, then the nearfield is in fact a commutative field.

In any sharply 2-transitive group, one naturally defines an associated near-domain, and conversely in any near-domain one can easily define a sharply 2-transitive group, see e.g. [BN] p. 225. A near-domain D is a nearfield if and only if the associated group G is split, i.e. $G = A \rtimes H$, in which case $H = G_x$ is the stabilizer of a point $x \in X$ and A is an abelian normal subgroup acting regularly on X. Then H acts regularly on $A \backslash \{0\}$, so A has the structure of a K-vector space for some field K. We call a subgroup of $\mathrm{GL}(n, K)$ *regular* if it acts regularly on $K^n \backslash \{0\}$.

1.2 Examples. Typical examples of sharply 2-transitive groups are the general affine groups over any field K. The *Kalscheuer nearfields* $(D, +, *, 0, 1)$ (s. [Ka]), which are defined on the quaternions as so-called *Dickson nearfields* are less obvious examples. Here we have $(D, +) \cong (\mathbb{R}^4, +)$ (so we consider $(D^*, \cdot)$ as a subgroup of $\mathrm{GL}(4, \mathbb{R})$) and the nearfield multiplication is defined by $x * y = x^{\exp(ir \log|y|)} \cdot y$ for fixed $r \in \mathbb{R}$, where the right-hand side is evaluated in the quaternions and the identification is via the regular action on $\mathbb{R}^4$. For $r = 0$, this just yields the quaternions (s. [S. et al.] 64.20 for a description of these examples).

Finite sharply 2-transitive groups split, and all finite nearfields have been classified by Zassenhaus [Za]. Continuous nearfields over the reals were classified by Kalscheuer [Ka], but it turns out that the (proper) Kalscheuer nearfields are nevertheless not compatible with o-minimality. Locally compact nearfields in characteristic 0 were classified by Tits and Grundhöfer (see [Ti2, Ti3, Gr87]). The close connection between Lie algebras and groups definable in o-minimal structures developed in [PPS1] allows us to transfer results from [Gr87] to the o-minimal situation.

Cherlin, Grundhöfer et al. [CGNV] showed that the only nearfields of finite Morley rank in characteristic 0 are algebraically closed fields. In other characteristics the problem is open.

In the o-minimal context, related results about groups acting definably on sets of dimension 1 were obtained by Mosley in his Ph.D. thesis, see Sec. 6–8 of [MMT].

One of the main results in [MMT], which were obtained after this paper has first been submitted, is that any definably primitive permutation group in an o-minimal structure is semialgebraic. However, the proof of this fact is rather involved and does not in general simplify the arguments of this paper significantly. So we prefer giving direct proofs wherever possible. Under the additional assumption of uniform elimination of imaginaries, [NPR] classified connected groups of dimension 2 and 3 definable in o-minimal structures.

I would like to thank Th. Grundhöfer and L. Kramer for helpful discussions. Also I thank the referee for suggestions concerning the organization of the paper and simplifying some arguments.

2 Sharply 2-transitive groups

In this section we prove the following

2.1 Theorem. *Let G be a group definable in an o-minimal structure $\mathcal{M}$, acting definably and sharply 2-transitively on an infinite definable set X. Then G is definably isomorphic as a permutation group to the general affine linear groups $G \cong K^+ \rtimes K^*$ acting on the affine line of K where K is either a real closed field $\mathcal{R}$, its algebraic closure $\mathcal{R}(i)$ or the skew field $\mathbb{H}(\mathcal{R})$ of quaternions over $\mathcal{R}$. In particular, any near-domain definable in an o-minimal structure $\mathcal{M}$ is isomorphic to $\mathcal{R}$, $\mathcal{R}(i)$ or $\mathbb{H}(\mathcal{R})$ for a real closed field $\mathcal{R}$.*

We will use the following results from [PPS1]:

2.2 Fact (i) ([PPS1], Thm. 4.1). *Any connected centerless group in an o-minimal structure either has a nontrivial abelian normal definable subgroup, or is a direct product of definably simple groups.*

(ii) ([PPS1], Cor. 4.4). *Any definably simple group in an o-minimal structure is definably isomorphic to a semialgebraic linear group over some real closed field.*

We will also make repeated use of the following fact:

2.3 Proposition. *A connected centerless group definable in an o-minimal structure does not contain infinite subgroups of bounded exponent.*

Proof. By [PPS1] Thm. 3.2 we may assume that such a group is a direct product of subgroups of $\mathrm{GL}(n_i, \mathcal{R}_i)$ for real closed fields $\mathcal{R}_i$ and numbers n_i. But by Burnside's theorem [Ro], 8.1.11, $\mathrm{GL}(n_i, \mathcal{R}_i)$ does not contain infinite subgroups of bounded exponent. □

2.4 Lemma. *An effective 2-transitive group G in an o-minimal structure cannot be a direct product of non-trivial definable subgroups.*

Proof. If $G = H_1 \times H_2$, then both H_1 and H_2 act transitively by 1.1 (ii). Since elements of different factors commute, the stabilizers in each factor have to be trivial (for if $a^h = a$ for some $a \in X$ and $h \in H_1$, then for all $h' \in H_2$ we have $a^{h'h} = a^{hh'} = a^{h'}$, so h fixes all of X), hence each factor has to act regularly. So both factors are isomorphic via the map $f_a : G_1 \ni g \mapsto g' \in G_2$ with $a^{gg'} = a$ which is easily seen to be an isomorphism. So we can identify X and $H = H_1$, say, so that G acts on H via $(g_1, g_2) : h \mapsto g_1^{-1} h g_2$. Note that the stabilizer of $1 \in H$ is $G_1 = \{(g, g) : g \in H\}$. Since G_1 acts transitively on $H \backslash \{1\}$ this implies that all elements of H are conjugate. On the other hand, H must be simple as any normal subgroup of H is normal in G and therefore must be transitive. By [PPS2] Thm. 1.1., H is isomorphic to the k-rational points of a k-simple algebraic group defined for some real closed field or its algebraic closure k. Thus, either H is k-isotropic, in which case it contains unipotent and semisimple elements which cannot be conjugate, or it is anisotropic and then contains elements of any finite order. □

Unless stated otherwise, **throughout this section** we assume G and X to satisfy the assumptions of Theorem 2.1.

In [PPS1] it was shown that if G acts transitively on X, then X can be equipped in a unique way with a manifold structure such that the action of G on X is continuous. The topologies that are being referred to below are these manifold topologies, as opposed to the topology induced from the order topology on the ambiente structure $\mathcal{M}$.

2.5 Lemma. *X is definably connected. If* $\dim X > 1$, *then G and the stabilizer G_x of any $x \in X$ are definably connected as well.*

Proof. By general properties of o-minimal structures, X has only finitely many definably connected components. Let $A \subset X$ be a connected component, and $x \neq y \in A$. For any $z \in X$, there is some $g \in G_x$ with $y^g = z$. Since the action of g is continuous, it takes connected components to connected components, and $x \in A^g \cap A$, so $A^g = A$. Hence we must have $z \in A$, so X is definably connected.

If $\dim X = k > 1$, then by the transitive action of G every $x \in X$ has an open neighbourhood of dimension k, and hence $X \backslash \{x\}$ is still definably connected. Since G_x acts regularly on $X \backslash \{x\}$, this easily implies that G_x is definably connected. Now $G = G^0 \cdot G_x$, so G is definably connected as well. □

The following lemma I learned from K. Peterzil. I thank him for allowing me to include it here.

2.6 Lemma. *If G is definably simple, then any maximal definable subgroup H of G is semi-algebraic.*

Proof. Any definably simple group is semi-algebraic, so we may assume that the o-minimal structure $\mathcal{M}$ is in fact an o-minimal expansion of a real closed field and we can consider the Lie algebra structure of G. Let H^0 be the $\mathcal{M}$-connected component of H, i.e. the minimal $\mathcal{M}$-definable subgroup of finite index in H. Note that H^0 is non-trivial since otherwise H and hence X would be finite. The Lie algebra $\mathfrak{g}$ of G is definable in the field structure of $\mathscr{R}$, and the Lie algebra $\mathfrak{h}$ of H^0 is a Lie subalgebra of $\mathfrak{g}$, hence also definable in the field structure. For any $g \in G$, $a(g) : x \mapsto g^{-1}xg$ is an $\mathscr{R}$-definable automorphism of G, hence $adj(g) = d_e(a(g))$ is an $\mathscr{R}$-definable automorphism of $\mathfrak{g}$, by [PPS1], Claim 2.28. By [PPS1], 2.19 and 2.30, the normalizer of H^0 is $\mathscr{R}$-definable as $N(H^0) = \{g \in G : adj(g)\mathfrak{h} \subseteq \mathfrak{h}\}$. Since H is a maximal definable subgroup and H^0 is not normal in G, we must have $N(H^0) = H$, so H is indeed $\mathscr{R}$-definable. □

2.7 Lemma. *If G is a sharply 2-transitive group definable in an o-minimal structure $\mathscr{M}$, then G is split.*

Proof. First assume that $\dim X = 1$, so $\dim G = \dim G^0 = 2$. Then G^0 must be centerless, since its center would be an abelian normal subgroup in G and hence regular on X contradicting the sharp 2-transitivity of G. So G^0 can neither be definably simple nor a product of definably simple groups because semi-algebraic groups of dimension at most 2 are solvable. So G^0 must have an abelian normal subgroup A of dimension 1, which must be normal in G and hence acts regularly on X, showing G to be split.

Quoting [MMT] we could alternatively argue as follows: G^0 is isomorphic to $\mathscr{R}^+ \rtimes \mathscr{R}^*_{>0}$ for some real closed field $\mathscr{R}$. As G_x acts transitively on $\mathscr{R}\setminus\{0\}$, there must be some element $a \in G_x$ taking 1 to -1. This element induces a definable automorphism of the additive group of $\mathscr{R}$ and therefore must be $\mathscr{R}$-linear, so it must be multiplication by -1. This shows that G_x is the multiplicative group of $\mathscr{R}$ acting on the additive group of $\mathscr{R}$. The argument is a special case of the more general Lemma 2.9 below.

Suppose now that $\dim X > 1$. Then G is connected and thus either definably simple by Lemma 2.4 or split. Assume towards a contradiction that G is definably simple and hence semialgebraic. Using the previous lemma, we may therefore assume that $\mathscr{M}$ carries only the field structure of some real closed field $\mathscr{R}$. But by [Ti2, Ti3], all continuous locally compact not totally disconnected sharply 2-transitive groups are split. Hence in the reals the following is true for any formula φ in the language of real closed fields: If φ defines a sharply 2-transitive group G, then $I^2 = \{ij; i, j \in G, i^2 = j^2 = 1, i, j \neq 1\}$ is a subgroup; hence G is split. By the completeness of the theory of real closed fields, the same is true in any real closed field $\mathscr{R}$, so there are no definably simple sharply 2-transitive groups definable in an o-minimal structure. □

2.8 Remark. We could have used the transfer argument immediately after quoting from [MMT] that all primitive permutation groups in o-minimal structures are semialgebraic. However this direct argument is much shorter than the argument for the general primitive case in [MMT].

2.9 Lemma. *There is a definable real closed field $\mathscr{R}$ and some n such that G is definably isomorphic to a semidirect product $A \rtimes H$ of an n-dimensional vector space A and a subgroup $H \leq \mathrm{GL}(n, \mathscr{R})$.*

Proof. By Lemma 2.7, we know that $G = A \rtimes H$ for some abelian normal subgroup A on which H acts regularly as a group of automorphisms. By [PPS1] Thm. 3.2, we may assume that $G^0 = A \rtimes H^0 \leq \mathrm{GL}(m, \mathscr{R})$ for some m and some real closed field $\mathscr{R}$. Hence A is either an elementary abelian p-group, which is impossible by Proposition 2.3, or a torsion-free divisible abelian group. Thus A is a vector space over the rationals, which is irreducible as an H-module. By Schur's Lemma, the centralizer K of H in the ring of H-endomorphisms of A is a division ring, and A is a K-vector space with $H \leq \mathrm{GL}_K(A)$. For fixed $v \in A \setminus \{0\}$ and $\alpha \in K$ with $\alpha(v) = w$ we must have $\alpha(v^g) = v^{\alpha g} = w^g$ for all $g \in H$. Thus K can be (definably) identified with the set of $w \in A$ for which the map $\tilde{w} : A \to A$ defined by $v^g \mapsto w^g$ is an H-endomorphism. The center of K is an infinite definable field containing the rationals, so K is isomorphic either to $\mathscr{R}$, its algebraic closure $\mathscr{R}(i)$ or the skew field $\mathbb{H}(\mathscr{R})$ of quaternions over $\mathscr{R}$ by [OPP]. Since the o-minimal dimension of A is finite, A is a finite dimensional K-vector space, so also a finite dimensional $\mathscr{R}$-vector space with $H \leq \mathrm{GL}_{\mathscr{R}}(A)$. □

Thus we may now assume that $\mathscr{M}$ is in fact an o-minimal expansion of a real closed field, and hence quotients of definable groups by definable subgroups are again definable (see [vdD], Ch. 8).

The additive group of the quaternions $\mathbb{H}(\mathscr{R})$ being isomorphic to a 4-dimensional $\mathscr{R}$-vector space, its multiplicative group $\mathbb{H}(\mathscr{R})^*$ has a standard embedding into $\mathrm{GL}(4, \mathscr{R})$, as described in the Kalscheuer examples in the introduction. Clearly, a real closed field $\mathscr{R}$ definable in an o-minimal structure cannot have non-trivial definable automorphisms as the fixed field always contains the rationals. Therefore any definable $\mathscr{R}$-semilinear transformation of a definable $\mathscr{R}$-vector space must be $\mathscr{R}$-linear. Hence the only non-trivial definable field automorphism of $\mathscr{R}(i)$ is complex conjugation and all definable automorphisms of the quaternions $\mathbb{H}(\mathscr{R})$ must be $\mathscr{R}$-linear. It follows as over the reals that the group of $\mathscr{R}$-linear automorphisms of $\mathbb{H}(\mathscr{R})$ consists of inner automorphisms and is isomorphic to $\mathrm{SO}_3(\mathscr{R})$ (see [S. et al.] 11.29). Therefore, the group $\Gamma\mathrm{L}_R(\mathbb{H}(\mathscr{R}))$ of $\mathscr{R}$-linear $\mathbb{H}(\mathscr{R})$-semilinear transformations of the 1-dimensional quaternionic vector space $\mathbb{H}(\mathscr{R})$ is a semidirect product $\mathbb{H}(\mathscr{R})^* \rtimes \mathrm{SO}_3(\mathscr{R})$ and thus itself has a natural embedding into $\mathrm{GL}(4, \mathscr{R})$. Note that if $\mathscr{R}$ is a real closed field with non-trivial automorphisms, then $\Gamma\mathrm{L}_R(\mathbb{H}(\mathscr{R}))$ is a proper subgroup of $\Gamma\mathrm{L}(\mathbb{H}(\mathscr{R}))$. Thus $\Gamma\mathrm{L}_R(\mathbb{H}(\mathscr{R}))$ is exactly the group of semialgebraic $\mathbb{H}(\mathscr{R})$-semilinear transformations of the 1-dimensional $\mathbb{H}(\mathscr{R})$-vectorspace $\mathbb{H}(\mathscr{R})$.

As usual, for any skewfield K we let $\mathrm{GL}(1, K)$ denote the group of invertible K-linear transformations of the 1-dimensional K-vectorspace K. Of course, this is exactly K^*.

The proof of Theorem 2.1 will use the following result of [Gr87]:

2.10 Theorem ([Gr87] Thm. 2 and 3). *Let R be a real closed field and let L be a Lie subalgebra of $\mathfrak{gl}(n, \mathscr{R})$ which has dimension n over $\mathscr{R}$ and such that every nonzero element of L is invertible. Then $n = 1, 2$ or 4. Moreover, if D is the unique division ring over $\mathscr{R}$ of dimension n, then $L \leq \mathrm{Lie}(\Gamma\mathrm{L}_{\mathscr{R}}(1, D))$, $L = Z(L) \oplus L'$ and $L' = D'$ where D' denotes the commutator algebra of $D^* \leq \mathrm{GL}(4, \mathscr{R})$.*

We will also make repeated use of the following fact:

2.11 Fact ([PPS1], 2.19). If H_1 and H_2 are definably connected definable subgroups of a definable group G with respective Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, then $H_1 \subseteq H_2$ if and only if $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$.

Proof of Theorem 2.1. By Lemma 2.7, G splits as $A \rtimes H$ and by 2.9 A is a finite dimensional $\mathscr{R}$-vector space with $H \leq \mathrm{GL}(n, \mathscr{R})$ where $n = \dim_{\mathscr{R}} A$. Since H acts regularly on $A \backslash \{0\}$, the dimension of H and its Lie algebra $\mathfrak{h}$ must equal n. The stabilizer in H of any element in A is trivial, so it follows from [PPS1], Theorem 2.18.1 that $\mathfrak{h} \backslash \{0\}$ consists of invertible elements, so $\mathfrak{h} \backslash \{0\} \subseteq \mathrm{GL}(n, \mathscr{R})$. We can apply 2.10, to conclude that $\mathfrak{h}$ has a direct sum decomposition as $\mathfrak{h} = Z(\mathfrak{h}) \oplus \mathfrak{h}'$ where $\mathfrak{h}' = D'$ is the commutator algebra of some skew field $D \subseteq \mathrm{End}(\mathscr{R}^n)$ with $\dim_{\mathscr{R}} D = n$. In particular, $n = 1, 2$ or 4.

If $n = 1$, clearly the only regular subgroup of $\mathrm{GL}(1, \mathscr{R})$ is $H = \mathrm{GL}(1, \mathscr{R})$, so $G \cong R^+ \rtimes \mathscr{R}^*$ and the nearfield associated to G is the real closed field $\mathscr{R}$.

Let now $n \geq 2$. We know that H is definably connected. If $\mathfrak{h}' = 0$, then $\mathfrak{h}$ is commutative, so H is abelian by [PPS1], Claim 2.31.1. Hence the nearfield associated to G is the commutative field $K = \mathscr{R}$ or $\mathscr{R}(i)$, and $G \cong K^+ \rtimes K^*$.

If $\mathfrak{h}' = D' \neq 0$, then we must have $n = 4$ and D is the skew field of quaternions $\mathbb{H}(\mathscr{R})$ over $\mathscr{R}$. By 2.11 and 2.10 we have $H \leq \Gamma\mathrm{L}_R(1, D) = \mathrm{GL}(1, \mathrm{D}) \rtimes \mathrm{SO}_3(\mathscr{R})$. We have to show that in fact $H = \mathrm{GL}(1, D)$.

It is well-known that $\mathfrak{h}' = \mathfrak{su}(2, \mathscr{R}) \subseteq \mathrm{GL}(4, \mathscr{R}) \cup \{0\}$, and that the commutator subgroup $(D^*)'$ of the multiplicative group $D^* = \mathrm{GL}(1, D)$ is canonically isomorphic to the semialgebraic group $\mathrm{SU}(2, \mathscr{R}(i))$. So $(D^*)'$ is a definable group with Lie algebra $\mathfrak{h}' = D'$. By 2.10 and 2.11 again, H contains $(D^*)'$ and is an almost direct product of $(D^*)'$ and $Z(H)^0$. Therefore $Z(H)^0$ is abelian of dimension 1 and must be torsion-free as otherwise $Z(H)^0$, thus H and consequently $A \backslash \{0\}$ would be definably compact which is impossible.

The projection of $H \leq \Gamma\mathrm{L}_R(1, D) = \mathrm{GL}(1, D) \rtimes \mathrm{SO}_3(\mathscr{R})$ onto $\mathrm{SO}_3(\mathscr{R})$ is a definable homomorphism $\varphi : H \to \mathrm{SO}_3(\mathscr{R})$ which is trivial on $(D^*)'$ as $(D^*)'$ is contained in $\mathrm{GL}(1, D)$. Since we have definable quotients, we therefore obtain a definable homomorphism $\varphi' : Z(H)^0 \to \mathrm{SO}_3(R)$. As $Z(H)^0$ has no finite subgoups, φ' is either trivial or an embedding. Assume towards a contradiction that $\varphi'(Z(H)^0) \neq 0$, so it is a definably connected abelian subgroup of $\mathrm{SO}_3(\mathscr{R})$. Then its Lie algebra $\mathfrak{j}$ is a definable 1-dimensional subalgebra of the 3-dimensional Lie algebra $\mathfrak{so}(3, \mathscr{R})$, on which $\mathrm{SO}(3, \mathscr{R})$ acts by the adjoint representation. The only 3-dimensional non-trivial representation of $\mathrm{SO}(3, \mathscr{R})$ is the standard representation (see [S. et al.], p. 623, [Gh]) and this action is transitive on the 1-dimensional subspaces of $\mathscr{R}^3$. Hence $\mathrm{SO}(3, \mathscr{R})$ acts transitively and definably on the 1-dimensional Lie subalgebras of $\mathfrak{so}(3, \mathscr{R})$, which means that $\mathfrak{j}$ is definably isomorphic to $\mathfrak{so}(2, \mathscr{R})$ under the adjoint map of some element $g \in \mathrm{SO}(3, \mathscr{R})$. The Lie algebra of the conjugate of $\mathrm{SO}(3, \mathscr{R})$ under g is $\mathfrak{j}$, hence $\varphi'(Z(H)^0) \cong \mathrm{SO}(2, \mathscr{R})$ by Fact 2.11. But $\mathrm{SO}_2(\mathscr{R})$ contains elements of finite order so that $\ker \varphi'$ is an infinite proper definable subgoup of $Z(H)^0$ which is impossible.

Hence $H \leq \mathrm{GL}(1, D)$ and since both groups act regularly on $\mathscr{R}^4$ we must have equality, showing that definably $G \cong D^+ \rtimes D^*$. □

2.12 Corollary. *Suppose that a group G acts definably and freely with finitely many orbits on a definably connected abelian group A in some o-minimal structure* $\mathcal{M}$. *Then* $A \rtimes G$ *is isomorphic to* $R^+ \rtimes R^*_{>0}$ *or to one of the sharply* 2*-transitive groups in Theorem 2.1. This applies in particular, if G is a regular subgroup of* $\mathrm{GL}(n, \mathcal{R})$ *definable in an o-minimal structure.*

Proof. By considering $A \rtimes G^0$, we define a real closed field $\mathcal{R}$ as in Lemma 2.9 by fixing representatives for each orbit and a finite dimensional $\mathcal{R}$-vector space structure on A such that $G^0 \leq \mathrm{GL}(n, \mathcal{R})$. Hence the Lie algebra of G^0 is n-dimensional without eigenvalue 0, and as above we can apply [Gr89] Thm. 3. If $n = 1$, then $G \cong \mathcal{R}^*_{>0}$ or $G \cong \mathcal{R}^*$. The rest follows as in Theorem 2.1. □

2.13 Corollary. *All nearfield planes definable in o-minimal structures are desarguesian.*

Proof. This follows immediately from Theorem 2.1 since any nearfield plane defines a nearfield through coordinatization.

Note that this is not true any more for the next weaker class of projective planes: There are proper semifields definable in o-minimal structures (i.e. structures $(A, +, \circ)$ where $(A, +)$ is a group, $a \circ x = b$ and $x \circ a = b$ have unique solutions in x for all $a, b \in A$ and both distributive laws hold, but $\circ$ need not be associative.) Take for example $(A, +) = (\mathbb{H}(\mathbb{R}), +)$ and for $t \neq \frac{1}{2} \in \mathbb{R}$ define $\circ_t$ by $a \circ_t b = tab + (1 - t)ba$ where the right hand-side is evaluated in the quaternions.

3 Sharply 3-transitive groups

Assume now that G **is a sharply 3-transitive group definable in an o-minimal structure** M, **acting definably and effectively on some infinite set** X **in some o-minimal structure** $\mathcal{M}$. Again for $\dim X > 1$, since G_x acts sharply 2-transitively on $X \setminus \{x\}$, G_x is definably connected and hence $G^0 = G$.

If $\dim X = 1$, then $X \setminus \{x\}$ is definably connected since G_x acts sharply 2-transitively. So G^0_x is still transitive on $X \setminus \{x\}$, implying that G^0 is a Zassenhaus group. As before, G^0 is centerless and cannot be a direct product of definably simple subgroups by 2.4.

3.1 Lemma. *G does not split.*

Proof. Assume towards a contradiction that G splits as $G = A \rtimes H$. But if H acted 2-transitively on $A \setminus \{0\}$, any element of A would have to have order 2. Since A is connected, and $G^0 = A \rtimes H^0$ is centerless we contradict Proposition 2.3. □

3.2 Lemma. *If* $\dim X > 1$, *then G is definably simple.*

Proof. By the previous lemma, G does not have a nontrivial abelian normal subgroup. By Lemma 2.4 and 2.2 (i), G must be definably simple. □

3.3 Proposition. *If G acts sharply 3-transitively on a 1-dimensional set X, then G is definably isomorphic to* $\mathrm{PGL}(2, \mathscr{R})$ *for some real closed field* $\mathscr{R}$.

Proof. If $\dim X = 1$, then it is not hard to see that G^0 is a semi-algebraic Zassenhaus group acting on the projective line of a real closed field, i.e. G^0 acts 2-transitively and only the identity fixes three distinct elements, hence using transfer from the reals, $G^0 \cong \mathrm{PSL}(2, \mathscr{R})$ acting on the projective line of $\mathscr{R}$ (alternatively, this follows again from [MMT]). Now let $H < G$ be the stabilizer of ∞. Then H acts sharply 2-transitively on the affine line of $\mathscr{R}$ and after fixing $0 \in \mathscr{R}$ acts as multiplication. Therefore $\mathrm{PGL}(2, \mathscr{R}) = \mathrm{PSL}(2, \mathscr{R}) \cdot H \leq G$, but since $\mathrm{PGL}(2, \mathscr{R})$ is 3-transitive we must have equality. □

3.4 Theorem. *If G is sharply 3-transitive, then G is definably permutation equivalent to* $\mathrm{PGL}_2(K)$ *acting on the projective line of K for some real closed field $\mathscr{R}$ or its algebraic closure* $\mathscr{R}(i)$.

Proof. For $\dim X = 1$ this was proved in Proposition 3.3. For $\dim X > 1$, we may assume by Lemma 2.6 that $\mathscr{M}$ is just a real closed field. The stabilizer G_x acts sharply 2-transitively on $X \backslash \{x\}$, so by Theorem 2.1 it induces on $X\{x\}$ the structure of the affine line of some skew field K. As before, the group action is continuous and X not totally disconnected. Thus, over the reals any such group satisfies the assumptions of [Ti3], Thm. 1, so G acts on X as the group $\mathrm{PGL}_2(K)$ of transformations $y \mapsto (a \cdot y + b)/(c \cdot y + d)$ with the structure of a commutative field induced by G_x. This is first-order expressible, so the result follows from the completeness of the theory of real closed fields. □

3.5 Proposition. *There are no definable* 4-*transitive groups in o-minimal structures.*

Proof. Suppose towards a contradiction that G acts 4-transitively on some infinite set X inside some o-minimal structure $\mathscr{M}$. Then for any $x, y \in X$ the stabilizer $G_{x,y}$ is 2-transitive on $X \backslash \{x, y\}$, and hence $G^0_{x,y}$ is primitive on $X \backslash \{x, y\}$ by [DM] 7.2D. Thus, G^0 is 2-primitive, and if G^0 contained an abelian normal subgroup H, then H would be an infinite elementary abelian 2-group by [DM] 7.2A. This is impossible by Proposition 2.3. As G^0 is 3-transitive, it must therefore be definably simple by Lemma 2.4, and hence semi-algebraic. Using transfer it follows that we must have $G^0 \cong \mathrm{PO}'_{n+1}(\mathscr{R}, 1)$ for some real closed field $\mathscr{R}$ (see [S. et al.] 96.18 and [Ti1] IV.F). But in $\mathrm{PO}'_{n+1}(\mathscr{R}, 1)$, the stabilizer of three suitably chosen points fixes a projective line over $\mathscr{R}$ through these points pointwise. Since $G^0_{x,y,z}$ has finite index in $G_{x,y,z}$, it follows that $G_{x,y,z}$ cannot act transitively on X. So G is not 4-transitive after all. □

3.6 Remark. Alternatively we could argue as follows: By [MMT] Thm 1.1, all 4-transitive groups in o-minimal structures are semi-algebraic, but over the reals there are no semi-algebraic 4-transitive groups by [Ti1] IV.F.1.3. Using transfer we get a contradiction.

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