A TRICHOTOMY THEOREM FOR o-MINIMAL STRUCTURES

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Abstract

Let $\mathscr{M} = \langle M, <, \dots \rangle$ be a linearly ordered structure. We define $\mathscr{M}$ to be *o-minimal* if every definable subset of M is a finite union of intervals. Classical examples are ordered divisible abelian groups and real closed fields. We prove a trichotomy theorem for the structure that an arbitrary o-minimal $\mathscr{M}$ can induce on a neighbourhood of any a in M. Roughly said, one of the following holds:

(i) a is trivial (technical term), *or*

(ii) a has a convex neighbourhood on which $\mathscr{M}$ induces the structure of an ordered vector space, *or*

(iii) a is contained in an open interval on which $\mathscr{M}$ induces the structure of an expansion of a real closed field.

The proof uses 'geometric calculus' which allows one to recover a differentiable structure by purely geometric methods.

1. *Introduction*

Let R be a real closed field. Then R can be linearly ordered as a field; the *semi-algebraic* sets are the subsets of R^n, with $n \geqslant 1$, which can be written as finite boolean combinations of solution sets to polynomial inequalities over R. Tarski showed that the only definable sets in $\mathscr{R} = \langle R, <, +, \cdot, 0, 1 \rangle$ are the semi-algebraic sets, which amounts to showing that the collection of semi-algebraic sets is closed under projections. It follows that in the structure $\mathscr{R}$ the only definable subsets of R are finite unions of intervals whose endpoints lie in $R \cup \{\pm\infty\}$. A linearly ordered structure for which the latter property holds is called *order-minimal*, or *o-minimal.*

Our basic object of investigation here is an arbitrary linearly ordered o-minimal structure $\mathscr{M} = \langle M, <, \dots \rangle$. Three basic examples are:

(i) $\langle D, < \rangle$, where $<$ is either a discrete or a dense linear ordering;

(ii) $\langle V, <, +, d \rangle_{d \in D}$, an ordered vector space over an ordered division ring (the scalars of the division ring D are considered as functions in one variable, by, say, left multiplication);

(iii) $\langle R, <, +, \cdot \rangle$, with R a real closed field, and more generally, expansions of R to richer structures which are still o-minimal, such as $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$ (see [**22**]).

As we show here, the above three types exhaust in some sense all examples of o-minimal structures. Because of the special nature of ordered structures, we can

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analyse the structures only locally. Given a point a in an o-minimal structure $\mathcal{M}$, we characterize the structure that $\mathcal{M}$ induces on some neighbourhood of a as one of the above three types. (We always refer to the order topology on M and the product topology on M^n.)

For the next three theorems we take $\mathcal{M}$ to be an o-minimal structure. A point $a \in \mathcal{M}$ is *non-trivial* if there are an infinite open interval I containing a and a definable continuous function F: $I \times I \to M$ such that F is strictly monotone in each variable. A point which is not non-trivial is called *trivial*. As is shown in [**10**], if every point in M is trivial then the definable sets in $\mathcal{M}$ are just boolean combinations of binary relations. If $\langle G, <, \star \rangle$ is an ordered group definable on some (infinite) interval in M then every point in G is non-trivial, as is witnessed by the group operation. Our first theorem states roughly that every non-trivial point arises in this fashion.

A set $G \subseteq M$ is *convex* if for every $a < b \in G$, the interval (a, b) is contained in G. A group $\langle G, * \rangle$ is called a *convex $\wedge$-definable group* if $G \subseteq M$ is convex and the graph of $*$ is obtained by the intersection of a definable set with G^3.

THEOREM 1.1. *Let $\mathcal{M}$ be ω_1-saturated. If a is non-trivial in M then there is a convex $\wedge$-definable infinite group $G \subseteq M$ such that $a \in G$ and G is a divisible ordered abelian group.*

It follows from the theorem above that given a non-trivial $a \in M$, there is a closed interval I containing a on which a group-interval is definable (see § 2 for a definition). This latter property holds without any saturation assumption on $\mathcal{M}$. In order to analyse the structure around non-trivial points it is thus left to investigate the possible expansions of group-intervals.

Given an A-definable set $D \subseteq M^n$, we let $\mathcal{M}|D$ denote the first order structure whose universe is D and whose 0-definable sets are those of the form $D^k \cap U$ for $U \subseteq M^{nk}$ A-definable in $\mathcal{M}$. (As Lemma 2.3 shows, if I is a closed interval then every M-definable subset of I^k is definable in $\mathcal{M}|I$.)

THEOREM 1.2. *Assume that $\langle I, <, +, 0 \rangle$ is a 0-definable group-interval in an ω_1-saturated $\mathcal{M}$. Then one and only one of the following holds:*

(1) *there are an ordered vector space $\mathcal{V} = \langle V, <, +, c, d(x) \rangle_{d \in D, c \in C}$ (with C a set of constants) over an ordered division ring D, an interval $[-p, p]$ in V, and an order-preserving isomorphism of group-intervals σ: $I \to [-p, p]$, such that $\sigma(S)$ is 0-definable in $\mathcal{V}$ for every 0-definable $S \subseteq I^n$ (abusing language we say that $\mathcal{M}|I$ is a* reduct *of $\mathcal{V}|[-p, p]$);*

(2) *a real closed field R is definable in $\mathcal{M}|I$, with the underlying set a sub-interval of I and the ordering compatible with $<$.*

As is shown in [**9**], the reduct of $\mathcal{V}|[-p, p]$ which is mentioned in Case (1) of the theorem arises as follows. If $F \subseteq [-p, p] \times [-p, p]$ is the intersection of the graph of $d(x)$ with $[-p, p]^2$ then F might not have a definable counterpart in $\mathcal{M}|I$. Instead, there could be a subinterval $J \subseteq [-p, p]$ such that the graph of $d(x)|J$ has such a definable counterpart in $\mathcal{M}|I$. However, there are no other restrictions for the identification of $\mathcal{V}|[-p, p]$ and $\mathcal{M}|I$. Moreover, the division ring D is determined in $\mathcal{M}$ and does not change when we move to an elementarily equivalent structure. Hence, by taking a sufficiently small convex neighbourhood $J \subseteq I$ of a non-trivial point, the structure that $\mathcal{M}$ induces on J is that of an ordered vector space over D (identified with an 'infinitesimal' neighbourhood of 0 in $\mathcal{V}$).

Theorem 1.2 can also be formulated without the saturation assumption but the statement (1) becomes more complicated and we omit it here. The two theorems together give a local trichotomy for the possible structure of definable sets around any point in M.

TRICHOTOMY THEOREM. *Let $\mathcal{M}$ be an ω_1-saturated structure. Given $a \in M$, one and only one of the following holds:*

(T1) *a is trivial;*

(T2) *the structure that $\mathcal{M}$ induces on some convex neighbourhood of a is an ordered vector space over an ordered division ring;*

(T3) *the structure that $\mathcal{M}$ induces on some open interval around a is an o-minimal expansion of a real closed field.*

We should note that there is more than one possibility for the local structure around trivial points, where the term trivial could be misleading. For example, if I is a group-interval in a structure $\mathcal{M}$ then its endpoints might be trivial although at least on one side of each point there is a 'non-trivial' structure. However, if a is generic in M (see below), then the term trivial seems appropriate, since there is then an open interval I around a where all points are trivial and the result from [**10**] mentioned earlier can be applied to the structure induced on I.

1.1. *The Zil'ber Principle for geometric structures*

We assume here that $\mathcal{M}$ is an ω_1-saturated structure which is not necessarily *o*-minimal. Given $A \subseteq M$, we note that $a \in M$ is in the (model-theoretic) *algebraic closure of* A, or $a \in \operatorname{acl}(A)$, if a lies in a finite A-definable set. The following definition is taken from [**6**].

DEFINITION 1.3. The structure $\mathcal{M}$ is *a geometric structure* if

(i) $\operatorname{acl}(-)$ satisfies the Exchange Principle: if $a, b \in M$, $A \subseteq M$ and $b \in \operatorname{acl}(A, a)$ then either $b \in \operatorname{acl}(A)$ or $a \in \operatorname{acl}(A, b)$;

(ii) for any formula $\varphi(x, \bar{y})$ there exists $n \in \mathbb{N}$ such that for any $\bar{b}$ in M^r, either $\varphi(x, \bar{b})$ has fewer than n solutions in $\mathcal{M}$ or it has infinitely many.

EXAMPLE 1.4. If $\mathcal{M}$ is an algebraically closed field, or a real closed field, or a pseudo-finite field, or the field of p-adics, then the model-theoretic algebraic closure is the same as the field-theoretic one, and hence it satisfies the Exchange Principle. All those fields satisfy the second condition as well; hence they are geometric structures. Real closed rings are geometric structures as well.

If $\mathcal{M}$ is strongly minimal (every definable subset of M is finite or co-finite) or o-minimal then $\mathcal{M}$ is a geometric structure (see [**19**] for the latter). Every reduct of a geometric structure is itself a geometric structure.

We assume now that $\mathcal{M}$ is a geometric structure. For $\bar{a} \in M^n$, $A \subseteq M$, $\dim(\bar{a}/A)$ is the minimal cardinality of $\bar{a}' \subseteq \bar{a}$ such that $\operatorname{acl}(\bar{a}' \cup A) = \operatorname{acl}(\bar{a} \cup A)$

(in the field examples this is just the transcendence degree of $\bar{a}$ over the field generated by A). The dimension of an A-definable set $U \subseteq M^n$ is defined to be $\max\{\dim(\bar{a}/A)\colon \bar{a} \in U\}$ (in the field examples this is the same as the algebro-geometric dimension of the Zariski closure of U); $\bar{a} \in U$ is *generic in U over A* if $\dim(\bar{a}/A) = \dim(U)$.

For our purposes, a curve is any definable 1-dimensional subset of M^2. A family of curves $\mathscr{F}$ is said to be definable if there are definable $U \subseteq M^k$ and $F \subseteq U \times M^2$ such that $\mathscr{F} = \{\{\langle x, y\rangle\colon \langle \bar{u}, x, y\rangle \in F\}\colon \bar{u} \in U\}$. For $\bar{u} \in U$ let $C_{\bar{u}} = \{\langle x, y\rangle\colon \langle \bar{u}, x, y\rangle \in F\}$. We say that $C_{\bar{u}}$ is generic in $\mathscr{F}$ if $\bar{u}$ is generic in U over the parameters defining F. The family $\mathscr{F}$ is said to be *interpretable* if U is replaced by U/E, where E is a definable equivalence relation on U.

DEFINITION 1.5. A definable (or interpretable) family of curves $\mathscr{F} = \{C_{\bar{u}}\colon \bar{u} \in U\}$ is *normal of dimension n* if $\dim(U) = n$ (or $\dim(U/E) = n$) and for $\bar{u} \neq \bar{v}$ from U (or U/E), $C_{\bar{u}}$ and $C_{\bar{v}}$ intersect in at most finitely many points.

Given a geometric structure $\mathscr{M}$, one and only one of the following properties holds:

(Z1) for every interpretable infinite normal family of curves $\mathscr{F}$, if $\mathscr{C}$ is a generic curve in $\mathscr{F}$ and $\langle a, b\rangle$ is generic in $\mathscr{C}$, then either $\dim(\mathscr{C} \cap (\{a\} \times M)) = 1$ or $\dim(\mathscr{C} \cap (M \times \{b\})) = 1$ (in particular, every normal $\mathscr{F}$ is of dimension at most 1);

(Z2) every interpretable normal family of curves is of dimension at most 1, but (Z1) does not hold;

(Z3) there is an interpretable normal family of curves of dimension greater than 1.

B. Zil'ber (see [**23**]) suggested a correspondence between the above trichotomy and the interpretability of certain algebraic structures in $\mathscr{M}$. We formulate this correspondence as follows.

DEFINITION 1.6. A class $\mathscr{K}$ of geometric structures is said to *satisfy the Zil'ber Principle*, (ZP), if the structures in $\mathscr{K}$ which satisfy (Z1) are those with no interpretable groups, the structures in $\mathscr{K}$ which satisfy (Z2) are those whose definable sets arise from an interpretable vector space (or more generally a module), and the structures in $\mathscr{K}$ which satisfy (Z3) are those in which a field can be interpreted.

Some of these connections are easy to establish. If (Z1) holds then no group is interpretable in $\mathscr{M}$, for otherwise, after fixing some parameters, we can obtain elements a, b, c in M which are pairwise independent but such that $a \in \operatorname{acl}\{b, c\}$. We can now find a definable family of curves $\mathscr{F}$ with a generic curve $\mathscr{C}$ in $\mathscr{F}$ defined over c and $\langle a, b\rangle$ generic in $\mathscr{C}$ over c. Then $\mathscr{F}$ illustrates the failure of (Z1).

If a field F is interpretable in $\mathscr{M}$ then the family $\{y = ax + b\colon a, b \in F\}$ helps illustrate (Z3). By quantifier elimination, if $\mathscr{M}$ is a module, then (Z2) holds in $\mathscr{M}$. Various formulations of the converse were established for certain classes of structures (see [**5**] for stable structures, and [**9**] for o-minimal structures).

Using this terminology, we note that Zil'ber's original conjecture was that (ZP) holds for the class $\mathscr{S}$ of strongly minimal structures. Hrushovski (see [**4**]) disproved the conjecture by constructing a structure in $\mathscr{S}$ satisfying (Z3), without a field or even a group interpretable in it. A subclass of $\mathscr{S}$, called the class of Zariski structures, was later shown by Hrushovski and Zil'ber to satisfy (ZP) (see [**7**]).

Let $\mathscr{M}$ be an o-minimal structure, and $a \in M$ be non-trivial. We say that a is *of type* (Z2) if there is an open interval I containing a such that $\mathscr{M}|I$ satisfies (Z2); otherwise a is *of type* (Z3). We state the following without proof:

(1) $\mathscr{M}$ satisfies (Z1) if and only if every point in M is trivial;

(2) $\mathscr{M}$ satisfies (Z2) if and only if the set of non-trivial points is non-empty, all of type (Z2);

(3) $\mathscr{M}$ satisfies (Z3) if and only if there is a point in $\mathscr{M}$ of type (Z3).

The Trichotomy Theorem together with the last comments are easily seen to establish the Zil'ber Principle for the class of o-minimal structures, with one restriction. In the case that $\mathscr{M}$ satisfies (Z2) we can only say that locally, around each non-trivial point, the definable sets arise from an ordered vector space structure. No such global result can be obtained unless there is a definable interaction between different parts of $\mathscr{M}$.

However, as a result of the proof we obtain a sharper, local result.

THEOREM 1.7. *Let $\mathscr{M}$ be an o-minimal structure, and let $a \in M$ be non-trivial. Then*

(1) *a is of type* (Z2) *if and only if it satisfies* (T2) *from the Trichotomy Theorem;*

(2) *a is of type* (Z3) *if and only if it satisfies* (T3).

REMARKS. (1) Zil'ber also conjectured that a field F, interpretable in a strongly minimal structure $\mathscr{M}$, must be *pure*, namely that every definable set in $\mathscr{M}|F$ is already definable in the field structure alone. This was disproved by Hrushovski in [**3**] but later was shown to be true for Zariski geometries (see [**7**]). Since we now know of proper o-minimal expansions of real closed fields, clearly definable fields in o-minimal structures need not be pure.

Question. Is there a class of o-minimal structures, characterized 'geometrically', in which every definable real closed field is pure?

(2) In [**16**] and [**9**] the main dividing line was the CF property. We chose to replace it here with the more general characterizations of (Z1), (Z2), (Z3). It may be shown that an o-minimal structure has the CF property if and only if it is of type (Z1) or (Z2). Hence, Theorem 1.7 implies Zil'ber's conjecture for o-minimal structures as formulated in [**15**].

The structure of the paper

Section 2 provides some basic preliminaries for model theory and o-minimal structures. It also includes basic notation for the possible ways curves may intersect each other. In § 3 we introduce some general machinery, called *q-relations*, for constructing groups in o-minimal structures. In § 4 we discuss some good properties of definable families of functions and how to modify a given family of functions to a well behaved one. In § 5 we show how to use germs of functions and their composition to obtain q-relations and hence to prove Theorem 1.1. In § 6 we repeat a similar argument for addition of germs of functions. In § 7 we use the two constructions to define a field and prove Theorem 1.2. Section 8 gives some examples regarding the possible global structure, while in § 9 we give some applications of the main theorems, including an example of an o-minimal structure without proper o-minimal expansions.

The proofs of Theorems 1.1 and 1.2 use a fine analysis of how curves may intersect in o-minimal structures. The ideas are inspired by the differentiable structure of an ordered field and some basic calculus theorems. In the Appendix, 'Geometric Calculus', we demonstrate the strength of this approach and show how one can formulate and prove basic calculus theorems on purely geometric grounds.

For a preliminary announcement describing the main ideas of this paper, see [**17**].

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2. *Preliminaries*

NOTATION. We use $\mathscr{M}$, $\mathscr{N}$ to denote structures and M, N for their underlying sets. We use $a, b, c, \ldots$ to denote elements of the underlying set of a structure, say M, and $\bar{a}, \bar{b}, \bar{c}, \ldots$ to denote elements in some cartesian product, say M^r. We mostly use $\mathscr{F}$, $\mathscr{G}$ to denote families of functions and take f, g to denote functions in these families.

2.1. *Model-theoretic preliminaries*

DEFINITION 2.1. A structure $\mathscr{M} = \langle M, \{S \in \mathscr{D}\}\rangle$ is a set M together with a family $\mathscr{D}$ of subsets of M, $M^2, \ldots$, closed under intersections, complements, cartesian products and projections and containing the diagonals. The sets in $\mathscr{D}$ are called the 0-*definable sets in* $\mathscr{M}$. For $\bar{a} \in A^r$, $A \subseteq M$, and S a 0-definable subset of M^{r+s}, the set

$$S(\bar{a}, M^s) = \{\bar{b} \in M^s \colon \langle \bar{a}, \bar{b}\rangle \in S\}$$

is called A-*definable.* A subset of M^s is *definable* in $\mathscr{M}$ if it is A-definable for some $A \subseteq M$. A (partial) function from M^n into M said to be A-definable if its graph is. A family $\mathscr{F}$ of subsets of M^n is *definable* if there is a definable subset D of M^{n+k} such that the sets in $\mathscr{F}$ are exactly the fibres of D over parameters in M^k.

In Model Theory, one associates with every 0-definable $S \subseteq M^n$ an n-ary relation symbol $\hat{S}$ and thinks of S as the set of solutions to $\hat{S}$ in $\mathscr{M}$. The closure properties guarantee that to every basic logical operation on $\hat{S}$, with $S \in \mathscr{D}$, there corresponds a 0-definable set of solutions in $\mathscr{D}$. That is, we assume that $\mathscr{M}$ has a relational language and quantifier elimination. Abusing notation, we still use function symbols such as $+$ instead of a relational symbol for its graph. We say that a group (a field) is definable in $\mathscr{M}$ if its universe and operation(s) are definable in $\mathscr{M}$. For other notions from model theory see [**1**].

2.2. *O-minimality*

The main general references for basic o-minimal properties, including the ones below, are [**2**, **19**, **8**]. On the notion of dimension and generic points see [**18**].

A structure $\mathcal{M} = \langle M, <, \dots \rangle$ is *order-minimal* (o-minimal for short) if $<$ is a linear ordering of M and every definable subset of M is a finite union of intervals whose endpoints are in $M \cup \{\pm\infty\}$. Then $M, M^2, \dots$ are equipped with the product topology induced by $<$. We use the term 'interval' to denote convex sets in M whose end points lie in $M \cup \{\pm\infty\}$.

It is easy to see that for $\mathcal{M}$ o-minimal, $M = C \cup D \cup A$ where C, D, A are 0-definable and pairwise disjoint, C and D are open, A is finite, C is densely ordered by $<$, and D is discretely ordered by $<$. As is shown in [**21**], there is no interaction between the dense and discrete parts. Hence it is sufficient to analyse the structures that $\mathcal{M}$ induces on C and D separately. In [**20**], a complete analysis of the discretely ordered part is given and the authors show that the definable sets all arise just from translates in one variable. It follows that every point in D is trivial. We are then left with the analysis of the densely ordered part.

For the rest of the paper we assume that $\mathcal{M}$ is an o-minimal structure and $<$ is a dense linear order on $\mathcal{M}$, with or without endpoints.

For $a \in M$, $B \subseteq M$, a is in *the definable closure of* B, $a \in \mathrm{dcl}(B)$, if the set $\{a\}$ is B-definable. Since $\mathcal{M}$ is linearly ordered, algebraic closure equals definable closure. We define dimensions of tuples and definable sets as in § 1.1.

The dimension formula

We use the following dimension formula:

$$\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A).$$

Generic points

If $U \subseteq M^n$ is an A-definable set, $\bar{u} \in U$ and $\dim(\bar{u}/A) = \dim(U)$, then $\bar{u}$ is called *generic in U over A*. If $\mathcal{M}$ is ω-saturated then every definable set contains a generic point over its defining parameters.

Dimension of definable sets, as defined above, can be evaluated only in sufficiently saturated structures. However, given a definable set in any o-minimal structure, we can go to an elementary extension which is sufficiently saturated and evaluate the dimension there (this can be seen to be independent of the particular extension). As (2) of the following theorem shows, the dimension of definable sets also has a topological characterization and therefore can be evaluated in any model.

THEOREM 2.2. (1) *Let $f\colon I \to M$ be a definable function on an interval I. Then I can be partitioned into finitely many open intervals and points such that f is continuous and monotone on each interval.*

(2) *Let $U \subseteq M^n$ be an A-definable set, $\bar{u}$ generic in U over A. Then $\dim(U) = k$ if and only if there are an open rectangular neighbourhood $V \subseteq M^n$ of $\bar{u}$ and a projection map π onto k of the coordinates which gives a homeomorphism between $U \cap V$ and an open subset of M^k. In particular, if $\bar{u}$ is generic in M^n over A and $\bar{u} \in U \subseteq M^n$ for some A-definable set U, then $\bar{u}$ is in the interior of U.*

It follows from the theorem that dimension of sets is a definable notion. Namely, for $U \subseteq M^{r+s}$ an A-definable set and $k \in \mathbb{N}$, the set

$$\{\bar{a} \in M^r \colon \dim(U(\bar{a}, M^s)) = k\}$$

is A-definable.

The induced structure on a subset

Since the main results of this paper are about the structure that $\mathscr{M}$ induces on a neighbourhood of a point, we want to clarify this notion.

For the purpose of the following theorem only, we assume that the reader is familiar with the definition of a cell and the cell-decomposition theorem.

LEMMA 2.3. *Let $I \subseteq M$ be a closed interval. Then for every definable $S \subseteq I^n$, S is definable over I.*

Proof. First note that by going to an elementary extension we may assume that $\mathscr{M}$ is ω-saturated. By the cell decomposition theorem we may assume that S is a cell. Since I is closed, the boundary of S lies in I^n. By induction, it is sufficient to handle the case where

$$S = \{\langle \bar{a}, b\rangle \in M^{k+1} \colon b = f(\bar{a}) \ \& \ \bar{a} \in C\},$$

where $C \subseteq I^k$ is a definable cell and $f \colon C \to M$ is a definable continuous function. By induction, C is definable over I.

If f is definable over $\bar{w}$, we let $\bar{w} = \bar{u}\bar{v}$, where $\bar{u} \in I^r$ for some r, $\bar{v} = v_1, \ldots, v_m$ and $v_i \notin I$ for $i = 1, \ldots, m$. We may assume that f is not definable over any sub-tuple of $\bar{w}$. We use induction on m.

Consider the type $q(\bar{x}, y)$, defined as $\{\langle \bar{x}, y\rangle \in I^{k+1}\}$ together with the formulas

$$\{f(\bar{x}) = y\} \cup \{g(\bar{x}) \neq y \colon \ g \colon \ I^k \to M \text{ is a partial function definable over } \bar{u}v_1 \ldots v_{m-1}\}.$$

If q is inconsistent then by compactness there are finitely many definable functions $g_1, \ldots, g_r$, each definable over $\bar{u}v_1 \ldots v_{m-1}$, such that, for every $\langle \bar{a}, b\rangle \in I^{k+1}$, if $f(\bar{a}) = b$ then $g_i(\bar{a}) = b$ for some i. By the induction assumption on k, we may assume that the domains of the g_i are pairwise disjoint and that the graph of each g_i is definable over I; hence so is the graph of f.

We assume now that q is consistent and let $\langle \bar{a}, b\rangle$ realize it. Then, $b \in \operatorname{dcl}(\bar{a}, \bar{u}, \bar{v})$ but $b \notin \operatorname{dcl}(\bar{a}, \bar{u}, v_1, \ldots, v_{m-1})$. By the exchange principle, $v_m \in \operatorname{dcl}(\bar{a}, b, \bar{u}, v_1, \ldots, v_{m-1})$. Hence f is definable over the set $\{\bar{a}, b, \bar{u}, v_1, \ldots, v_{m-1}\}$ and $\bar{a}$, b, $\bar{u}$ in I. We can now finish by induction on m.

It follows from the lemma that for a closed interval I the M-definable subsets of I^n are definable in $\mathscr{M}|I$. Hence 'the structure that $\mathscr{M}$ induces on I' is an unambiguous notion. (The above lemma also follows from Lemma 1.2 in [**13**] and, moreover, it then holds when I is replaced by any 0-definable subset of M).

DEFINITION 2.4. An element $a \in M$ is *non-trivial over $A \subseteq M$* if there exists an open interval I containing a and an A-definable function $F \colon I \times I \to M$ such that F is continuous and strictly monotone in both coordinates. We say that a is *non-trivial* if it is non-trivial over some $A \subseteq M$.

LEMMA 2.5. *Let $\mathcal{M}$ be ω-saturated. If a is non-trivial then for every open interval J containing a, a is non-trivial over J.*

Proof. By assumption, there is a map $F: J \times J \to M$ which is continuous and strictly monotone in both variables. We may assume that $F(a,a) = a$ for if not, we can compose F on the left with the inverse function of $F(a,x)$. Hence there are closed intervals $J_1, J_2 \subseteq J$ such that $a \in \mathrm{Int}(J_1)$ and $f: J_1 \times J_1 \to J_2$. We can now apply the last lemma and conclude that $f | J_1 \times J_1$ is definable over J.

Non-orthogonality

DEFINITION 2.6. We say that $a, b \in M$ are *non-orthogonal to each other* if there is an order-preserving or order-reversing definable continuous map which sends an open neighbourhood of a onto an open neighbourhood of b.

The following technical lemma will be used in several places in the paper.

LEMMA 2.7. *Assume that a is non-trivial. Then there is an open interval I around a such that every two points in I are non-orthogonal to each other. Moreover, the map which shows the non-orthogonality can be taken to be either order-preserving or order-reversing, uniformly for every two points in I.*

Proof. Assume that $F: I \times I \to M$ is a definable, continuous map which is strictly monotone in both variables, $a \in \mathrm{Int}(I)$. By the continuity of F, $F(a', y)$ is either strictly increasing in y for all $a' \in I$, or strictly decreasing in y for all $a' \in I$. A similar result holds for $F(x, a')$.

Given $\langle b, c \rangle \in I \times I$, let $d = F(b,c)$. The curve $\{\langle x, y \rangle \in I \times I: F(x,y) = d\}$ is the graph of a continuous map $g_{b,c}$ from a neighbourhood of b onto a neighbourhood of c. Then $g_{b,c}$ is either order-preserving or order-reversing.

If it is order-reversing, then F is either strictly increasing in both variables or strictly decreasing. In both cases the map $g_{c,c}$ gives an order-reversing continuous bijection between two neighbourhoods of c. The map $g_{c,c} \circ g_{b,c}$ gives an order-preserving map between neighbourhoods of b and c.

We deal similarly with the case that $g_{b,c}$ is order-preserving.

$\bigwedge$-definable groups and rings. Group-intervals

If $I \subseteq M$ is not a definable set (for example, if it is a convex $\bigwedge$-definable ordered group) then we will not consider the structure which $\mathcal{M}$ induces on I as a standard first ordered structure. However, there is still a natural notion of a definable subset of I^n.

DEFINITION 2.8. Assume that I is a subset of M. We say that $S \subseteq I^n$ is *definable* in $\mathcal{M}|I$ if $S = I^n \cap D$ for D an $\mathcal{M}$-definable set. A map $h: I^n \to I^k$ is *definable* in $\mathcal{M}|I$ if the graph of h is definable in $\mathcal{M}|I$.

As mentioned in [**16**, p. 99] we have the following lemma.

LEMMA 2.9. *If G is a convex $\bigwedge$-definable ordered group then the only subgroups of G definable in $\mathcal{M}|G$ are G and $\{0\}$. Furthermore, G is divisible and abelian.*

For $p > 0$ in a convex $\bigwedge$-definable ordered group G, the structure $\langle [-p,p], <, +, 0 \rangle$ (where $+$ here denotes the partial function on $[-p,p] \times [-p,p]$) is called *a group-interval*. As was shown in [**9**, 6.3], group-intervals in o-minimal structures eliminate quantifiers in the proper language and all are elementarily equivalent to each other. Clearly, every convex $\bigwedge$-definable group contains a group-interval.

A *convex $\bigwedge$-definable ordered ring* is a convex $\bigwedge$-definable ordered group $\langle R, <, +\rangle$ together with a map $\bullet$: $R^2 \to R$ which is definable in $\mathscr{M}|R$ and makes R into an ordered ring. Note that by [**19**] every o-minimal ordered ring is a real closed field, but since a $\bigwedge$-definable ordered ring is not necessarily o-minimal as an independent structure, it might not even contain an identity.

LEMMA 2.10. (1) *If G is a convex $\bigwedge$-definable ordered group then the group operation is continuous on G.*

(2) *If R is a convex $\bigwedge$-definable ordered ring then the ring operations are continuous on R.*

Proof. (1) It is easy to verify that if $P(x,y)$ is a function which is strictly monotone and continuous in each variable then P is continuous. Clearly, $+$ is monotone in each variable. It is continuous in each variable because it is surjective and monotone in each variable and hence takes intervals to intervals.

(2) By (1), we only need to check that the functions $l_a(x) = ax$ and $r_a(x) = xa$ are continuous for each $a \in R$. But by Theorem 2.2, there is a point $x_0 \in R$ at which l_a is continuous. It follows that $l_a(x) = l_a(x + x_0) - ax_0$ is continuous at $x = 0$, and similarly l_a is continuous at any point in R. Similarly, r_a is continuous.

Preorders

For a point $p \in M$ we will say that a property P holds for $x \in (p)$, or for $x \in (p)^+$, or for $x \in (p)^-$, if there exist $a < p < b$ so that P holds for all $x \in (a,b)$, or for all $x \in (p,b)$, or for all $x \in (a,p)$, respectively.

DEFINITION 2.11. A binary relation R on a set S is called a *preorder relation* (or just a preorder) on S if R is transitive, reflexive and total on I.

Let R be a preorder on an open convex set I. We say that R *is positive*, or *negative*, *on* I, if for all $v, u \in I$, $v < u$ implies $R(v,u)$, or $R(u,v)$, respectively. The following lemma is used extensively throughout this work.

LEMMA 2.12. *Let $R(x,y)$ be an A-definable preorder on an open convex set $I \subseteq M$, and $a \in I$ be generic in M over A. Then there is an open interval $I_0 \subseteq I$ containing a such that R is positive or negative on I_0.*

Proof. Since R is a preorder, by o-minimality, $R(a,b)$ holds for $b \in (a)^+$, or $R(b,a)$ holds for $b \in (a)^+$. We assume that $R(a,b)$ holds for $b \in (a)^+$ and show that there is an interval $I_0 \subseteq I$ containing a such that $R(x,y)$ is positive on I_0.

Since $R(a,b)$ holds for $b \in (a)^+$, we can find $a^1 > a$ in I such that $R(a,b)$ holds for all $b \in (a, a^1)$. Decreasing a^1, if needed, we can assume that a^1 is generic over aA and therefore a is generic over a^1A. Thus there is an open interval $I' \subseteq I$, containing a, such that $R(c,b)$ holds for all $c \in I'$ and all $b \in (c, a^1)$. It is easy to see that R is positive on the interval $I_0 = \{x \in I' \colon x < a^1\}$.

Intersection of curves

DEFINITION 2.13. Let $f(x)$, $g(x)$ be definable functions, and let $p \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. We will use the following notation:

$f \preccurlyeq_p^+ g$ if $f(p) = g(p)$ and $f(x) \leqslant g(x)$ for $x \in (p)^+$;

$f \preccurlyeq_p^- g$ if $f(p) = g(p)$ and $f(x) \leqslant g(x)$ for $x \in (p)^-$;

$f \succcurlyeq_p^+ g$ if $g \preccurlyeq_p^+ f$;

$f \succcurlyeq_p^- g$ if $g \preccurlyeq_p^- f$;

$f \succcurlyeq_p g$ if $f \succcurlyeq_p^+ g$ and $f \preccurlyeq_p^- g$;

$f \preccurlyeq_p g$ if $g \succcurlyeq_p f$.

We also use:

$f \prec_p^+ g$ if $f \preccurlyeq_p^+ g$ and $f(x) \neq g(x)$ for $x \in (p)^+$;

$f \prec_p^- g$ if $f \preccurlyeq_p^- g$ and $f(x) \neq g(x)$ for $x \in (p)^-$;

We define $f \succ_p^+ g$, $f \succ_p^- g$, $f \succ_p g$ and $f \prec_p g$ similarly.

Notice that by o-minimality, if $f(p) = g(p)$ then $f \prec_p^+ g$ or $f \succ_p^+ g$ or $f(x) = g(x)$ for $x \in (p)^+$. Similarly, $f \prec_p^- g$ or $f \succ_p^- g$ or $f(x) = g(x)$ for $x \in (p)^-$.

We say that a function $f(x)$ *touches from above*, *or touches from below*, a function $g(x)$ at a point $p \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ if $f(p) = g(p)$ and $f(x) \geqslant g(x)$, or $f(x) \leqslant g(x)$, respectively, for $x \in (p)$.

We say that f *touches* g *at* p if it touches from above or below (see Fig. 1).

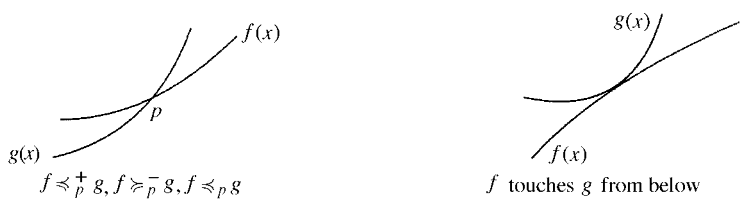

FIG. 1.

Notice that if f is strictly increasing, or strictly decreasing, on a neighbourhood of q and $h(p) = g(p) = q$ then $g \prec_p h$ if and only if $fg(x) \prec_p fh(x)$, or $fg(x) \succ_p fh(x)$, respectively.

3. *Q-relations and groups*

The main objective of this paper is, given a non-trivial point a in M, to define a one-dimensional group (or a field) containing a. In the context of stable structures this problem is often reduced to a certain combinatorial configuration of geometric dependencies which is called the group (or field) configuration. Our goal in this section is to reduce the problem of defining one-dimensional ordered groups in o-minimal structures to that of defining 4-ary relations with certain properties, which we call *quotient relations*, or q-relations. Roughly said, if R is a q-relation and $a, b, c, d \in M$ then in the ordered group which we are going to define, written multiplicatively, we have $\langle a, b\rangle R \langle c, d\rangle$ whenever $ab^{-1} < cd^{-1}$ (but not necessarily the converse).

DEFINITION 3.1. Let R be a 4-ary definable relation, and $\mathscr{A}$ be a non-empty open convex subset of M. We say that R *is a quotient-relation*, or *a q-relation*, on

$\mathscr{A}$ if the following properties hold (we will write $\langle a,b\rangle R\langle c,d\rangle$ instead of $R(a,b,c,d)$):

(q1) R is a preorder on $\mathscr{A}^2$, that is,
  (a) $\langle a_1,b_1\rangle R\langle a_2,b_2\rangle$ and $\langle a_2,b_2\rangle R\langle a_3,b_3\rangle$ implies $\langle a_1,b_1\rangle R\langle a_3,b_3\rangle$,
  (b) $\langle a,b\rangle R\langle a,b\rangle$ for all $a,b\in\mathscr{A}$,
  (c) for all $a,b,c,d\in\mathscr{A}$, $\langle a,b\rangle R\langle c,d\rangle$ or $\langle c,d\rangle R\langle a,b\rangle$;

(q2) for all $b,a,a^1\in\mathscr{A}$, $\langle a,b\rangle R\langle a^1,b\rangle$ if and only if $a\leqslant a^1$;

(q3) $\langle a_1,b_1\rangle R\langle a_2,b_2\rangle$ if and only if $\langle b_2,a_2\rangle R\langle b_1,a_1\rangle$;

(q4) if $\langle a,b\rangle R\langle c,d\rangle$ and $\langle b,a_1\rangle R\langle d,c_1\rangle$ then $\langle a,a_1\rangle R\langle c,c_1\rangle$;

(q5) for all $a,a_1,a_2,b\in\mathscr{A}$ with $a_1<a_2$ there are $b_1,b_2\in\mathscr{A}$ so that
$$\langle a_1,a\rangle R\langle b,b_1\rangle R\langle a_2,a\rangle \quad\text{and}\quad \langle a_1,a\rangle R\langle b_2,b\rangle R\langle a_2,a\rangle.$$

Axiom (q5) is different from the others in its logical form. It is usually the hardest to verify and guarantees that R is 'non-degenerate', as the next example shows.

EXAMPLE 3.2. Let $\langle A,<\rangle$ be a dense linear ordering. We let R be the 4-ary transitive relation defined by the following conditions:

(i) if $a<b$, $c_1=d_1$, $c_2=d_2$ and $e>f$ then $\langle a,b\rangle R\langle c_1,d_1\rangle R\langle c_2,d_2\rangle R\langle e,f\rangle$;

(ii) if $a_i<b_i$ for $i=1,2$ or if $a_i>b_i$ for $i=1,2$ then $\langle a_1,b_1\rangle R\langle a_2,b_2\rangle$ if and only if we have $a_1<a_2$ or we have $a_1=a_2$ and $b_1>b_2$.

The relation R is just a modified version of the lexicographic ordering on A^2. It is easily seen to satisfy (q1)–(q4) but not (q5). Since R is definable in $\langle A,<\rangle$ and this structure is trivial, no group can be definably recovered in $\langle A,<,R\rangle$.

We now consider two examples of q-relations and discuss the associated 1-dimensional order groups.

EXAMPLE 3.3. Let $\mathscr{G}$ be an ordered group. The quotient relation R, defined as $\langle a,b\rangle R\langle c,d\rangle$ if and only if $ab^{-1}\leqslant cd^{-1}$, is a q-relation on G. In order to recover a group in the structure $\langle G,<,R\rangle$ we need to consider first the equivalence relation
$$\langle a,b\rangle\sim\langle c,d\rangle \quad\Leftrightarrow\quad \langle a,b\rangle R\langle c,d\rangle \text{ and } \langle c,d\rangle R\langle a,b\rangle.$$
We can then easily define a group on $G^2/\sim$.

In practice, such an R could arise, for example, as follows. We are given the family of real-valued functions $f_a(x)=ax$, for $a\in\mathbb{R}^+$. We then define R by the condition $\langle a,b\rangle R\langle c,d\rangle$ if and only if $f_af_b^{-1}\preccurlyeq_0^+ f_cf_d^{-1}$. It is easy to see that we obtain precisely R as above with G being the multiplicative group of the positive real numbers.

EXAMPLE 3.4. Consider the real functions $f_a(x)=x^2+ax$, for $a\in\mathbb{R}^+$. It is easy to check that the relation R defined as $\langle a,b\rangle R\langle c,d\rangle$ if and only if $f_af_b^{-1}\preccurlyeq_0^+ f_cf_d^{-1}$ is a q-relation on the set of the positive elements of $\mathbb{R}$. Thus R has the property that if $a/b<c/d$ then $\langle a,b\rangle R\langle c,d\rangle$, but the converse does not hold.

Consider the structure $\langle\mathbb{R}^+,<,R\rangle$. The difference from the previous example is that in this case, if we define $\sim$ as above, then $\langle a,b\rangle\sim\langle c,d\rangle$ if and only if $a=b$

and $c = d$, or $a = c$ and $b = d$. Thus $\sim$ is definable in the structure $\langle \mathbb{R}^+, < \rangle$, and therefore no group can be defined from $\sim$. To get a group, one needs to define first an equivalence relation E such that $\langle a, b\rangle E \langle c, d\rangle$ if and only if $f_a f_b^{-1}(x)$ and $f_c f_d^{-1}(x)$ have the same derivatives at 0; and then, using properties of derivatives, one has no difficulty defining a group on $(\mathbb{R}^+)^2/E$. Now E is clearly definable if the field structure is present (since derivatives are definable) but it is also definable in $\langle \mathbb{R}^+, <, R\rangle$ alone, by

$$\langle a, b\rangle E \langle c, d\rangle \quad \Leftrightarrow \quad \langle c_1, d\rangle R \langle a, b\rangle R \langle c^1, d\rangle \text{ for all } c_1 < c < c^1.$$

As we show in this section, every q-relation on a convex set $\mathscr{A}$ has an associated equivalence relation E defined as above and a natural group structure on $\mathscr{A}^2/E$. In the subsequent sections, the q-relations will arise in a fashion similar to Example 3.4 and E will mimic the definition of tangency between two functions at a point.

We fix an open convex set $\mathscr{A} \subseteq M$ and a 0-definable relation $R \subseteq M^4$ such that R is a q-relation on $\mathscr{A}$.

LEMMA 3.5. *For $a, b, b_1 \in \mathscr{A}$,*

(1) $\langle a, a\rangle R \langle b, b\rangle$,

(2) $\langle a, b\rangle R \langle a, b_1\rangle$ *if and only if* $b_1 \leqslant b$.

Proof. Part (1) follows from (q1)(c) and (q3). Part (2) follows from (q2) and (q3).

DEFINITION 3.6. For $a, b, c, d \in \mathscr{A}$, let

$$\langle a, b\rangle E \langle c, d\rangle \quad \Leftrightarrow \quad \langle c_1, d\rangle R \langle a, b\rangle R \langle c^1, d\rangle \text{ for all } c_1 < c < c^1.$$

Now E is definable in $\mathscr{M}|\mathscr{A}$, for if we let

$$\langle a, b\rangle E' \langle c, d\rangle \quad \Leftrightarrow \quad \langle c_1, d\rangle R \langle a, b\rangle R \langle c^1, d\rangle \text{ for } c_1 \in (c)^- \text{ and } c^1 \in (c)^+,$$

then E' is definable and by properties (q1), (q2), $E = E' \cap \mathscr{A}^4$.

LEMMA 3.7. *Let $a, b, c, d \in \mathscr{A}$ be such that $\langle a, b\rangle E \langle c, d\rangle$.*

(1) *If $c' \in \mathscr{A}$ and $\langle c', d\rangle R \langle a, b\rangle$ then $c' \leqslant c$.*

(2) *If $c' \in \mathscr{A}$ and $\langle a, b\rangle R \langle c', d\rangle$ then $c \leqslant c'$.*

Proof. (1) Suppose that this is not true, and $c' > c$. Take $c < c'' < c'$. Since $\langle a, b\rangle E \langle c, d\rangle$, we have $\langle a, b\rangle R \langle c'', d\rangle$; by (q1)(a), $\langle c', d\rangle R \langle c'', d\rangle$, and, by (q2), $c' \leqslant c''$. We have a contradiction.

(2) This proof follows the same idea as in (1).

LEMMA 3.8. *Let $a, b, c, d \in \mathscr{A}$. The following hold:*

(1) $\langle a, b\rangle E \langle c, d\rangle$ *if and only if* $c = \inf\{c' \in \mathscr{A} \colon \langle a, b\rangle R \langle c', d\rangle\}$;

(2) $\langle a, b\rangle E \langle c, d\rangle$ *if and only if* $c = \sup\{c' \in \mathscr{A} \colon \langle c', d\rangle R \langle a, b\rangle\}$.

Proof. This is easy.

LEMMA 3.9. *The relation E is an equivalence relation on $\mathscr{A}^2$.*

Proof. *Reflexivity.* This follows from (q1)(b).

Symmetry. Assume that $\langle a,b\rangle E\langle c,d\rangle$, for $a,b,c,d\in\mathscr{A}$. Let $a_1<a<a^1\in\mathscr{A}$. We need to show that

$$\langle a_1,b\rangle R\langle c,d\rangle R\langle a^1,b\rangle.$$

By (q5), there are $c_1,c^1\in\mathscr{A}$ such that

$$\langle a_1,b\rangle R\langle c_1,d\rangle R\langle a,b\rangle R\langle c^1,d\rangle R\langle a^1,b\rangle.$$

By Lemma 3.7, $c_1\leqslant c\leqslant c^1$ and thus $\langle a_1,b\rangle R\langle c,d\rangle R\langle a^1,b\rangle$, by (q2).

Transitivity. Let $a,b,c,d,e,f\in\mathscr{A}$ be such that $\langle a,b\rangle E\langle c,d\rangle E\langle e,f\rangle$. We will prove that $\langle a,b\rangle E\langle e,f\rangle$.

Let $e'<e<e''\in\mathscr{A}$. We need to show that $\langle e',f\rangle R\langle a,b\rangle R\langle e'',f\rangle$. Choose $e_1,e^1\in\mathscr{A}$ such that

$$e'<e_1<e<e^1<e''.$$

Since $\langle c,d\rangle E\langle e,f\rangle$, we have $\langle e_1,f\rangle R\langle c,d\rangle R\langle e^1,f\rangle$. By (q5), there are $a_1,a^1\in\mathscr{A}$, such that

$$\langle e',f\rangle R\langle a_1,b\rangle R\langle e_1,f\rangle \quad\text{and}\quad \langle e^1,f\rangle R\langle a^1,b\rangle R\langle e'',f\rangle.$$

Thus we have

$$\langle e',f\rangle R\langle a_1,b\rangle R\langle c,d\rangle R\langle a^1,b\rangle R\langle e'',f\rangle.$$

Since we have proved already that E is symmetric, $\langle c,d\rangle E\langle a,b\rangle$; hence, by Lemma 3.7, $a_1\leqslant a\leqslant a^1$; and thus $\langle e',f\rangle R\langle a,b\rangle R\langle e'',f\rangle$.

NOTATION. For $a,b\in\mathscr{A}$, we let a/b denote the E-class of $\langle a,b\rangle$.

LEMMA 3.10. (1) *We have $a/a=b/b$ for all $a,b\in\mathscr{A}$.*

(2) *Let $a,b,c,d\in\mathscr{A}$. Then $a/b=c/d$ if and only if $b/a=d/c$.*

(3) *For all $a,b,c\in\mathscr{A}$ there are unique $d,e\in\mathscr{A}$ such that $a/b=d/c$ and $a/b=c/e$.*

Proof. (1) This follows from Lemma 3.5 and (q2).

(2) Obviously it suffices to prove the 'only if' part. Suppose that $a/b=c/d$ and $d_1<d<d^1$. We will show that $\langle d_1,c\rangle R\langle b,a\rangle R\langle d^1,c\rangle$. By (q3), it is equivalent to

$$\langle c,d^1\rangle R\langle a,b\rangle R\langle c,d_1\rangle.$$

Choose $d_2,d^2\in\mathscr{A}$ such that

$$d_1<d_2<d<d^2<d^1.$$

By (q5), there are c_1,c_2 so that

$$\langle d_1,c\rangle R\langle d,c^1\rangle R\langle d_2,c\rangle \quad\text{and}\quad \langle d^2,c\rangle R\langle d,c_1\rangle R\langle d^1,c\rangle. \tag{1}$$

Since $d_2<d<d^2$, by (q2), (q3) and Lemma 3.5, $c_1<c<c^1$; so $\langle c_1,d\rangle R\langle a,b\rangle R\langle c^1,d\rangle$, and, by equation (1) and (q3),

$$\langle c,d^1\rangle R\langle a,b\rangle R\langle c,d_1\rangle.$$

(3) Uniqueness follows from Lemma 3.8. By (2), we need to show only the existence of d.

Since $\mathscr{A}$ is open, we can choose $a_1, a^1 \in \mathscr{A}$ such that $a_1 < a < a^1$. By (q5), there are $d_1, d^1 \in \mathscr{A}$ such that

$$\langle a_1, b\rangle R\langle d_1, c\rangle R\langle a, b\rangle R\langle d^1, c\rangle R\langle a^1, b\rangle.$$

Since R is definable and $\mathscr{M}$ is o-minimal, there exists $d = \inf\{d' \in [d_1, d^1]\colon \langle a, b\rangle R\langle d'c\rangle\}$. It is easy to see that $d \in \mathscr{A}$ and $\langle a, b\rangle E\langle d, c\rangle$.

LEMMA 3.11. *Let a, b, c, d, e, f be in $\mathscr{A}$.*

(1) *If $a/b = c/d$ and $b/e = d/f$ then $a/e = c/f$.*

(2) *If $a/b = c/d$, $a/b \neq e/f$ and $\langle a, b\rangle R\langle e, f\rangle$ then $\langle c, d\rangle R\langle e, f\rangle$.*

(3) *If $\langle c, d\rangle R\langle a, b\rangle$ and $\langle a, b\rangle R\langle c, d\rangle$ then $a/b = c/d$.*

Proof. (1) Let $c_1 < c < c^1 \in \mathscr{A}$. We need to show that $\langle c_1, f\rangle R\langle a, e\rangle R\langle c^1, f\rangle$. Choose $c_2, c_3, c^2, c^3 \in \mathscr{A}$ so that

$$c_1 < c_2 < c_3 < c < c^3 < c^2 < c^1.$$

By (q5), there are $d_1, d^1 \in \mathscr{A}$ such that

$$\langle c_2, d\rangle R\langle c_1, d_1\rangle R\langle c_3, d\rangle \quad \text{and} \quad \langle c^3, d\rangle R\langle c^1, d^1\rangle R\langle c^2, d\rangle.$$

By Lemma 3.5, $d_1 < d < d^1$. Since $\langle a, b\rangle E\langle c, d\rangle$,

$$\langle c_3, d\rangle R\langle a, b\rangle R\langle c^3, d\rangle$$

and therefore

$$\langle c_1, d_1\rangle R\langle a, b\rangle R\langle c^1, d^1\rangle. \tag{2}$$

Since $\langle b, e\rangle E\langle d, f\rangle$ and $d_1 < d < d^1$,

$$\langle d_1, f\rangle R\langle b, e\rangle R\langle d^1, f\rangle. \tag{3}$$

By equations (2) and (3) and (q4),

$$\langle c_1, f\rangle R\langle a, e\rangle R\langle c^1, f\rangle.$$

(2) Since $a/b \neq e/f$, by Lemma 3.8 there is $a^1 > a$ such that $\langle a^1, b\rangle R\langle e, f\rangle$. But $\langle c, d\rangle E\langle a, b\rangle$ and hence $\langle c, d\rangle R\langle a^1, b\rangle R\langle e, f\rangle$.

(3) For $c_1 < c < c^1$,

$$\langle c_1, d\rangle R\langle c, d\rangle R\langle a, b\rangle R\langle c, d\rangle R\langle c^1, d\rangle.$$

Hence $\langle a, b\rangle E\langle c, d\rangle$.

For $a, b, c \in \mathscr{A}$ we define $a/b \star b/c = a/c$. By Lemma 3.11(1), $\star$ is well defined and, by Lemma 3.10(3), it is a binary operation on $\mathscr{A}^2/E$. For $a/b \neq e/f$ we define $a/b < e/f$ if and only if $\langle a, b\rangle R\langle e, f\rangle$. By Lemma 3.11 and (q1), this is a well defined linear ordering on $\mathscr{A}^2/E$.

LEMMA 3.12. *The structure $\langle \mathscr{A}^2/E, <, \star\rangle$ is an ordered group, with the class a/a as the identity element.*

Proof. Obviously $\star$ is associative; by Lemma 3.10(1), the diagonal $\{\langle a, a\rangle\colon a \in \mathscr{A}\}$ is an E-class and acts as an identity element; b/a is the inverse of a/b. By (q4), the group is ordered by $<$.

Since E is definable in $\mathscr{M}|\mathscr{A}$, $<$ and $\star$ are both the traces of $\mathscr{M}$-definable sets on $\mathscr{A}^2/E$. If we now pick $e \in \mathscr{A}$ then, by Lemma 3.10(3), the map $f(x) = x/e$ gives an order-preserving bijection between $\langle \mathscr{A}, < \rangle$ and $\langle \mathscr{A}^2/E, < \rangle$ and $f(x)$ is definable in $\mathscr{M}|\mathscr{A}$. Using this map, we see that $\star$ induces a convex $\wedge$-definable ordered group structure on $\mathscr{A}$, with e as the unit element. Namely,

$$a \star b = c \quad \Leftrightarrow \quad a/e \star b/e = c/e.$$

By Lemma 2.9, the group must be divisible and abelian. Thus we have proved the following theorem.

THEOREM 3.13. *Let $R \subseteq M^4$ be B-definable, and $\mathscr{A} \subseteq M$ be a convex open set such that R is a q-relation on $\mathscr{A}$. Then, for every $e \in \mathscr{A}$, there is a Be-definable function $\star_e$ such that $\langle \mathscr{A}, <, \star_e \rangle$ is a convex $\wedge$-definable ordered group whose identity element is e. The group is divisible and abelian.*

As we remarked earlier, in our setting, for a convex set $\mathscr{A}$ and a relation $R \subseteq \mathscr{A}^4$, properties (q1)–(q4) are going to be fairly easy to verify. Most work will go towards proving (q5). As we are going to show below, in order to deduce (q5) for such a relation R it is sufficient to verify the following properties: for all $a, b, c, d \in \mathscr{A}$,

(R1) if $c > d$ then $\langle b, a \rangle R \langle c, d \rangle$ for $b \in (a)^+$,

(R2) if $c < d$ then $\langle c, d \rangle R \langle b, a \rangle$ for $b \in (a)^-$,

(R3) if $c < d$ then $\langle c, d \rangle R \langle d, a \rangle$ for $b \in (a)^-$,

(R4) if $c > d$ then $\langle d, a \rangle R \langle c, b \rangle$ for $b \in (a)^+$.

DEFINITION 3.14. If $a \in M$ and $B \subseteq M$ then the *$\mathscr{M}$-cut* of a over B is the set

$$\{m \in M\colon\ b_1 < m < b_2 \text{ for all } b_1, b_2 \in B \text{ with } b_1 < a < b_2\}.$$

DEFINITION 3.15. An element a is said to be *dcl-internal* in a set C if for any finite $C_0 \subseteq C$ there exist $c_1, c_2 \in C$ such that $c_1 < a < c_2$, $\dim(c_1 c_2 / C_0 a) = 2$ and $(c_1, c_2) \cap \mathrm{dcl}(C_0 a) = \{a\}$.

DEFINITION 3.16. Given a finite set $B \subseteq M$, we call an open convex set $\mathscr{A} \subseteq M$ *good over B* if there exist $a^* \in \mathscr{A}$ and an infinite set $C \subseteq M$ containing B such that

(i) $\dim(a^*/B) = 1$,

(ii) a^* is dcl-internal in C,

(iii) $\mathscr{A}$ is the $\mathscr{M}$-cut of a^* over C.

EXAMPLE 3.17. Let $\mathscr{N} \prec \mathscr{M}$, $a^* \in N$, and $\mathscr{A}$ be the $\mathscr{M}$-cut of a^* over N. If $\mathscr{N}$ is ω-saturated then a^* is dcl-internal in N, and if, in addition, $\mathscr{M}$ is $|\mathscr{N}|^+$-saturated then $\mathscr{A}$ is good over any finite set $B \subseteq N$ such that $\dim(a^*/B) = 1$. However, if $\mathscr{M}$ is not $|\mathscr{N}|^+$-saturated then $\mathscr{A}$ might not be open.

LEMMA 3.18. *For $\mathscr{M}$ an ω_1-saturated structure, $B \subseteq M$ finite and a^* generic over B in some open interval $I \subseteq M$, there exists $\mathscr{A} \subseteq I$ containing a^* which is good over B.*

Proof. We obtain C by applying successively the saturation of $\mathscr{M}$, at each step getting $c_n < a^* < c^n$ such that, for all $c, c' \in \mathrm{dcl}(c_1, \ldots, c_{n-1}, c^1, \ldots, c^{n-1}, B, a^*)$ for which $c < a^* < c'$, we have $c < c_n < a^* < c^n < c'$. Now $\mathscr{A}$ is the $\mathscr{M}$-cut of a^* over C.

THEOREM 3.19. *Let $A \subseteq M$ be an open interval and $B \subseteq M$ be finite. Assume that $R \subseteq A^4$ is a B-definable relation satisfying, on A,* (q1)–(q4) *and* (R1)–(R4). *If $\mathscr{A} \subseteq A$ is a good convex set over B then R is a q-relation on $\mathscr{A}$.*

Proof. The proof is accomplished in a series of lemmas. We assume that C and a^* are as in the definition of a good convex set.

LEMMA 3.20. *Given $\varphi(x, y)$ a formula over B, if $\varphi(a_1, a_2)$ holds for some $a_1 < a_2$, with $a_1, a_2 \in \mathscr{A}$, then $\varphi(a, b)$ holds for all $a < b$ with $a, b \in \mathscr{A}$.*

Proof. Assume first that $\varphi(a, a^*)$ holds for some $a < a^*$, with $a \in \mathscr{A}$. Then, since $\varphi(x, a^*)$ is a formula over C and a^* is dcl-internal, there exists $c_1 < a^*$, with $c_1 \in C$, such that $\varphi(a, a^*)$ holds for all $a \in (c_1, a^*)$. By the assumption on C we may choose c_1 generic over Ba^*, and hence a^* generic over Bc_1. It follows (again, by the assumption on C) that for all $b \in \mathscr{A}$, if $a \in (c_1, b)$ then $\varphi(a, b)$ holds. Hence for all $a < b$, with $a, b \in \mathscr{A}$, $\varphi(a, b)$ holds.

Now, if $\neg\varphi(a, a^*)$ holds for all $a < a^*$ in $\mathscr{A}$, we can apply the above argument to $\neg\varphi$, and hence, for all $a < b$ with $a, b \in \mathscr{A}$, $\neg\varphi(a, b)$ holds, which contradicts our assumption that $\varphi(a_1, a_2)$ holds for some $a_1 < a_2$ from $\mathscr{A}$.

LEMMA 3.21. *For all $a, a_1, b \in \mathscr{A}$ there are $b_1, b^1 \in \mathscr{A}$ such that*

$$\langle b_1, b \rangle R \langle a, a_1 \rangle R \langle b^1, b \rangle.$$

Proof. Since $\mathscr{A}$ is an open $\mathscr{M}$-cut over C, it suffices to show that

$$\langle c_1, b \rangle R \langle a, a_1 \rangle R \langle c^1, b \rangle$$

for all $c_1, c^1 \in C \cap A$ with $c_1 < a^* < c^1$.

Suppose that this is not so, and, for instance, $\langle a, a_1 \rangle R \langle c_1, b \rangle$ for some $c_1 \in C \cap A$ with $c_1 < a^*$. Since a^* is dcl-internal, we may increase c_1, if needed, and assume that a^* is generic over Bc_1. Let c_2 be any element in $C \cap (c_1, a^*)$ such that a^* is generic over Bc_1c_2. Since $c_2 < a^*$ and $c_2 \in C$, we have $c_2 < b$, and therefore $\langle a, a_1 \rangle R \langle c_1, c_2 \rangle$. Notice that $\mathscr{A}$ is still a good convex set over Bc_1c_2 and a_1 is generic over C.

Since $c_2 > c_1$, it follows from the above equation and (R2) that $a < a_1$, and thus, by Lemma 3.20,

$$\langle a_2, a_1 \rangle R \langle c_1, c_2 \rangle \quad \text{for } a_2 \in (a_1)^-.$$

But, by (R2), $\langle c_1, c_2 \rangle R \langle a', a_1 \rangle$ for $a' \in (a_1)^-$ and, by (q1)(c),

$$\langle a_2, a_1 \rangle R \langle a', a_1 \rangle \quad \text{for } a_2, a' \in (a_1)^-,$$

which contradicts (q2).

LEMMA 3.22. *For all $b, a, a_1, a_2 \in \mathscr{A}$ with $a_1 < a_2$ there exists $d \in \mathscr{A}$ such that*

$$\langle a_1, a \rangle R \langle d, b \rangle R \langle a_2, a \rangle.$$

Proof. By Lemma 3.21, there exist $d_1, d_2 \in \mathscr{A}$ such that $\langle d_1, b\rangle R\langle a_1, a\rangle R\langle d_2, b\rangle$. Let

$$d' = \sup\{d \in [d_1, d_2]\colon \langle d, b\rangle R\langle a_1, a\rangle\}.$$

Since R is a definable relation, such a d' exists.

Case 1: $\langle d', b\rangle R\langle a_1, a\rangle$. Since $d' \in \mathscr{A}$ and d' is generic over B, we can apply (R1) and obtain $d > d' \in \mathscr{A}$ such that $\langle d, d'\rangle R\langle a_2, a_1\rangle$. By (q4),

$$\langle d, b\rangle R\langle a_2, a\rangle$$

and, since $d > d'$, by the choice of d',

$$\langle a_1, a\rangle R\langle d, b\rangle.$$

Case 2: $\langle a_1, a\rangle R\langle d', b\rangle$. We want to show that $\langle d', b\rangle R\langle a_2, a\rangle$. If not, then

$$\langle a_2, a\rangle R\langle d', b\rangle. \tag{4}$$

By (R2), we can find $d < d' \in \mathscr{A}$ such that

$$\langle a_1, a_2\rangle R\langle d, d'\rangle.$$

Combining the above equation with (4), we obtain

$$\langle a_1, a\rangle R\langle d, b\rangle,$$

which contradicts the choice of d'.

LEMMA 3.23. *For all $b, a, a_1, a_2 \in \mathscr{A}$ with $a_1 < a_2$ there exists $d \in \mathscr{A}$ such that*

$$\langle a_1, a\rangle R\langle b, d\rangle R\langle a_2, a\rangle.$$

Proof. By (R3), there is an $a' < a$ in $\mathscr{A}$ such that

$$\langle a_1, a'\rangle R\langle a_2, a\rangle.$$

Since $a' < a$, we have

$$\langle a_1, a\rangle R\langle a_1, a'\rangle R\langle a_2, a\rangle.$$

By Lemma 3.22, we can find $d \in \mathscr{A}$, such that

$$\langle a', a_1\rangle R\langle d, b\rangle R\langle a, a_1\rangle.$$

Therefore, $\langle a_1, a\rangle R\langle b, d\rangle R\langle a_1, a'\rangle$, and hence

$$\langle a_1, a\rangle R\langle b, d\rangle R\langle a_2, a\rangle.$$

By Lemma 3.22 and 3.23, R satisfies (q5) and hence it is a q-relation on $\mathscr{A}$. This completes the proof of Theorem 3.19.

4. *Nice families of functions*

Unless otherwise stated we assume for the rest of this paper that $\mathscr{M}$ is an ω_1-saturated structure (in practice, most arguments will use only the existence of generic points).

DEFINITION 4.1. If $\mathscr{F} = \{f_{\bar{u}}(x)\colon \bar{u} \in U\}$ is a definable family of (partial) functions from M into M, we say that $\mathscr{F}$ *is normal of dimension n* if the family of graphs of the functions is normal of dimension n.

For $\langle x, y \rangle \in M^2$, we let

$$U_{xy} = \{\bar{u} \in U\colon\ f_{\bar{u}}(x) = y\}.$$

Our goal in this section is to replace a given normal family of functions of dimension greater than 1 with a well-behaved one, in order eventually to use it to define a q-relation.

LEMMA 4.2. *Let $\mathscr{G} = \{G(\bar{u}, x)\colon\ \bar{u} \in U\}$, be an A-definable normal family of functions on I of dimension n.*

(i) *Assume $n > 0$. For $\bar{u} \in U$ and $a \in I$, if* $\dim(\bar{u}/A) > 0$, $\dim(a/A\bar{u}) = 1$ *and* $G(\bar{u}, a) = b$, *then* $\dim(ab/A) = 2$.

(ii) *Assume $n > 1$. For $\langle \bar{u}, a_1, a_2 \rangle$ generic in $U \times I^2$ over A, if $b_i = G(\bar{u}, a_i)$, for $i = 1, 2$, then* $\dim(b_1 b_2/Aa_1 a_2) = 2$.

Proof. (i) If $b \in \operatorname{dcl}(aA)$ then there exist a neighbourhood I_1 of a and an A-definable function $i(x)$ which equals $G(\bar{u}, x)$ for all $x \in I_1$. Since $\dim(\bar{u}/A) > 0$, there are infinitely many functions from $\mathscr{G}$ which agree with $i(x)$ on I_1, contradicting the normality of $\mathscr{G}$.

(ii) By the dimension formula and the fact that $\dim(\bar{u}a_1/A) \geqslant 3$, we may conclude that $\dim(\bar{u}/Aa_1 b_1) > 0$. We also have

$$\dim(a_2/Aa_1 b_1 \bar{u}) = \dim(a_2/Aa_1 \bar{u}) = 1,$$

and so, applying (i) with $Aa_1 b_1$ playing the role of A, we have $\dim(a_2 b_2/Aa_1 b_1) = 2$.

THEOREM 4.3. *For $U \subseteq M^n$ a definable set and I an interval, let $\mathscr{G} = \{g_{\bar{u}}(x)\colon\ \bar{u} \in U\}$ be a definable normal family of functions on I of dimension $n > 1$. Then there exist an open interval $J \subseteq I$ and an open set $V \subseteq I^2$, together with a definable continuous family of functions on J, $\mathscr{F} = \{f_{\bar{u}}\colon\ \bar{u} \in V\}$, such that*

(i) *every function $f_{\bar{u}} \in \mathscr{F}$ is strictly increasing;*

(ii) *for every $\langle x, y \rangle \in M^2$, either V_{xy} is empty or the projection map $\pi_1\colon\ V \to M$ is a homeomorphism between V_{xy} and an open interval in M;*

(iii) *for every $\bar{u}_1 \neq \bar{u}_2$, both in V, there is at most one x in J such that $f_{\bar{u}_1}(x) = f_{\bar{u}_2}(x)$, in which case $\pi_1(\bar{u}_1) < \pi_1(\bar{u}_2)$ if and only if $f_{\bar{u}_1} <_x f_{\bar{u}_2}$.*

DEFINITION 4.4. A definable continuous family of functions which satisfies (i), (ii) and (iii) of the theorem is called a *p-nice* family on J. A family $\mathscr{F}$ is called *n-nice* if it satisfies (i), (ii) and

(iii)$'$ for every $\bar{u}_1 \neq \bar{u}_2$, both in V, there is at most one x in J such that $f_{\bar{u}_1}(x) = f_{\bar{u}_2}(x)$, in which case $\pi_1(\bar{u}_1) < \pi_1(\bar{u}_2)$ if and only if $f_{\bar{u}_1} >_x f_{\bar{u}_2}$.

A family $\mathscr{F}$ is called *nice* if it is either *p-nice* or *n-nice*.

Proof. By taking a definable subset of $U \times I$ and projecting onto $M^2 \times I$, if needed, we may assume that U is an open subset of M^2 and that $G(y_1, y_2, x) = g_{y_1, y_2}(x)$ is continuous and strictly monotone in all three variables. Using the same type of manipulations as in the proof of Lemma 2.7, we may assume that $G(y_1, y_2, x)$ is strictly increasing in all its variables on an open set $V \times J \subseteq U \times I$.

Moreover, for every $\langle y_1, y_2, a\rangle \in V \times J$ there are definable bijections between neighbourhoods of y_1, y_2 and a; therefore we may assume that $V \subseteq J^2$.

We use $\bar{u}$ for $\langle y_1, y_2\rangle$. Take $\langle \bar{u}_0, e\rangle$ generic in $V \times J$ and let $b = G(\bar{u}_0, e)$.

It is easy to see that $\dim(V_{eb}) = 1$. Since G is strictly increasing in all its arguments, both projections π_1, π_2: $V_{eb} \to M$ are one-to-one, and therefore there is an open subset $V_1 \subseteq V$ containing $\bar{u}_0$ so that π_1 and π_2 are homeomorphisms from $V_1 \cap V_{eb}$ onto open subintervals of M. We may assume that $V = V_1$. Using genericity of $\bar{u}_0$ and e we can cut down V and J so that for all $\bar{u} \in V$ and $x \in J$, if $y = G(\bar{u}, x)$, then $\dim(V_{xy}) = 1$. Moreover, by Theorem 2.2, we may assume that both projections are homeomorphisms between V_{xy} and open subintervals of M. Notice that since G is strictly increasing in all arguments, the composition $\pi_2\pi_1^{-1}$ is an order-reversing homeomorphism.

Consider now the relation $\prec_e^+$ on V_{eb}. By o-minimality, there is an open V' containing $\bar{u}_0$, definable over generic parameters, such that either

$$(\forall \bar{u} \in V' \cap V_{eb})(\pi_1(\bar{u}) < \pi_1(\bar{u}_0) \to G(\bar{u}, x) \prec_e^+ G(\bar{u}_0, x))$$

or

$$(\forall \bar{u} \in V' \cap V_{eb})(\pi_1(\bar{u}) < \pi_1(\bar{u}_0) \to G(\bar{u}, x) \succ_e^+ G(\bar{u}_0, x)).$$

Interchanging y_1, y_2 in $G(y_1, y_2, x)$ and replacing V by V^{-1}, if needed, we can assume that the first holds. Since $\langle \bar{u}_0, e\rangle$ is generic in $V' \times J$, it has a neighbourhood $V_1 \times J_1 \subseteq V \times J$ such that for every $(\bar{v}, e') \in V_1 \times J_1$ we have

$$(\forall \bar{u} \in V_1)(G(\bar{u}, e') = G(\bar{v}, e') \ \&\ \pi_1(\bar{u}) < \pi_1(\bar{v}) \to G(\bar{u}, x) \prec_{e'}^+ G(\bar{v}, x)). \tag{5}$$

We may replace V by V_1 and use V again to denote the set. By repeating the above argument we may also assume that for every $\langle \bar{u}, e'\rangle \in V \times J$, we have either

$$(\forall \bar{v} \in V)(G(\bar{v}, x) = G(\bar{u}, x) \ \&\ \pi_1(\bar{v}) < \pi_1(\bar{u}) \to G(\bar{v}, x) \prec_{e'}^- G(\bar{v}, x)) \tag{6}$$

or

$$(\forall \bar{v} \in V)(G(\bar{v}, x) = G(\bar{u}, x) \ \&\ \pi_1(\bar{v}) < \pi_1(\bar{u}) \to G(\bar{v}, x) \succ_{e'}^- G(\bar{v}, x)). \tag{7}$$

We choose $a_1 < a_2$ in J, $\dim(a_1a_2/\bar{u}_0) = 2$ and let $b_i = G(\bar{u}_0, a_i)$ for $i = 1, 2$. By Lemma 4.2, $\dim(b_1b_2/a_1a_2) = 2$ and therefore there are open intervals J_1, J_2, containing b_1 and b_2 respectively, so that for all $c_1 \in J_1$ and $c_2 \in J_2$ there exists a $\bar{v} \in V$ such that $G(\bar{v}, a_1) = c_1$ and $G(\bar{v}, a_2) = c_2$. Thus there are $c_1 < b_1$, $c_2 > b_2$ and $\bar{v} \in V$ such that $G(\bar{v}, a_i) = c_i$. Since $G(\bar{v}, a_1) < G(\bar{u}_0, a_1)$, $G(\bar{v}, a_2) > G(\bar{u}_0, a_2)$ and $a_1 < a_2$, there is $e' \in (a_1, a_2)$ such that

$$G(\bar{v}, x) \succ_{e'} G(\bar{u}_0, x).$$

By (5), $\pi_1(\bar{v}) > \pi_1(\bar{u}_0)$, and since $G(\bar{v}, x) \prec_{e'}^- G(\bar{u}_0, x)$, (6) does not hold.

It is not hard to see that the family $\{G(\bar{v}, x) \colon v \in V\}$ is a p-nice family on J.

5. *Getting a q-relation: the compositional case*

The goal of this section is to show how a nice family of functions can be used to define a q-relation on a convex subset of $\mathcal{M}$. First we prove a general lemma regarding composition of functions, a lemma that will be used later as well.

5.1. *The main lemma*

We fix some notation:

(a) $\mathscr{F} = \{f_{\bar{u}}: \bar{u} \in U\}$ and $\mathscr{G} = \{g_{\bar{v}}: \bar{v} \in V\}$ are nice families of functions on open intervals I and J, respectively, all definable over the empty set;

(b) $\langle i^*, j^*, k^* \rangle$ is generic in $I \times J \times M$ and $U_{i^*j^*}$, $V_{j^*k^*}$ are infinite (and hence 1-cells);

(c) $A = \pi(U)$ and $C = \pi(V)$, where π is the projection map on the first coordinate.

By genericity, changing I and J if needed, we may assume that there is an open interval K containing k^* such that for every $\langle i, j, k \rangle \in I \times J \times K$ and $\langle a, c \rangle \in A \times C$ there are (unique) $\bar{u} \in U$, $\bar{v} \in V$ such that $f_{\bar{u}}(i) = j$, $g_{\bar{v}}(j) = k$ and $\pi(\bar{u}) = a$, $\pi(\bar{v}) = c$. Shrinking I, J, A, C, if needed, we can assume that I, J, K, A and C are definable over a finite set S such that $\dim(i^*j^*k^*/S) = 3$; for simplicity we assume that $S = \emptyset$.

For $\langle i, j, a \rangle \in I \times J \times A$, we denote by $f[i, j; a](x)$ the (unique) function $f_{\bar{u}} \in \mathscr{F}$ such that $j = f_{\bar{u}}(i)$ and $\pi(u) = a$.

For $\langle j, k, c \rangle \in J \times K \times C$, we denote by $g[j, k; c](x)$ the (unique) function $g_{\bar{v}} \in \mathscr{G}$ such that $k = g_{\bar{v}}(j)$ and $\pi(\bar{v}) = c$.

REMARK 5.1. It follows immediately from the properties of a nice family that if $f[i_1, j_1; a_1](i) = f[i_2, j_2; a_2](i) = j$ for some $i \in I$ then

(1) $f[i_1, j_1; a_1] = f[i_2, j_2; a_2]$ (as functions) if and only if $a_1 = a_2$, which is true if and only if $f[i_1, j_1; a_1] = f[i, j; a_1] = f[i_2, j_2; a_2]$ (as functions),

(2) if $\mathscr{F}$ is p-nice then $f[i_1, j_1; a_1] \prec_i f[i_2, j_2; a_2]$ if and only if $a_1 < a_2$. If $\mathscr{F}$ is n-nice then $f[i_1, j_1; a_1] \succ_i f[i_2, j_2; a_2]$ if and only if $a_1 < a_2$.

A similar result holds for $\mathscr{G}$.

Because of the p-nice case in (2) we sometimes think of $f[i, j; a](x)$ as the $\mathscr{F}$-curve through $\langle i, j \rangle$ with slope a. This intuitive concept is made clearer in § 10.

To save notation we use $f_a(x)$ to denote $f[i^*, j^*; a](x)$ and $g_c(x)$ to denote $g[j^*, k^*; c](x)$.

Our eventual plan is to define a q-relation R by:

$$\langle a, b \rangle R \langle c, d \rangle \quad \Leftrightarrow \quad f_a f_b^{-1} \preccurlyeq^+_{j^*} f_c f_d^{-1}.$$

The following lemma is the main technical tool towards establishing the (crucial) property (q5). For later purposes we prove it in the more general context of arbitrary $\mathscr{F}$ and $\mathscr{G}$.

LEMMA 5.2. *For every $c, d \in C$ and $a \in A$,*

(1) *if $g_c \prec_{j^*} g_d$ and $\langle i^*, j^*, k^* \rangle$ is generic over a, then there exists $b \in A$, with $f_b \succ_{i^*} f_a$, such that $g_c f_b \prec_{i^*} g_d f_a$,*

(2) *if $g_c \prec_{j^*} g_d$ and $\langle i^*, j^*, k^* \rangle$ is generic over a, then there exists $b \in A$, with $f_b \prec_{i^*} f_a$, such that $g_c f_a \prec_{i^*} g_d f_b$.*

Proof. To simplify notation we assume that both $\mathscr{F}$ and $\mathscr{G}$ are p-nice (the other cases can be reduced to this case by re-parametrizing $\mathscr{F}$ and $\mathscr{G}$). Hence $a < b$ if and only if $f_a \prec_{i^*} f_b$, and $d < c$ if and only if $g_d \prec_{j^*} g_c$. Since the proofs of (1) and (2) are the same, we prove (1) only.

Suppose that (1) fails. Hence, for some $c < d$ in C and $a \in A$,

$$g_c f_b \succcurlyeq^+_{i^*} g_d f_a \quad \text{for all } b > a. \tag{8}$$

By monotonicity, if we replace d in (8) by any $d' < d$, then (8) still holds. Hence, we may assume that $\dim(i^* j^* k^*/ad) = 3$ and the functional inequality is strict.

Since $d > c$ and (8) holds if we replace c by any $c' > c$, there is an open interval C', with $d \in C' \subseteq C$, such that

$$(\forall \langle c', b' \rangle \in C' \times A)(b' > a \to g[j^*, k^*; c'] f[i^*, j^*; b'] \succ^+_{i^*} g[j^*, k^*; d] f[i^*, j^*; a]). \tag{9}$$

We will assume from now on that $a, d \in \operatorname{acl}(\emptyset)$.

The statement in (9) can be written as a first-order statement $\varphi(i^*, j^*, k^*)$, where $\varphi(x, y, z)$ is over the parameters used to define C'. We may choose C' to be definable over parameters which are generic over $\bar{w} = \langle i^*, j^*, k^* \rangle$; hence $\bar{w}$ is a generic tuple over those parameters. Therefore, there is an open set W containing $\bar{w}$ such that for every $\bar{w}_1 \in W$, $\varphi(\bar{w}_1)$ holds. Without loss of generality, we may assume that $W = I' \times J' \times K'$, an open rectangular box containing $\langle i^*, j^*, k^* \rangle$. Hence

$$\begin{gathered} \text{for every } \langle c', b' \rangle \text{ in } C' \times A \text{ and every } \langle i', j', k' \rangle \in I' \times J' \times K', \\ \text{if } b' > a \text{ then } g[j', k'; c'] f[i', j'; b'] \succ^+_{i'} g[j', k'; d] f[i', j'; a]. \end{gathered} \tag{10}$$

In the remainder of this section we show that (10) yields a contradiction.

For every $\langle i, k \rangle \in I' \times K'$ we define a binary relation $R_{ik}(x, y)$ on J' as follows:

$$R_{ik}(j_1, j_2) \quad \Leftrightarrow \quad g[j_1, k; d] f[i, j_1; a](x) \preccurlyeq^+_i g[j_2, k; d] f[i, j_2; a](x).$$

Clearly, for each $\langle i, k \rangle$, R_{ik} is a preorder on J' definable by a first-order formula using i, k as parameters. Since j^* is generic over $\{i^*, k^*\}$, by Lemma 2.12, there is an open interval J'' containing j^* such that $R_{i^*k^*}$ is positive or negative on J''. The two cases can be handled in a similar way, so we may assume that $R_{i^*k^*}$ is positive on J'', and hence, without loss of generality, on J'.

Since J' is definable with parameters generic over $\{i^*, k^*\}$, $\langle i^*, k^* \rangle$ is generic over those parameters, and therefore there is an open rectangular box $I'' \times K''$ such that R_{ik} is positive on J' for all $\langle i, k \rangle \in I'' \times K''$.

Without loss of generality, assume that $I'' = I'$ and $K'' = K'$. Thus

$$\begin{gathered} g[j_1, k; d] f[i, j_1; a](x) \succcurlyeq^+_i g[j_2, k; d] f[i, j_2; a](x) \\ \text{for all } j_1 \geqslant j_2 \text{ in } J' \text{ and } \langle i, k \rangle \in I' \times K'. \end{gathered} \tag{11}$$

To simplify notation we denote by $H(x)$ the function $g[j^*, k^*; d] f[i^*, j^*; a](x)$.

Using the continuity of $\mathscr{F}$ and $\mathscr{G}$, we can find open intervals $I_0 \subseteq I'$ and $A_0 \subseteq A$ containing i^* and a, respectively, so that $H(I_0) \subseteq K'$ and $f[i^*, j^*, a'](I_0) \subseteq J'$ for all $a' \in A_0$.

Lemma 5.3. *There exist $j_0 \in J'$, $k_0 \in K'$, $i_1 \in I_0$, $a_1 \in A_0$, and $d_1 \in C'$ such that $k_0 > k^*$, $a_1 > a$,*

$$g[j_0, k_0; d_1] f[i^*, j^*; a_1](i_1) = H(i_1)$$

and

$$g[j_0, k_0; d_1] f[i^*, j^*; a_1](x) \prec^+_{i_1} H(x).$$

Proof. Fix $i_0 > i^*$ in I_0. Let $j_0 = f[i^*, j^*; a](i_0)$ and $k_0 = H(i_0)$. By Remark 5.1, $g[j^*, k^*; d](x) = g[j_0, k_0; d](x)$; hence $H(x) = g[j_0, k_0; d]f[i^*, j^*; a](x)$. Take $d_1 \in C'$ with $d_1 < d$. By the properties of $\mathscr{F}$,

$$g[j_0, k_0; d_1]f[i^*, j^*; a](i_0) = k_0 = H(i_0)$$

and

$$g[j_0, k_0; d_1]f[i^*, j^*; a](x) \prec_{i_0} H(x).$$

We can therefore find x_1, x_2 in I_0, with $i^* < x_1 < i_0 < x_2$, such that

$$g[j_0, k_0; d_1]f[i^*, j^*; a](x_1) > H(x_1)$$

and

$$g[j_0, k_0; d_1]f[i^*, j^*; a](x_2) < H(x_2).$$

By the continuity of $\mathscr{F}$ and $\mathscr{G}$ there is an $a_1 > a$ in A_0 such that

$$g[j_0, k_0; d_1]f[i^*, j^*; a_1](x_1) > H(x_1)$$

and

$$g[j_0, k_0; d_1]f[i^*, j^*; a_1](x_2) < H(x_2).$$

Again, by the continuity of $\mathscr{F}$ and $\mathscr{G}$ (and by o-minimality), we can find i_1, with $x_1 < i_1 < x_2$ (hence $i_1 \in I_0$ and $i_1 > i^*$), such that

$$g[j_0, k_0; d_1]f[i^*, j^*; a_1](i_1) = H(i_1),$$

and moreover, if we take the maximum of all such i_1 then we also have

$$g[j_0, k_0; d_1]f[i^*, j^*; a_1](x) \prec^+_{i_1} H(x),$$

which completes the proof of Lemma 5.3.

Let $\langle i_1, d_1, a_1, j_0, k_0 \rangle$ be as in Lemma 5.3 and define

$$j_1 = f[i^*, j^*; a_1](i_1), \quad j_2 = f[i^*, j^*; a](i_1), \quad k_1 = g[j^*, k^*; d](j_2).$$

Since $i_1 > i^*$ and $a_1 > a$, we must have $j_1 > j_2$. Also, since

$$H(i_1) = k_1 = g[j_0, k_0; d_1]f[i^*, j^*; a_1](i_1),$$

we have $k_1 = g[j_0, k_0; d_1](j_1)$.

By Remark 5.1,

$$f[i^*, j^*; a_1](x) = f[i_1, j_1; a_1](x), \quad g[j_0, k_0; d_1](x) = g[j_1, k_1; d_1](x)$$

and

$$H(x) = g[j_2, k_1; d]f[i_1, j_2; a].$$

Therefore

$$g[j_1, k_1; d_1]f[i_1, j_1; a_1] \prec^+_{i_1} g[j_2, k_1; d]f[i_1, j_2; a]. \tag{12}$$

Since $a_1 \in A_0$ and $i_1 \in I_0$, we have $j_1, j_2 \in J'$ and $k_1 \in K'$. By (10), as $a_1 > a$,

$$g[j_1, k_1; d]f[i_1, j_1; a] \prec^+_{i_1} g[j_1, k_1; d_1]f[i_1, j_1; a_1].$$

Therefore, by (12) and the above equation,

$$g[j_1, k_1; d]f[i_1, j_1; a] \prec^+_{i_1} g[j_2, k_1; d]f[i_1, j_2; a],$$

which contradicts (11), since $j_1 > j_2$. This completes the proof of Lemma 5.2.

5.2. *Getting a q-relation*

We now assume that:

(a) $\mathscr{F} = \{f_{\bar{u}}\colon\ \bar{u} \in U\}$ is a p-nice family of functions on an open interval I' and both $\mathscr{F}$ and I' are definable over the empty set;

(b) $i^* \in I'$ is a generic point;

(c) $\bar{u}^* \in U$ is generic over i^*;

(d) $j^* = f_{\bar{u}^*}(i^*)$;

(e) $a^* = \pi(\bar{u}^*)$, the projection of a on the first coordinate.

LEMMA 5.4. *We have* $\dim(i^*j^*a^*) = 3$.

Proof. By the dimension formula, $\dim(\bar{u}^*/i^*j^*) = 1$. Since $\pi\colon\ U_{i^*j^*} \to M$ is one-to-one, $\dim(a^*/i^*j^*) = 1$, and hence $\dim(i^*j^*a^*) = 3$.

As in § 5.1, there are open intervals I, J and A containing i^*, j^* and a^*, respectively, such that for all $i \in I$, $j \in J$ and $a \in A$ there is a unique $\bar{u} \in U$ such that $j = f_{\bar{u}}(i)$ and $a = \pi(\bar{u})$. We may assume that I, J and A are definable over the empty set and that $\mathrm{dom}(f_{\bar{u}}) = I$ for all $\bar{u} \in U$.

For $\langle i, j, a\rangle \in I \times J \times A$, we still use $f[i, j; a](x)$ for the unique function $f_{\bar{u}} \in \mathscr{F}$ such that $j = f_{\bar{u}}(i)$ and $\pi(\bar{u}) = a$. We use $f_a(x)$ for $f[i^*, j^*; a](x)$.

LEMMA 5.5. *For every $a, c, d \in A$ with a generic over $\{i^*, j^*\}$,*

(1) *if $c > d$ then there exists $b > a$ in A such that $f_b f_a^{-1} \preccurlyeq^+_{j^*} f_c f_d^{-1}$,*

(2) *if $c < d$ then there exists $b < a$ in A such that $f_b f_a^{-1} \succcurlyeq^+_{j^*} f_c f_d^{-1}$,*

(3) *if $c < d$ then there exists $b < a$ in A such that $f_b^{-1} f_a \preccurlyeq^+_{i^*} f_c^{-1} f_d$,*

(4) *if $c > d$ then there exists $b > a$ in A such that $f_b^{-1} f_a \succcurlyeq^+_{i^*} f_c^{-1} f_d$.*

Proof. Since the proofs of all cases are almost identical, we will discuss only (1). All we do is change the setting slightly so we can use Lemma 5.2.

Suppose that (1) fails, and hence there are $c > d$ and a generic over $\{i^*, j^*\}$ such that

$$f_b f_a^{-1} \succ^+_{j^*} f_c f_d^{-1} \quad \text{for all } b > a. \tag{13}$$

Decreasing c and increasing d if needed, we may assume that $\dim(i^*j^*acd) = 5$; hence $\dim(i^*j^*/acd) = 2$. Let $R(x, y)$ be the following definable relation on I:

$$R(k,i) \quad \Leftrightarrow \quad f[k, j^*; b]f[k, j^*; a]^{-1}(x) \succcurlyeq^+_{j^*} f[i, j^*; b]f[i, j^*; a]^{-1}(x) \text{ for } b \in (a)^+.$$

Then R is definable over j^*, a and it is easy to verify that R is a preorder on I. Since i^* is generic over j^*, by Lemma 2.12, there is an open interval containing i^* such that R is positive or negative on it. In either case we can find $k^* \in I$ generic over $\{i^*, j^*\}$ such that $R(k^*, i^*)$ holds, that is,

$$f[k^*, j^*; b]f[k^*, j^*; a]^{-1} \succcurlyeq^+_{j^*} f[i^*, j^*; b]f[i^*, j^*; a]^{-1} \quad \text{for } b \in (a)^+.$$

Combining the above equation with (13) we obtain

$$f[k^*, j^*; b]f[k^*, j^*; a]^{-1}(x) \succ^+_{j^*} f_c f_d^{-1}(x) \quad \text{for } b \in (a)^+. \tag{14}$$

After rearranging terms in (14), we have

$$f[i^*,j^*;c]^{-1}f[k^*,j^*;b](x) \succ^+_{k^*} f[i^*,j^*;d]^{-1}f[k^*,j^*;a](x) \quad \text{for } b\in(a)^+. \tag{15}$$

Consider now the family $\mathscr{G}=\mathscr{F}^{-1}=\{f^{-1}: f\in\mathscr{F}\}$. Then $\mathscr{G}$ is a nice family of functions and, without loss of generality, we may assume that the domain of all functions in $\mathscr{G}$ is J. Notice that $f[i^*,j^*;c]^{-1}(x) \prec_{j^*} f[i^*,j^*;d]^{-1}(x)$; hence we can apply Lemma 5.2 to $\mathscr{G}$, $\mathscr{F}$ and the triple $\langle k^*,j^*,i^*\rangle$. Therefore, there exists $b>a$ such that

$$f[i^*,j^*;c]^{-1}f[k^*,j^*;b](x) \prec^+_{k^*} f[i^*,j^*;d]^{-1}f[k^*,j^*;a](x).$$

But then the last equation holds for $b\in(a)^+$, which contradicts (15).

We define the binary relation $R_\circ$ on A^4 as

$$\langle a,b\rangle R_\circ\langle c,d\rangle \quad\Leftrightarrow\quad f_af_b^{-1}\preccurlyeq^+_{j^*} f_cf_d^{-1}.$$

It is easy to check that $R_\circ$ satisfies (q1)–(q4) from the definition of a q-relation. By Lemma 5.5, $R_\circ$ satisfies properties (R1)–(R4) for $a,b,c,d\in A$ with a generic over $\{i^*,j^*\}$. Therefore, by cutting down A, if needed, we may assume that these hold for all $a,b,c,d\in A$. Putting together Lemma 3.18 and Theorem 3.19, we see that $R_\circ$ is a q-relation on some convex set.

We can thus formulate a general theorem.

THEOREM 5.6. *Let $\mathscr{F}=\{f_{\bar u}: \bar u\in U\}$ be a definable nice family of functions on an interval I, and let a^* be generic in $\pi_1(U)$. Then the binary relation $R_\circ$, defined above, is a definable q-relation on an infinite convex set $\mathscr{A}$ containing a^*.*

5.3. *Proof of Theorem* 1.1

We are now ready to prove Theorem 1.1. For the first part of the proof we will assume that the reader is familiar with the paper [**16**]; namely we will need the definition and properties of definable quotients.

Let $a\in\mathscr{M}$ be a non-trivial point, that is, there exist an open interval I containing a and a definable function $F: I\times I$ which is continuous and strictly monotone in both variables. By Lemma 2.7, we can assume that every two points in I are non-orthogonal. As in the beginning of the proof of Theorem 5.1 in [**16**], we may assume that F is strictly increasing in both variables.

The goal is to show that there is a convex $\wedge$-definable group in I containing a. However, since any two points a_2, a_2 in I are non-orthogonal to each other, it suffices to find such a group anywhere in I.

Following the proof of Theorem 5.1 in [**16**] up to the point where Lemma 4.4 is applied, we obtain an open subset $U_1\subseteq I\times I$ (and without loss of generality we can assume that U_1 is 0-definable), a generic point $e\in I$ and a definable family of functions f_y such that $e\in\operatorname{dom}(f_{y_1}f_{y_2})$ for all $\langle y_1,y_2\rangle\in U_1$. The function $G(y_1,y_2,x)=f_{y_1}f_{y_2}(x)$ is continuous and strictly increasing in all its arguments. Since e is a generic point, there is an open interval $I'\subseteq I$ such that $I'\subseteq\operatorname{dom}(f_{y_1}f_{y_2})$ for all $\langle y_1,y_2\rangle\in U_1$. We will assume that $U_1=U$ and $I'=I$. For $\bar u=\langle y_1,y_2\rangle$, we use both $G(\bar u,x)$ and $G(y_1,y_2,x)$, depending on context.

For $a\in I$, we denote by $\sim^+_a$ the equivalence relation on U defined by $\bar u_1\sim^+_a\bar u_2$ if and only if $G(\bar u_1,x)=G(\bar u_2,x)$ for $x\in(a)^+$. We use $[\bar u]_{\sim^+_a}$ to denote the $\sim^+_a$-class of $\bar u\in U$.

If $J \subseteq I$ is an open interval, then we define $\bar{u}_1 \sim_J \bar{u}_2$ if and only if $G(\bar{u}_1, x) = G(\bar{u}_2, x)$ for all $x \in J$.

Notice that since $G(y_1, y_2, x)$ is strictly increasing in all arguments, if we take $\langle y, y_1 \rangle, \langle y, y_2 \rangle \in U$ with $y_1 \neq y_2$ then $\langle y, y_1 \rangle \not\sim_a^+ \langle y, y_2 \rangle$ and therefore $\dim([\langle y_1, y_2 \rangle]_{\sim_a^+}) < 2$. Let $\bar{u}_0$ be generic in U over e.

LEMMA 5.7. *If* $\dim([\bar{u}_0]_{\sim_e^+}) = 1$ *then there are an open subset* $V \subseteq U$ *and an open interval* $J \subseteq I$ *containing* a *such that* $\dim(V/\sim_J) \leqslant 1$.

Proof. Let $\bar{u}_1$ be generic in $[\bar{u}_0]_{\sim_e^+}$ over $\{\bar{u}_0, e\}$. Then $\dim(\bar{u}_1/e\bar{u}_0) = 1$ and there is an $e_1 > e$ such that $G(\bar{u}_0, x) = G(\bar{u}_1, x)$ for $e < x < e_1$. Decreasing e_1, if needed, we can assume that $\dim(e_1/e\bar{u}_0\bar{u}_1) = 1$ and therefore $\dim(\bar{u}_1/ee_1\bar{u}_0) = 1$ and $\dim(\bar{u}_0/ee_1) = 2$. Let J be the interval (e, e_1). Since $\bar{u}_0 \sim_J \bar{u}_1$ and $\dim(\bar{u}_1/ee_1\bar{u}_0) = 1$, one of the projections, say π_1, of $[\bar{u}_1]_{\sim_J}$ contains an open interval around $\pi_1(\bar{u}_1)$. As $\bar{u}_1$ is generic in U over e, e_1, the dimension of the set

$$\{\bar{u} \in U\colon \pi_1([\bar{u}]_{\sim_J}) \text{ contains an open interval around } \pi_1(\bar{u})\}$$

is 2 and we can take any open subset of the set above as V.

Thus, if $\dim([\bar{u}_0]_{\sim_a^+}) = 1$, then instead of applying Lemma 4.4 in the proof of Theorem 5.1 in [**16**], we use Lemma 5.7, replacing e by any generic point in J and leaving the rest of the proof unchanged. We obtain the existence of a convex $\bigwedge$-definable ordered group.

Therefore the only case we need to consider is the case when $\dim([\bar{u}_0]_{\sim_e^+}) = 0$. Since $[\bar{u}_0]_{\sim_e^+}$ is a finite set, cutting down U, we can assume that it contains only $\bar{u}_0$, and thus

$$(\forall v \in U)(G(\bar{v}, x) \sim_e^+ G(\bar{u}_0, x) \to \bar{v} = \bar{u}_0)$$

holds. By genericity of $\bar{u}_0$ and e, there exist an open rectangular box $V \subseteq U$ and an open interval $J \subseteq I$, such that

$$(\forall \bar{v} \in V\ \forall b \in J\ \forall \bar{u} \in U)(G(\bar{v}, x) \sim_b^+ G(\bar{u}, x) \to \bar{v} = \bar{u}).$$

By o-minimality, it follows that for $\bar{u} \neq \bar{v}$ from V,

$$\{\langle x, y \rangle \in J \times M\colon G(\bar{u}, x) = y\} \cap \{\langle x, y \rangle \in J \times M\colon G(\bar{v}, x) = y\} \text{ is finite.}$$

If we restrict G to $V \times J$, then G defines a definable normal family of functions of dimension 2.

By Theorem 4.3, we can replace G by a nice family of functions $\mathscr{F}$ on a subinterval of J, parametrized over a subset of J^2. By Theorem 5.6 and Theorem 3.13, given a^* generic in J, there is in $\mathscr{M}$ a convex $\bigwedge$-definable ordered divisible and abelian group containing a^*.

This completes the proof of Theorem 1.1.

6. *Getting a q-relation: the additive case*

We assume now that $\langle I, +, <, \dots \rangle$ is an o-minimal expansion of a group-interval, $\mathscr{F}$, i^*, j^*, a^* as at the beginning of § 5.2. We still use $f_a(x)$ to denote the function $f[i^*, j^*; a](x)$. Define the relation

$$\langle a, b \rangle R_+ \langle c, d \rangle \quad \Leftrightarrow \quad f_a - f_b <_{i^*} f_c - f_d.$$

It is easily verified that R_+ satisfies (q1)–(q4) of the q-relation properties. To show that R_+ is a q-relation on some convex set, it remains to prove properties (R1)–(R4). The proof is very similar to that of Lemma 5.5, with the main difference that composition of functions is everywhere replaced by addition, and the compositional inverse, f^{-1}, replaced by the additive inverse $-f$. We therefore omit some details in the proof of this case.

LEMMA 6.1. *For every $c, d \in A$ and a generic over $\{i^*, j^*\}$,*
(1) *if $c > d$ then there exists $b > a$ in A such that $f_b - f_a \preccurlyeq^+_{i^*} f_c - f_d$,*
(2) *if $c < d$ then there exists $b < a$ in A such that $f_b - f_a \succcurlyeq^+_{i^*} f_c - f_d$.*

Proof. We prove (1) only. Suppose that (1) fails and there are $c > d$ such that

$$f_b - f_a \succ^+_{i^*} f_c - f_d \quad \text{for } b \in (a)^+. \tag{16}$$

Decreasing c, if needed, we can assume that c is generic over i^*, j^*, a, d, and thus $\langle i^*, j^* \rangle$ is generic over a, c, d. We will assume from now on that $a, d \in \operatorname{acl}(\emptyset)$.

Rewriting (16) we obtain

$$f[i^*, j^*; b] - f[i^*, j^*; a] \succ^+_{i^*} f[i^*, j^*; c] - f[i^*, j^*, d] \quad \text{for } b \in (a)^+.$$

We consider the function on the left-hand side. Since j^* is generic over $\{i^*, a, b\}$, we may replace j^* on the left, using a suitable preorder, with k^* generic over all other parameters. We obtain

$$f[i^*, k^*; b] - f[i^*, k^*; a] \succ^+_{i^*} f_c - f_d(x) \quad \text{for } b \in (a)^+.$$

We fix k^*, and after rearranging terms in the above equation we obtain

$$f[i^*, k^*; b] - f[i^*, j^*, c] \succ^+_{i^*} f[i^*, k^*; a] - f[i^*, j^*; d] \quad \text{for } b \in (a)^+.$$

Instead of using here an additive analogue of Lemma 5.2, as we did in the proof of Lemma 5.5, we proceed directly to derive a contradiction. As before, we may use the properties of $\mathscr{F}$ and genericity arguments, to find open intervals D', A', I', J', K' containing d, a, i^*, j^*, k^*, respectively, such that

$$\begin{aligned} &\text{for every } \langle c', b' \rangle \text{ in } D' \times A' \text{ and every } \langle i', j', k' \rangle \in I' \times J' \times K', \\ &\text{if } b' > a \text{ then } f[i', k'; b'] - f[i', j'; c'] \succ^+_{i'} f[i', k'; a] - f[i', j'; d]. \end{aligned} \tag{17}$$

For every $\langle i, l \rangle$ we define a binary relation $R_{il}(x, y)$ on J' as follows:

$$R_{il}(j_1, j_2) \quad \Leftrightarrow \quad f[i, j_1 + l; a] - f[i, j_1; d](x) \preccurlyeq^+_i f[i, j_2 + l; a] - f[i, j_2; d](x).$$

Notice that the functions on both sides of the inequality take the value l at i. If we let $l^* = k^* - j^*$ then, as before, we may assume that for some open interval L containing l^*, for all $\langle i, l \rangle \in I' \times L$, the relation R_{il} is uniformly either positive or negative on J'. Let us assume now, in contrast to the proof of Lemma 5.2, that R_{il} is negative on J'. We have then for all $\langle i, l \rangle \in I' \times L$,

$$\begin{aligned} &f[i, j_1 + l; a] - f[i, j_1; d](x) \preccurlyeq^+_i f[i, j_2 + l; a] - f[i, j_2; d](x) \\ &\text{for all } j_1 \geqslant j_2 \text{ in } J' \text{ and } \langle i, l \rangle \in I' \times L. \end{aligned} \tag{18}$$

We denote by $H(x)$ the function $f[i^*, k^*; a] - f[i^*, j^*; d]$. Using the continuity of $\mathscr{F}$, we can find open intervals $I_0 \subseteq I'$ and $A_0 \subseteq A'$ containing i^* and a^*, respectively, so that $H(I_0) \subseteq L$ and $f[i^*, k^*, a'](I_0) \subseteq K'$ for all $a' \in A_0$. Moreover, we may assume that $K' - J' \subseteq L$.

LEMMA 6.2. *There are* $\langle i_1, a_1, d_1 \rangle \in I_0 \times A_0 \times D'$ *and* j_0, i_0 *such that* $i_1 > i^*$, $a_1 < a$ *and*

$$f[i^*, k^*; a_1] - f[i_0, j_0; d_1](i_1) = H(i_1)$$

and

$$f[i^*, k^*; a_1] - f[i_0, j_0; d_1](x) >^+_{i_1} H(x).$$

Proof. This is similar to the proof of Lemma 5.3 and we omit it.

Let $\langle i_1, d_1, a_1, i_0, j_0 \rangle$ be as in Lemma 6.2 and define

$$l = H(i_1) = f[i^*, k^*; a_1] - f[i_0, j_0; d_1](i_1),$$

and

$$j_1 = f[i^*, k^*; a_1](i_1) - l, \quad j_2 = f[i^*, k^*; a](i_1) - l.$$

Since $i_1 > i^*$ and $a_1 < a$, we must have $j_1 > j_2$, and clearly $f[i_0, j_0; d_1](i_1) = j_1$ and $f[i^*, j^*; d](i_1) = j_2$.

By Remark 5.1,

$$f[i^*, k^*; a_1](x) = f[i_1, j_1 + l; a_1](x), \quad f[i_0, j_0; d_1](x) = f[i_1, j_1; d_1](x)$$

and

$$H(x) = f[i_1, j_2 + l; a](x) - f[i_1, j_2; d](x).$$

Therefore

$$f[i_1, j_1 + l; a_1] - f[i_1, j_1; d_1] >^+_{i_1} f[i_1, j_2 + l; a] - f[i_1, j_2; d]. \tag{19}$$

Since $a_1 \in A_0$ and $i_1 \in I_0$, we have $l \in L$ and $j_1 + l, j_2 + l \in K'$; hence $j_1, j_2 \in J'$. By (17), as $a_1 < a$,

$$f[i_1, j_1 + l; a] - f[i_1, j_1; d] >^+_{i_1} f[i_1, j_1 + l; a_1] - f[i_1, j_1; d_1],$$

and together with (19), we have

$$f[i_1, j_1 + l; a] - f[i_1, j_1; d] >^+_{i_1} f[i_1, j_2 + l; a] - f[i_1, j_2; d],$$

which contradicts (18) (since $j_1 > j_2$).

Notice that by Lemma 6.1, R_+ satisfies (R1)–(R4) on A. (For (R1) and (R2) this is immediate; (R3) and (R4) follow by rearranging terms.) Let $\mathscr{A} \subseteq A$ be a convex set containing a^*, good over the parameters defining R_+. By Theorem 3.19, we may conclude that the following theorem holds.

THEOREM 6.3. *The relation* R_+ *is a q-relation on* $\mathscr{A}$.

REMARKS. 1. Assume that M is an o-minimal expansion of a real closed field R. Then the differentiable structure of R can be used to give a fast proof of Lemma 5.5 and Lemma 6.1. The proof involves a uniqueness theorem for some differential equations, and basic calculus properties (see [**15**] for related work). The argument above might suggest an alternative, purely geometric approach, to some of these calculus questions. In the Appendix we show how to prove some basic calculus results in this manner.

2. The standing assumption in Lemma 5.5 and Lemma 6.1 is that $\langle i^*, j^* \rangle$ is a generic point. As the next example shows, at least in the additive case the lemma would have failed if we omitted the genericity assumption.

EXAMPLE 6.4. We work in $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$. For $0 < a < 1$, $0 < b < 1$, $0 \leqslant x \leqslant 1$, define

$$f_{a,b}(x) = x^{1/a} + b.$$

For any $r < 1$ we get a nice family of functions on the interval $[0, r]$. Take $i^* = 0$, $j^* = 0$ and for $a \in (0,1)$ denote by $f_a(x)$ the function $x^{1/a}$ (in our previous notation this is $f[i^*, j^*; a](x)$).

If we now define R by $\langle a, b \rangle R \langle c, d \rangle$ if and only $f_a - f_b \preccurlyeq_0^+ f_c - f_d$, then we obtain precisely the relation which was defined in Example 3.2. It satisfies (q1)–(q4) but not (q5).

In particular, if we take any $d < c < a$ then *for every* $b > a$,

$$f_b - f_a \succ_0^+ f_c - f_d.$$

QUESTION. Find a similar example that shows that Lemma 5.5 would have failed without the assumption that $\langle i^*, j^* \rangle$ is generic.

7. *Getting a field*

We are now ready to prove Theorem 1.2.

We assume that a is non-trivial in M. By Theorem 1.1, there is a definable group-interval $\langle I, <, +, a \rangle$, with a as the identity element. For simplicity we use 0 instead of a and assume that I and $+$ are 0-definable.

DEFINITION 7.1. (i) For two groups H, G, a *presentation of* H *on* G is a homomorphism σ: $H \to \mathrm{Aut}(G)$. We say that σ is *faithful* if $\mathrm{Ker}(\sigma) = \{1\}$. For $h \in H$ we use σ_h to denote the corresponding automorphism of G.

(ii) For H, G two convex $\wedge$-definable groups, we say that a presentation σ of H on G is *definable* if there is a definable set $D \subseteq M^3$ such that

$$D \cap H \times G^2 = \{\langle h, g_1, g_2 \rangle \in H \times G^2 \colon \sigma_h(g_1) = g_2\}.$$

We say that σ is *continuous* if the map $\langle h, g_1 \rangle \mapsto \sigma_h(g_1)$ is continuous from $H \times G$ into G.

For the proof of the proposition below we need the notion of *a partial endomorphism* which is taken from [**9**].

DEFINITION 7.2. (i) A partial definable function λ: $M \to M$ is called a *partial endomorphism*, or *p.e.*, if its domain, $\mathrm{dom}(\lambda)$, is an interval $(-a, a)$ around 0 and it is linear where defined, that is, $\lambda(x+y) = \lambda(x) + \lambda(y)$, if $x, y, x+y \in \mathrm{dom}(\lambda)$.

(ii) For λ a p.e., we say that *the germ of* λ *at* 0 *is* A*-definable* if there are an A-definable function $f(x)$ and an open neighbourhood J of 0 such that $f|J = \lambda|J$.

PROPOSITION 7.3. *Assume that $\mathcal{M}|I$ does not satisfy clause* (1) *of Theorem* 1.2. *Then there are convex* $\wedge$*-definable groups* $H \subseteq I$ *and* $G \subseteq I$ *and a definable, faithful and continuous presentation* $\sigma \colon H \to \mathrm{Aut}(G)$.

Proof. By Proposition 4.2 in [**9**], the theory of $\mathcal{M}|I$ is not linear. Namely, one of the following two statements holds (as we show below, (1) implies (2); the converse fails).

(1) There is a definable p.e. h whose germ at 0 is not 0-definable.

(2) There is a definable function h on an interval $J \subseteq I$ which is not linear on any subinterval of J. (Recall that f is called *linear* on an interval J if for every $a, b \in J$, $h(a+x) - h(a) = h(b+x) - h(b)$ for $x \in (0)$.)

As we now show, (1) gives the conclusion of the proposition directly.

LEMMA 7.4. *Assume that there is a definable p.e. whose germ at* 0 *is not definable. Then there exist convex* $\wedge$*-definable groups* $H \subseteq I$ *and* $G \subseteq I$ *and a definable, faithful and continuous presentation* $\sigma \colon H \to \mathrm{Aut}(G)$. *Moreover, G is definably embedded in* $\langle I, <, + \rangle$.

Proof. Assume that the graph of $h(x)$ is definable via a formula $\varphi(x, y, \bar{a})$. We may assume that for every $\bar{b}$, the formula $\varphi(x, y, \bar{b})$ defines a p.e., which we denote by $h_{\bar{b}}$. Since the germ of h at 0 is not 0-definable, as $\bar{b}$ varies we obtain infinitely many partial endomorphisms, whose germs at 0 are pairwise distinct. Without loss of generality, we may assume that there is an open interval $I_1 \subseteq I$ such that $I_1 \subseteq \mathrm{dom}(h_{\bar{b}})$ for all $\bar{b}$. As was pointed out in [**9**], if two partial endomorphisms agree on a non-zero point, then they agree on all of their common domain. Therefore, by fixing a non-zero generic $q \in I_1$ we can reparametrize the family of germs defined by φ so that we get $h_c(q) = c$ for all c in some interval J. By taking q sufficiently close to 0 and cutting down J if needed, we may also assume that $J \subseteq I_1$.

We now fix $p \in J$ generic over all mentioned parameters and replace every h_c by $h_p^{-1} h_c$ (clearly, a p.e. as well). After cutting down J and I_1, if needed, we have $q \in J$. If we reparametrize again, as above, we obtain $h_q(q) = q$ and hence h_q is the identity map, where defined. Since p was generic over q, the map $\langle a, b \rangle \mapsto h_a(b)$ is continuous on a neighbourhood of $\langle q, q \rangle$. Hence, for every a, b close to q there is $c \in J$ such that $h_a h_b(q) = h_c(q)$ and by the above remarks $h_a h_b(x) = h_c(x)$ for every x in their common domain. We can now define a partial operation on J:

$$a \star b = c \quad \Leftrightarrow \quad h_a h_b(q) = h_c(q).$$

Let B be the definable closure of all parameters mentioned thus far and let H be the $\mathcal{M}$-cut of q over B (see Definition 3.14). It is easy to see that $\star$ makes H into a convex $\wedge$-definable ordered group, with q as its identity element.

Let G be the $\mathcal{M}$-cut of 0 over B. For every $a \in H$, h_a defines an automorphism of G. By the definition of $\star$ we have $h_{a \star b} = h_a h_b$, and clearly $h_a = \mathrm{id}$ if and only if $a = q$. Hence, the map $a \mapsto h_a$ gives a definable faithful presentation of H on G. The continuity of this action is proved similarly to Lemma 2.10(2).

We have therefore reduced Proposition 7.3 to the following. We have a (fixed) definable function $h(x)$ on an interval J which is not linear on any subinterval.

LEMMA 7.5. *Let $f_{u,v}(x) = h(u+x)+v$. Then there are an open set $U \subseteq M^2$ and an open interval $I_1 \subseteq J$ such that:*

(i) *$\{f_{u_1,v_1}(x)\colon \langle u_1, v_1\rangle \in U\}$ is a p-nice family on I_1;*

(ii) *if $u_1 < u_2$ then for every v_1, v_2 the function $f_{u_2,v_2}(x) - f_{u_1,v_1}(x)$ is strictly increasing on I_1.*

Proof. For a' generic in J, we may assume that $h(x)$ is strictly increasing on a neighbourhood of a' (if not, replace h with $-h$). By the assumptions on the non-linearity of $h(x)$, for $b \in (a')^+$, we have either

$$h(a'+x) - h(a') > h(b+x) - h(b) \quad \text{for } x \in (0)^+$$

or

$$h(a'+x) - h(a') < h(b+x) - h(b) \quad \text{for } x \in (0)^+. \tag{20}$$

We may assume that (20) holds (if not replace h with h^{-1}). Take $b' > a'$ sufficiently close and generic over a'. By (20), there are open intervals I_1 containing a', and I_2 containing b', such that for all $a_1 < a_2$ in I_1 and $b_1 < b_2$ in I_2, if $a_2 - a_1 = b_2 - b_1$ then

$$h(a_2) - h(a_1) < h(b_2) - h(b_1).$$

Hence

$$h(a_1) - h(a_2) > h(b_1) - h(b_2),$$

and in particular,

$$h(a'+x) - h(a') > h(b'+x) - h(b') \quad \text{for } x \in (0)^-. \tag{21}$$

Putting together (20) and (21) we obtain

$$h(a'+x) - h(a') <_0 h(b'+x) - h(b').$$

The above holds for $b' \in (a')^+$.

We can now use an appropriate preorder as we have done several times before, and by the genericity of a' we will assume that, for every $a_1 < a_2$ in J,

$$h(a_1+x) - h(a_1) <_0 h(a_2+x) - h(a_2). \tag{22}$$

Take $J_1 \subseteq J$ a proper subinterval and let $\varepsilon > 0$ be such that for every $a \in J_1$, $a - \varepsilon$ and $a + \varepsilon$ are still in J. We let $I_1 = (-\varepsilon, \varepsilon)$, $f_{u,v}(x) = h(u+x)+v$, and define

$$\mathscr{F} = \{f_{u,v}\colon u, v \in J_1, x \in I_1\}.$$

We take $\langle u_1, v_1\rangle \neq \langle u_2, v_2\rangle$. We may assume that $u_1 \neq u_2$, for if $u_1 = u_2$ then the two functions agree at a point if and only if $v_1 = v_2$. Without loss of generality, assume that $u_1 < u_2$. By o-minimality, it is sufficient to show that if $f_{u_1,v_1}(i) = f_{u_2,v_2}(i)$ for some $i \in I_1$ then $f_{u_1,v_1} <_i f_{u_2,v_2}$ (for then f_{u_1,v_1} and f_{u_2,v_2} can agree at most once on I_1). This is immediate, for if $f_{u_1,v_1}(i) = f_{u_2,v_2}(i)$ then $f_{u_1+i,v_1}(0) = f_{u_2+i,v_2}(0)$ and by (22), $f_{u_1+i,v_1} <_0 f_{u_2+i,v_2}$. It follows that $f_{u_1,v_1} <_i f_{u_2,v_2}$.

For $\mathscr{F}$ to be p-nice it remains only to find $U \subseteq J_1^2$ such that $U_{x,y}$ is either empty or a 1-cell for every $\langle x, y\rangle \in I_1 \times M$. By Theorem 2.2, we can find such a U, thus proving (i).

Given $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in U$ and $i \in I_1$, let $j_1 = h(u_1+i)+v_1$ and $j_2 = h(u_2+i)+v_2$. Then

$$f_{u_2,v_2}(x) - f_{u_1,v_1}(x) = j_2 - j_1 + f_{u_2,v_2}(x) - f_{u_1,v_1+j_2-j_1}(x).$$

But $f_{u_1,v_1+j_2-j_1}(i)=f_{u_2,v_2}(i)$; hence, by part (i), for $\langle u_1,v_1\rangle,\langle u_2,v_2\rangle\in U$ and $i\in I_1$,

$$\text{if } u_1<u_2 \text{ then } f_{u_2,v_2}(x)-f_{u_1,v_1}(x) \text{ is strictly increasing at } i. \tag{23}$$

We use $f[i,j;a]$ as in previous sections. Let $\langle i^*,j^*\rangle$ be a fixed generic point in $I_1\times M$ and let $f_a(x)=f[i^*,j^*;a](x)$.

For J_1 as in the last proof, we let R_+ be the relation on $J_1\times J_1$ defined by

$$\langle a,b\rangle R_+\langle c,d\rangle \quad\Leftrightarrow\quad f_a-f_b\prec^+_{i^*} f_c-f_d,$$

and let $R_\circ$ be the relation on $J_1\times J_1$ defined by

$$\langle a,b\rangle R_\circ\langle c,d\rangle \quad\Leftrightarrow\quad f_af_b^{-1}\prec^+_{j^*} f_cf_d^{-1}.$$

Let $\mathscr{I}\subseteq J_1$ be a good convex set over the parameters defining $R_\circ$ and R_+ (i^* and j^* are of course among those). By Theorem 5.6 and Theorem 6.3, $R_\circ$ and R_+ are q-relations on $\mathscr{I}$. They induce two equivalence relations, $E_\circ$ and E_+, on $\mathscr{I}\times\mathscr{I}$. We first show that $E_\circ$ and E_+ are different.

Lemma 7.6. *For c,d in $\mathscr{I}$, let $[c,d]$ and c/d denote the E_+ and $E_\circ$ classes of $\langle c,d\rangle$, respectively. If* $\dim(c,d/i^*,j^*)=2$ *then* $[c,d]\cap c/d=\{\langle c,d\rangle\}$.

Proof. Suppose that $\langle a,b\rangle E_+\langle c,d\rangle$, $\langle a,b\rangle E_\circ\langle c,d\rangle$ and $\langle a,b\rangle\neq\langle c,d\rangle$.

Then, by Lemma 3.10(3), $a\neq c$ and $b\neq d$. Without loss of generality, assume that $b<d<c$ (all other cases can be handled similarly).

Take $c'>c$. By the properties of $E_\circ$, we can find $a'>a$ such that

$$f_{a'}f_b^{-1}\prec_{j^*} f_{c'}f_d^{-1}.$$

By subtracting the identity function from both sides we see that

$$(f_{a'}-f_b)f_b^{-1}\prec_{j^*}(f_{c'}-f_d)f_d^{-1}.$$

By the definition of E_+, we have $f_c-f_d\prec_{i^*} f_{a'}-f_b$. Therefore for all $c'>c$,

$$(f_c-f_d)f_b^{-1}\prec_{j^*}(f_{c'}-f_d)f_d^{-1}. \tag{24}$$

We will use Lemma 5.2 to derive a contradiction. Rewriting (24) we see that for every $c'>c$,

$$(f[i^*,j^*;c]-f[i^*,j^*;d])f_b^{-1}\prec_{j^*}(f[i^*,j^*;c']-f[i^*,j^*;d])f_d^{-1}. \tag{25}$$

By Lemma 7.6, since $c>d$, we may assume that $f_{c'}-f_d$ is increasing at i^* for $c'\geqslant c$. From (25), for all $c'>c$,

$$(f[i^*,j^*;c']-f[i^*,j^*;d])^{-1}(f[i^*,j^*;c]-f[i^*,j^*;d])\prec_{i^*} f_d^{-1}f_b.$$

If we consider the function on the left-hand side, we may use a suitable preorder (as was done in previous sections) to replace j^* on the left by k^* generic over all parameters mentioned.

After rearranging terms we find that for $c'\in(c)^+$,

$$(f[i^*,k^*;c]-f[i^*,k^*;d])f_b^{-1}\prec_{j^*}(f[i^*,k^*;c']-f[i^*,k^*;d])f_d^{-1}.$$

Rearranging terms again, we see that for $c'\in(c)^+$,

$$f[i^*,k^*;c]f_b^{-1}-f[i^*,k^*;c']f_d^{-1}\prec_{j^*} f[i^*,k^*;d]f_b^{-1}-f[i^*,k^*;d]f_d^{-1}.$$

We consider the function on the left. Since k^* is generic over all other parameters mentioned there, we can replace it on the left, using a suitable preorder, by k^{**} generic over all previously-mentioned parameters (including k^*). Rearranging terms back we find the following: for $c' \in (c)^+$,

$$(f[i^*, k^{**}; c] - f[i^*, k^*; d])f[i^*, j^*; b]^{-1} <_{j^*} (f[i^*, k^{**}; c'] - f[i^*, k^*; d])f[i^*, j^*, d]^{-1}. \tag{26}$$

Let $\bar{u}_0, \bar{u}_1 \in U$ be such that $f_{\bar{u}_0}(x) = f[i^*, k^{**}; c](x)$ and $f_{\bar{u}_1}(x) = f[i^*, k^*; d](x)$. Since $\dim(i^*k^*k^{**}cd) = 5$, we have $\dim(i^*\bar{u}_0\bar{u}_1) = 5$.

By Lemma 7.5, $f_{\bar{u}_0} - f_{\bar{u}_1}(x)$ is increasing on some neighbourhood of i^*. By genericity, there are an open set V containing $\bar{u}_0$ and an open interval $J_2 \subseteq \mathscr{I}$ containing i^* such that $(f_{\bar{u}} - f_{\bar{u}_1})(x)$ is increasing on J_2 for every $\bar{u} \in V$. We let

$$\mathscr{G} = \{f_{\bar{u}} - f_{\bar{u}_1} \colon \bar{u} \in V\}.$$

It is easy to verify that $\mathscr{G}$ is a p-nice family on J_2. Since $\langle i^*, j^*, k^*, k^{**}\rangle$ is of dimension 4, the triple $\langle i^*, j^*, k^{**} - k^*\rangle$ is of dimension 3.

We can now apply Lemma 5.2 to $\mathscr{G}$, $\mathscr{F}^{-1}$ and the triple $\langle j^*, i^*, k^{**} - k^*\rangle$. Since $f_b^{-1} >_{j^*} f_d^{-1}$, there is a $c' > c$ such that

$$(f[i^*, k^{**}; c] - f[i^*, k^*; d])f[i^*, j^*; b]^{-1} >_{j^*} (f[i^*, k^{**}; c'] - f[i^*, k^*; d])f[i^*, j^*, d]^{-1}.$$

This contradicts (26) and thus completes the proof.

By Lemma 3.12, if E is an equivalence relation induced by a q-relation, then the map $\langle [\langle a,b\rangle], [\langle b,c\rangle]\rangle \to [\langle a,c\rangle]$ is a definable group operation on the E-classes of $\mathscr{I}$. Furthermore, the group is ordered by the ordering

$$[\langle a,e\rangle] < [\langle b,e\rangle] \quad \Leftrightarrow \quad a < b.$$

We take $\langle H, \star, <\rangle$ to denote the group induced on $\mathscr{I}^2/E_\circ$ by $E_\circ$, and let $\langle G, \oplus, <, \rangle$ denote the group induced on $\mathscr{I}^2/E_+$ by E_+. We still use a/b and $[a,b]$ to denote the classes of $\langle a,b\rangle$ with respect to $E_\circ$ and E_+, respectively.

We define $k\colon \mathscr{I}^3 \to \mathscr{I}$ by

$$k(a,p,q) = b \quad \Leftrightarrow \quad \langle a,p\rangle E_\circ \langle b,q\rangle.$$

By Lemma 3.10, k is a well-defined map and if we fix any two of the three variables, the function in the third variable that we obtain is a strictly monotone permutation of $\mathscr{I}$. It follows that k is strictly monotone and continuous in each variable; hence k is a continuous map from $\mathscr{I}^3$ into $\mathscr{I}$. Notice that $k(x,y,z)$ is definable in $\mathscr{M}|\mathscr{I}$.

By the commutativity of H, $a/p \star p/b = p/b \star a/p$. Hence

$$k(a,p,q) = b \quad \Leftrightarrow \quad a/p = b/q \quad \Leftrightarrow \quad a/b = p/q. \tag{27}$$

By Lemma 3.10 and (27), for every $a,p,q \in \mathscr{I}$, $k(a,p,q) = k(a,p',q')$ if and only if $p/q = p'/q'$. Also by (27), $k(k(x,p,q),r,s) = k(x,pr,qs)$ and therefore we have the following lemma.

LEMMA 7.7. *The map $p/q \mapsto h(x,p,q)$ is a well-defined faithful action of the group H on the set $\mathscr{I}$.*

As we show below, this action induces a faithful group presentation $\hat{k}\colon H \to \operatorname{Aut}(G)$.

LEMMA 7.8. *For every* $a_1, a_2, p, q \in \mathcal{I}$,

$$a_1/a_2 = k(a_1,p,q)/k(a_2,p,q).$$

Proof. We have

$$a_1/p = k(a_1,p,q)/q, \quad a_2/p = k(a_2,p,q)/q.$$

By the group properties

$$\langle p, a_2\rangle E_\circ \langle q, k(a_2,p,q)\rangle,$$

and by Lemma 3.11(1), $a_1/a_2 = k(a_1,p,q)/k(a_2,p,q)$.

LEMMA 7.9. *For all* $a,b,c,d \in \mathcal{I}$ *and* $p,q \in \mathcal{I}$,

$$\langle a,b\rangle E_+ \langle c,d\rangle \quad \Rightarrow \quad \langle k(a,p,q), k(b,p,q)\rangle E_+ \langle k(c,p,q), k(d,p,q)\rangle.$$

Proof. To simplify notation we use $k(x)$ instead of $k(x,p,q)$ here. Assume that $\langle a,b\rangle E_+\langle c,d\rangle$ and that it is not true that $\langle k(a),k(b)\rangle E_+\langle k(c),k(d)\rangle$. Then, without loss of generality, we may assume that there exists $c'' > k(c)$ such that

$$f_{c''} - f_{k(d)} <^+_{i^*} f_{k(a)} - f_{k(b)}.$$

We compose both sides of the inequality with f_q^{-1} on the right and obtain

$$f_{c''}f_q^{-1} - f_{k(d)}f_q^{-1} <^+_{j^*} f_{k(a)}f_q^{-1} - f_{k(b)}f_q^{-1}.$$

By property (5) of q-relations, for $k(c) < c' < c''$ there is a $\hat{c}$ such that $f_{c'}f_q^{-1} <_{j^*} f_{\hat{c}}f_p^{-1} <_{j^*} f_{c''}f_q^{-1}$. Since $\langle k(c),q\rangle E_\circ\langle c,p\rangle$, we have $c < \hat{c}$. By the definitions of k and $E_\circ$, for every $d^1 > d$, $a^1 > a$, $b_1 < b$,

$$f_{\hat{c}}f_p^{-1} - f_{d^1}f_p^{-1} <_{j^*} f_{\hat{c}}f_p^{-1} - f_{k(d)}f_q^{-1} <^+_{j^*} f_{k(a)}f_q^{-1} - f_{k(b)}f_q^{-1} <^+_{j^*} f_{a^1}f_p^{-1} - f_{b_1}f_p^{-1}.$$

Hence, after composing both sides of the above inequality with f_p, we find that $\langle \hat{c}, d^1\rangle R_+\langle a^1, b_1\rangle$. We have thus established that

$$\text{for all } d^1 > d, a^1 > a, b_1 < b, \text{ and all } c^1 \in (c)^+, \langle c^1, d^1\rangle R_+\langle a^1, b_1\rangle. \tag{28}$$

Since $\langle a,b\rangle E_+\langle c,d\rangle$, it is sufficient to prove the following in order to get a contradiction:

$$\text{there exists } c^1 > c \text{ such that } \langle c^1, d\rangle R_+\langle a,b\rangle. \tag{29}$$

To prove (29), we first show that

$$\text{for all } d^1 > d, b_1 < b, \text{ and } c^1 \in (c)^+, \langle c^1, d^1\rangle R_+\langle a, b_1\rangle. \tag{30}$$

For if not, there are $d^1 > d$ and $b_1 < b$ such that for $c^1 \in (c)^+$, $\langle a, b_1\rangle R_+\langle c^1, d^1\rangle$, and therefore, by (28) for $c^1 \in (c)^+$, $\langle c^1, d^1\rangle E_+\langle a, b_1\rangle$. This yields a contradiction since, by Lemma 3.10, there is a unique c_0 such that $\langle c_0, d^1\rangle E_+\langle a, b_1\rangle$.

Similarly, we can use (30) to show that

$$\text{for all } b_1 < b \text{ and } c^1 \in (c)^+, \langle c^1, d\rangle R_+\langle a, b_1\rangle. \tag{31}$$

And finally, using (31), we can prove that

$$\text{for } c^1 \in (c)^+, \langle c^1, d\rangle R_+\langle a,b\rangle.$$

Hence (29) follows, and hence the lemma is true.

We showed that for every p, $k(x,p,q)\colon \mathcal{I} \to \mathcal{I}$, as a function of x, is an automorphism of $\langle \mathcal{I}, E_+\rangle$. It follows from the definition of $\oplus$ on G that $[x,y] \mapsto [k(x,p,q), k(y,p,q)]$ is a group-endomorphism of $\langle G, \oplus\rangle$. Hence,

$$p/q \mapsto ([a,b] \mapsto [k(a,p,q), k(b,p,q)])$$

is a group presentation; call it $\hat{k}\colon H \to \operatorname{Aut}(G)$. We now show that the presentation is faithful.

LEMMA 7.10. *If $p \neq p'$ in $\mathcal{I}$ then $\hat{k}(p/q) \neq \hat{k}(p'/q)$.*

Proof. We assume that $E_\circ$ and E_+ are defined over the empty set. Let a_1, a_2 be in $\mathcal{I}$, and $\dim(a_1, a_2/p, p', q) = 2$. We show that $\langle k(a_1,p,q), k(a_2,p,q)\rangle$ is not E_+-equivalent to $\langle k(a_1,p',q), k(a_2,p',q)\rangle$.

Assume that $\langle k(a_1,p,q), k(a_2,p,q)\rangle E_+ \langle k(a_1,p',q), k(a_2,p',q)\rangle$. By the group properties and the definition of k we also have

$$a_1/a_2 = a_1/p \star p/a_2 = k(a_1,p,q)/q \star q/k(a_2,p,q) = k(a_1,p,q)/k(a_2,p,q).$$

Similarly, $a_1/a_2 = k(a_1,p',q)/k(a_2,p',q)$; hence

$$\langle k(a_1,p,q), k(a_2,p,q)\rangle E_\circ \langle k(a_1,p',q), k(a_2,p',q)\rangle.$$

Since a_i and $k(a_i,p',q)$ are interdefinable over p', q, for $i = 1,2$, we have

$$\dim(k(a_1,p',q), k(a_2,p',q)/p',q) = 2.$$

We can now apply Lemma 7.6 to $\langle k(a_1,p,q), k(a_2,p,q)\rangle$ and $\langle k(a_1,p',q), k(a_2,p',q)\rangle$, and conclude that $k(a_1,p,q) = k(a_1,p',q)$ and $k(a_2,p,q) = k(a_2,p',q)$. It follows that $p/q = p'/q$ and therefore $p = p'$, in contradiction to our assumption.

We now return to the proof of Proposition 7.3. By Lemma 3.10 and property (q2), the map $x \mapsto \langle x, e\rangle$ induces an order-preserving bijection (and hence a homeomorphism) between $\mathcal{I}$ and the linearly ordered set $\mathcal{I}^2/E_\circ$ or $\mathcal{I}^2/E_+$. Therefore, we can definably equip $\mathcal{I}$ with the ordered group structures of H and G and assume that H and G are convex $\wedge$-definable groups. Since k was a continuous map, the map $\langle h, g\rangle \mapsto \hat{k}(h)(g)$ is a continuous map from $H \times G$ into G.

By Lemma 7.10 and the preceding remarks, $\hat{k}\colon H \to \operatorname{Aut}(G)$ is a definable continuous faithful presentation of H on G. This completes the proof of Proposition 7.3.

In order to define a real closed field we are going to need the following lemma.

LEMMA 7.11. *Let $R = \langle \mathcal{I}, <, +, \bullet, 0\rangle$ be a convex $\wedge$-definable commutative ordered ring in $\mathcal{M}$. Then K, the fraction field of R, is a real closed field definable in $\mathcal{M}$, and R is definably embedded onto a convex subring of K.*

Proof. This argument stems from an earlier observation of A. Wilkie.

We let $\mathcal{I}^2_* = \{\langle a,b\rangle \in \mathcal{I}\colon b \neq 0\}$. For $\langle a,b\rangle, \langle c,d\rangle \in \mathcal{I}^2_*$, let $\langle a,b\rangle \sim \langle c,d\rangle$ if and only if $a \bullet d = b \bullet c$. We take $\langle \hat{K}, +, \cdot\rangle$ to be the standard field of fractions of R on $\mathcal{I}^2_*/\sim$ and let a/b denote the $\sim$-class of $\langle a,b\rangle$. Now K is ordered as a field by making a_1/a_2 positive in K if and only if $a_1 > 0$ and $a_2 > 0$ in R. The graphs of $+$ and $\cdot$ in $\hat{K}$ are the images under the quotient map of definable sets in $\mathcal{M}|\mathcal{I}$. We want to show that $\hat{K}$ is definable (not only $\wedge$-definable) in $\mathcal{M}$.

By Lemma 2.10, $\bullet$ is continuous on $\mathscr{I}$. Since $0 \bullet 0 = 0$, we can show that for every $\langle a, b \rangle \in \mathscr{I}_*^2$ and every neighbourhood U of $\langle 0, 0 \rangle$, there is a $\langle c, d \rangle \in U$ such that $a/b = c/d$. It follows that for every open interval I around 0, every $\sim$-class has a representative in I^2. Moreover, there are many definable sets of representatives for $\mathscr{I}_*^2/\sim$. For instance, if $\beta \in \mathscr{I}$ is positive in R then the set

$$\{\langle a, -|a| + \beta \rangle : -\beta < a < \beta\}$$

is a (definable) set of representatives for $\mathscr{I}_*^2/\sim$. The map $x \mapsto x/-|x| + \beta$ is an order-preserving bijection between $K = (-\beta, \beta)$ (with the ordering induced from $\mathscr{M}$) and the ordered field $\hat{K}$. Therefore, we can definably equip K with an ordered field structure isomorphic to $\hat{K}$, whose ordering is compatible with that of $\mathscr{M}$. By [**18**], K is a real closed field.

As usual, if we fix any $b \in \mathscr{I}$, the map $\tau(x) = x \bullet b/b$ is an order-preserving embedding of R in its field of fractions $\hat{K}$. Hence we get a definable embedding of R into K. Since τ is continuous, the image of R in $\hat{K}$ is a convex subring.

It remains to prove the following proposition.

PROPOSITION 7.12. *Let $\langle H, <, \star, 1 \rangle$ and $\langle G, <, +, 0 \rangle$ be two convex infinite $\wedge$-definable ordered groups in $\mathscr{M}$, and let σ: $H \to \mathrm{Aut}(G)$ be a definable continuous faithful presentation. Then there is a definable real closed field K whose underlying set is an interval in M and its ordering is compatible with $<$. Moreover, there is a convex subgroup of G which is definably isomorphic to a convex subgroup of $\langle K, + \rangle$.*

Proof. By Lemma 2.9, H and G are both commutative and divisible.

Let Λ be the collection of all definable endomorphisms of G. It is easily verified that Λ, with addition and composition of functions, is a ring, with $\mathrm{id}(x)$ as the identity element and the zero map as the zero element. We can make it into an ordered ring by letting $\lambda > 0$ if and only if λ is an increasing function. Since no proper subgroup of G is definable in $\mathscr{M}|G$, if two definable endomorphisms agree on any non-zero point of G then they agree everywhere on G. Moreover, for $\sigma_1, \sigma_2 \in \Lambda$, $\sigma_1 < \sigma_2$ if and only if $\sigma_1(p) < \sigma_2(p)$ for some $p > 0$ in G. For $h \in H$, we let σ_h denote the corresponding automorphism of G.

It follows from our hypothesis that $\hat{H} = \{\sigma_h : h \in H\}$ is isomorphic to H and furthermore it is a subgroup of the multiplicative group of units in Λ.

Step 1: $\hat{H}$ is a convex subset of the ordered set $\langle \Lambda, < \rangle$. For $g, h \in H$, assume that $\sigma_h < \sigma < \sigma_g$, where $\sigma \in \Lambda$. Then for $p > 0$ in G, $\sigma_h(p) < \sigma(p) < \sigma_g(p)$. By the continuity of the presentation, we can find $k \in H$ between h and g such that $\sigma_k(p) = \sigma(p)$, but then, by the above comments, $\sigma(x) = \sigma_k(x)$ for all $x \in G$; hence $\sigma \in \hat{H}$.

Step 2: for every $g, h, k \in H$, $\sigma_g + \sigma_h - \sigma_k \in \hat{H}$. Assume first that $\sigma_k \leqslant \sigma_g$ and $\sigma_k \leqslant \sigma_h$. Without loss of generality, let $\sigma_g \leqslant \sigma_h$. Then on one hand, since $\sigma_{kg^{-1}} < \mathrm{id}$,

$$\sigma_g + \sigma_h - \sigma_k \geqslant \sigma_g + \sigma_h - \sigma_g = \sigma_h,$$

while on the other hand,

$$\sigma_{g \star k^{-1} \star h} = \sigma_g + \sigma_{g \star k^{-1} \star h} - \sigma_g \geqslant \sigma_g + \sigma_{k \star g^{-1}}(\sigma_{g \star k^{-1} \star h} - \sigma_g) = \sigma_g + \sigma_h - \sigma_k.$$

Since $\hat{H}$ is convex, $\sigma_g + \sigma_h - \sigma_k$ is in $\hat{H}$.

If $\sigma_g \leqslant \sigma_k \leqslant \sigma_h$ then

$$\sigma_g \leqslant \sigma_g + \sigma_h - \sigma_k \leqslant \sigma_h;$$

hence $\sigma_g + \sigma_h - \sigma_k$ is in $\hat{H}$. The case where $\sigma_k \geqslant \sigma_g$ and $\sigma_k \geqslant \sigma_h$ is handled similarly.

Step 3: the set $\hat{R} = \{\sigma_h - \sigma_g : h, g \in H\}$ is a convex commutative subring of Λ. We first show closure under addition. Given $g, h, k, l \in H$, by Step 2 there exists $l_1 \in H$ such that $\sigma_k - \sigma_l = \sigma_h - \sigma_{l_1}$; hence $\sigma_g - \sigma_h + \sigma_k - \sigma_l = \sigma_g - \sigma_{l_1}$.

The closure under multiplication follows immediately from closure under addition. The convexity and commutativity follow from that of $\hat{H}$.

Notice that if $\mathscr{I}$ is the underlying convex set of H then we can equip $\mathscr{I}$ with a definable ring structure making it into a convex $\wedge$-definable ordered ring isomorphic to $\hat{R}$. For $p > 0$ in G, the additive group of $\hat{R}$ is definably isomorphic to the convex subgroup of G whose underlying set is $\{\sigma_h(p) - \sigma_g(p) : g, h \in H\}$. By Lemma 7.11, we can complete the proof of Proposition 7.12 and with it the proof of Theorem 1.2.

8. *Some examples on the global picture*

The Trichotomy Theorem gives a characterization of the local structure in a neighbourhood of each non-trivial point in $\mathscr{M}$ as either an ordered vector space or an o-minimal expansion of a real closed field. Since $\mathscr{M}$ is an ordered structure, there are inherent difficulties in trying to give a global characterization of the structure of definable subsets of M^n. Below are some basic examples to demonstrate the difference between the local and global analysis.

EXAMPLE 8.1. Let I be a linearly ordered set and assume that for each $i \in I$, $\mathscr{M}_i$ is an o-minimal structure in the language L_i such that L_i and L_j are disjoint for $i \neq j$. If we let $L = \bigcup_{i \in I} L_i$ then there is an o-minimal L-structure which is made up by 'patching' the $\mathscr{M}_i$ in the right order, and inserting new points at the ends of each $\mathscr{M}_i$. Clearly, there is no interaction between the different pieces of the structure.

We denote by $\mathscr{T}1$, $\mathscr{T}2$, $\mathscr{T}3$ the sets of points satisfying (T1), (T2) and (T3) from the Trichotomy Theorem, respectively. Every non-orthogonality class is contained in $\mathscr{T}1$, $\mathscr{T}2$ or $\mathscr{T}3$ and by Lemma 2.7, if a is in either $\mathscr{T}2$ or $\mathscr{T}3$ its non-orthogonality class is an open set.

EXAMPLE 8.2. Let $P(x, y)$ be the restriction to $[-1, 1]^2$ of the standard real multiplication function. Consider the structure $\mathscr{M} = \langle \mathbb{R}, <, +, P(x, y) \rangle$. Then $\mathscr{M}$ consists of a single non-orthogonality class, contained in $\mathscr{T}3$. Hence every point lies in a definable real closed field. However, as implied by [**13**], no real closed field whose universe is $\mathbb{R}$ is definable in $\mathscr{M}$.

EXAMPLE 8.3. Let $\langle K, <, +, \cdot \rangle$ be a non-standard elementary extension of the structure $\langle \mathbb{R}, <, +, \cdot \rangle$. For $\alpha \in K$ infinitesimally close to 0, define $P(x, y, z)$ to be the partial function $x + y - z$ whenever the distance between any two of x, y, z is no more than α. Consider the structure $\mathscr{M} = \langle K, <, P \rangle$. For every $a \in K$, P induces the structure of a group-interval (after fixing the parameter a) on the interval $[a - \frac{1}{2}\alpha, a + \frac{1}{2}\alpha]$. This group-interval can be extended to the interval

$[a - n\alpha, a + n\alpha]$, for every $n \in \mathbb{N}$. In $\mathscr{M}$ every point is in $\mathscr{T}2$ and a and b are non-orthogonal to each other if and only if $|a - b| < n\alpha$ for some $n \in \mathbb{N}$. By Theorem 1.1 every point lies in a definable group-interval (or a convex $\wedge$-definable group) but there are no definable ordered groups in $\mathscr{M}$.

9. *Some corollaries*

A large body of work has been developed for o-minimal expansions of ordered groups and real closed fields, which can now be applied, at least locally, for o-minimal structures of types (Z2) and (Z3). We mention here a few such applications.

O-minimal expansions of ordered groups (or ordered group-intervals) are known to have definable Skolem functions (see [**2**]). The same is true for o-minimal expansions of group-intervals. We can conclude that the following corollary holds.

COROLLARY 9.1. *If a is a non-trivial point in an o-minimal $\mathscr{M}$ then there is a closed interval I, with a in the interior of I, such that $\mathscr{M}|I$ has definable Skolem functions.*

QUESTION. Does (some version of) elimination of imaginaries hold globally in an arbitrary o-minimal structure?

Even though the main theorems of this paper are local in nature there are some cases in which global results can be proved. Consider an o-minimal expansion $\mathscr{M}$ of an ordered group. If the structure is of type (Z2) (or 'linear' as it was called in [**9**]), it eliminates quantifiers and can be embedded onto an elementary substructure of a reduct of a vector space over a division ring (see [**9**]). When $\mathscr{M}$ is of type (Z3) we can still prove global results in some cases.

For $\mathscr{M}$ an o-minimal expansion of an ordered group, we say that $f\colon M \to M$ is *not eventually linear* if there is no $c \in M$ such that f is linear on $[c, +\infty)$.

COROLLARY 9.2. *Let $\mathscr{M} = \langle M, <, +, \dots \rangle$ be an o-minimal expansion of an ordered group. Then the following are equivalent.*

(1) *There is in $\mathscr{M}$ a definable function which is not eventually linear.*

(2) *There is in $\mathscr{M}$ a definable bijection between bounded and unbounded intervals (sometimes referred to as '$\mathscr{M}$ has poles').*

(3) *There is an $\mathscr{M}$-definable real closed field whose underlying set is M and whose ordering is compatible with $<$.*

Proof. By Proposition 4.4 in [**11**], (1) implies (2). With Theorem 1.2, the rest of the argument is identical to [**14**].

An important property of o-minimal expansions of real closed fields is that definable sets are locally n-differentiable manifolds with respect to the field structure and topology. In [**12**], this was used to show that if H is an n-dimensional definable group in an o-minimal expansion of a real closed field R then $H/Z(H)$ can be definably embedded in $\mathrm{GL}_n(R)$. Moreover, it was shown there that the group of definable automorphisms of H can be embedded (not definably though) in $\mathrm{GL}_n(R)$. We can now conclude that the following corollary is true.

COROLLARY 9.3. *Let $\mathscr{M}$ be an o-minimal expansion of an ordered group of type* (Z3). *Let R be the real closed field obtained by Theorem* 1.2. *If H is an n-dimensional group definable in $\mathscr{M}$ then the group of definable automorphisms of H can be embedded in* $\mathrm{GL}_n(R)$.

Proof. We may assume that H has an element of the form $(a, a, \ldots, a)$ as its identity. Since all points in M are non-orthogonal to each other, we may assume that a is the zero element of R. As the proof in [**12**] only used the structure of the field in a neighbourhood of the group identity, we can repeat the proof there to get the desired result.

The last corollary provides us with an interesting family of examples of o-minimal structures which cannot be properly expanded while still preserving o-minimality. Let $\langle D, <, +, \cdot \rangle$ be an ordered division ring which is not a field and let V be an ordered vector space over D. By standard quantifier elimination methods, this structure can be shown to be o-minimal.

Assume now that $\hat{V}$ is an o-minimal expansion of V of type (Z3). By Corollary 9.3, the ring of $\hat{V}$-definable endomorphisms of $\langle V, + \rangle$, call it $\hat{D}$, can be embedded in a real closed field R. But D is a subring of $\hat{D}$, which contradicts the commutativity of R. It follows that every expansion of $\hat{V}$ is of type (Z2), that is, there is an ordered division ring $\hat{D}$ extending D such that $\hat{V}$ is a reduct of an ordered vector space over $\hat{D}$.

We may consider D above as an ordered vector space over itself by restricting ourselves to the language containing $<$, $+$ and a unary function $\lambda_a(x) = ax$ for every $a \in D$. It is not too difficult to show that if D_1 is an o-minimal expansion of D then every linear function definable in D_1 is already definable in D; hence for D_1 to be a proper expansion, it must be of type (Z3). However, by the above this is impossible. We have thus proved our final corollary.

COROLLARY 9.4. *Let D be an ordered division ring which is not a field. Then D, considered as an ordered vector space over itself, has no proper o-minimal expansions.*

10. *Appendix: geometric calculus*

Let $\mathscr{M}$ be an o-minimal structure with a nice family of functions. By the main theorem of this paper, a real closed field is definable in $\mathscr{M}$. It is known (see [**2**] for example), that definable functions in such structures possess good differentiability properties and many theorems from basic calculus, for instance, the uniqueness theorem for solutions of differential equations (see [**12**]), hold for them.

In this section we want to show how one can recover a differentiable structure of $\mathscr{M}$ from its geometry only, without using a field structure at all. We call this approach 'Geometric Calculus'. Although we do not use it directly in proving the main results of this paper, almost all ideas of the proofs were inspired by this approach.

We assume from now on that $\mathscr{M}$ is an o-minimal structure with a p-nice family of functions $\mathscr{F} = \{f(x, \bar{u}) \colon \bar{u} \in U\}$ on an open interval I. We are going to use this family to define the notion of tangency in the same way as the family of linear functions $\{y = ax + b \colon a, b \in \mathbb{R}\}$ is used to define the standard derivative over the field of reals.

DEFINITION 10.1. Let $h(x)$ be a definable function with $I \subseteq \mathrm{dom}(h)$ and $p \in I$. We say that h is *$\mathscr{F}$-bounded at p* if there are $f^1(x), f^2(x) \in \mathscr{F}$ such that $f^1(p) = f^2(p) = h(p)$ and $f^1(x) \preccurlyeq_p h(x) \preccurlyeq_p f^2(x)$.

We fix a point $p \in I$ and a function $h(x)$ which is $\mathscr{F}$-bounded at p. For simplicity, we assume that h, I and U are 0-definable. Let $q = h(p)$. As before, $U_{pq} = \{\bar{u} \in U \colon f(p, \bar{u}) = q\}$. Let $J_{pq} = \pi_1(U_{pq})$ and, for $a \in J_{pq}$, we let $f_a(x)$ denote the unique function $f(x, \bar{v}) \in \mathscr{F}$ such that $\bar{v} \in U_{pq}$ and $a = \pi_1(\bar{v})$.

Let

$$S^+_{\succcurlyeq}(p) = \{a \in J_{pq} \colon f_a(x) \succcurlyeq^+_p h(x)\}, \quad S^+_{\preccurlyeq}(p) = \{a \in J_{pq} \colon f_a(x) \preccurlyeq^+_p h(x)\},$$

$$S^-_{\succcurlyeq}(p) = \{a \in J_{pq} \colon f_a(x) \succcurlyeq^-_p h(x)\}, \quad S^-_{\preccurlyeq}(p) = \{a \in J_{pq} \colon f_a(x) \preccurlyeq^-_p h(x)\}.$$

By o-minimality, there are a^+ and a^- in M such that

$$a^+ = \inf(S^+_{\succcurlyeq}) = \sup(S^+_{\preccurlyeq})$$

and

$$a^- = \inf(S^-_{\preccurlyeq}) = \sup(S^-_{\succcurlyeq}).$$

We denote this a^+ and a^- by $d^+_{\mathscr{F}}h(p)$ and $d^-_{\mathscr{F}}h(p)$, and call them the *right $\mathscr{F}$-derivative of h* at p and the *left $\mathscr{F}$-derivative of h* at p, respectively.

DEFINITION 10.2. We say that a function $h(x)$ is *$\mathscr{F}$-differentiable at p* and a is the *$\mathscr{F}$-derivative of h at p* if $d^+_{\mathscr{F}}h(p) = d^-_{\mathscr{F}}h(p) = a$. The $\mathscr{F}$-derivative of h at p is denoted by $d_{\mathscr{F}}h(p)$, and the function $f_a(x)$ is denoted by $\tau_{\mathscr{F}}h(p)$.

LEMMA 10.3. *If p is generic then there is at most one $a \in J_{pq}$ such that f_a touches h at p.*

Proof. Suppose that the lemma is not true and there are $a_1 < a_2 \in J_{pq}$ such that both f_{a_1} and f_{a_2} touch h at p. By properties of nice families then for every $a \in (a_1, a_2)$, f_a touches $h(x)$ at p. Going to an elementary extension, if needed, we can assume that $\mathscr{M}$ is ω-saturated.

Let $a \in (a_1, a_2)$ be generic over p. We will assume that f_a touches h at p from above, and therefore there are $p_1 < p < p_2$ such that $f_a(x) \geqslant h(x)$ for all $x \in (p_1, p_2)$. By o-minimality, since p is generic, h is continuous on an open interval containing p, and we can choose p_1, p_2 so that h is continuous on $[p_1, p_2]$. Also, since a is generic over p, f_a and h cannot be equal on any open interval containing p and therefore we can assume that $f_a(x) > h(x)$ for all $x \neq p$ in (p_1, p_2).

Changing p_1 and p_2, if needed, we can assume that p_1 and p_2 are generic over $\{p, a\}$, and thus $\dim(p, a, p_1, p_2) = 4$. Let $q_1 = f_a(p_1)$. Since f_a is the unique curve in $\mathscr{F}$ passing through $\langle p_1, q_1 \rangle$ and $\langle p, h(p) \rangle$, $a \in \mathrm{dcl}(p_1, q_1, p)$; therefore $\dim(p_1, q_1, p, p_2) = 4$ and p is generic over p_1, q_1, p_2.

Thus there is an open interval containing p such that for all p' in this interval the following holds:

$$\exists \bar{w} \in U \; f(p_1, \bar{w}) = q_1 \;\&\; f(p', \bar{w}) = h(p') \;\&\; \forall x \in (p_1, p') f(x, \bar{w}) > h(x).$$

We choose p' such that $p < p' < p_2$, p' is generic over p, p_1, q_1, a, and the above statement holds for p'. Let $\bar{w} \in U$ be such that $f(p_1, \bar{w}) = q_1$, $f(p', \bar{w}) = h(p')$, and $(\forall x \in (p_1, p'))(f(x, \bar{w}) > h(x))$. Since $f_a(p) = h(p)$ and $f(p, \bar{w}) > h(p)$, we have $f_a(p) < f(p,\bar{w})$; and for the same reason $f_a(p') > f(p',\bar{w})$. As both $f_a(x)$ and $f(x,\bar{w})$ are continuous, there must be a $p_0 \in (p,p')$ such that $f_a(p_0) = f(p_0,\bar{w})$. But $f_a(p_1) = f(p_1,\bar{w})$, and thus by the properties of a nice family, $f_a(x) = f(x,\bar{w})$ for all $x \in I$. This is a contradiction, since $f_a(p) < f(p,\bar{w})$.

LEMMA 10.4. *If p is generic, then h is $\mathscr{F}$-differentiable at p and, moreover, for $a = d_{\mathscr{F}}h(p)$, the function $f_a(x)$ touches $h(x)$ at p.*

Proof. Suppose that $h(x)$ is not differentiable at p, that is, $a^+ = d^+_{\mathscr{F}}h(p) \neq d^-_{\mathscr{F}}h(p) = a^-$. Assume that, for instance, $a^+ > a^-$. Then for any a in the interval (a^-,a^+) we have $f_a(x) \preccurlyeq^-_p h(x)$ since $a > a^-$, and $f_a(x) \preccurlyeq^+_p h(x)$ since $a < a^+$. Therefore, for every $a \in (a^-,a^+)$, the function f_a touches $h(x)$ at p, contradicting Lemma 10.3.

Thus $h(x)$ is differentiable at p. Write $a = d_{\mathscr{F}}h(p)$. We want to show that f_a touches h at p. Suppose that it does not, and, for instance, $h(x) \prec_p f_a(x)$. Let $p_1 < p < p_2$ be such that $h(x) > f_a(x)$ for all $x \in (p_1,p)$ and $f_a(x) > h(x)$ for all $x \in (p,p_2)$.

Since p is generic, we can choose p_1 and p_2 so that for any $p' \in (p_1,p_2)$ the function $h(x)$ is $\mathscr{F}$-differentiable at p' and if $f = \tau_{\mathscr{F}}(h(p'))$ then $f(x) < h(x)$ for $x \in (p_1,p')$ and $f(x) > h(x)$ for $x \in (p',p_2)$.

Considering an elementary extension of $\mathscr{M}$, if needed, we can assume that $\mathscr{M}$ is ω-saturated, and the interval (p_1, p_2) does not contain any elements of $\mathrm{acl}(p,a)$ except p itself.

Let $a' < a$ be an element of J_{pq} (recall that $q = h(p)$). Then $f_{a'} \prec^+_p h(x)$ and we can choose a' and $y \in (p, p_2)$ so that $f_{a'}(y) < h(y)$. Since $f_a(y) > h(y)$, by continuity of $f(x,\bar{u})$, we can find $b < a \in J_{pq}$ such that $f_b(y) = h(y)$. As $b < a$, $f_b(x) \prec^+_p h(x)$ and hence we can find $p' \in (p,y]$ such that $f_b(p') = h(p')$ and $f_b(x) \prec^-_{p'} h(x)$. Namely let p' be the first point to the right side of p where $f_b(p') = h(p')$.

Since $p' \in (p, p_2)$, $h(x)$ is $\mathscr{F}$-differentiable at p', and let $f = \tau_{\mathscr{F}} h(p')$. As $f_b(x) \prec^-_{p'} h(x)$, we have $f_b(x) \preccurlyeq^-_{p'} f(x)$. Since $p \in (p_1, p')$, $f(p) < h(p) = f_b(p)$, and therefore, by continuity of functions f and f_b, there must be a point in (p,p') at which these functions are equal. But $f(p') = f_b(p')$ and thus, by the properties of nice families, $f(x)$ and $f_b(x)$ must be equal, contradicting the fact that $f(p) < f_b(p)$.

Summarizing all of the above for an arbitrary function $g(x)$ we obtain the following theorem.

THEOREM 10.5. *Let $\mathscr{F} = \{f(x,\bar{u})\colon \bar{u} \in U\}$ be a nice family on an open interval I, $g(x)$ a definable function with $I \subseteq \mathrm{dom}(g)$, and p a point in I generic over the parameters needed to define f and g. If $g(x)$ is $\mathscr{F}$-bounded at p then it is $\mathscr{F}$-differentiable at p, and moreover $\tau_{\mathscr{F}} g(p)$ is the unique function in $\mathscr{F}$ which touches $g(x)$ at p.*

Assume that $\mathscr{F} = \{f(x,\bar{u})\colon \bar{u} \in U\}$ is a nice family on I. Notice that by clause (ii) of Definition 4.4, for every $\langle a,b_1\rangle, \langle a,b_2\rangle \in U$, $f_{a,b_1}(x) = f_{a,b_2}(x)$ for some $x \in I$ if and only if $b_1 = b_2$.

The theorem below is a generalization of the calculus theorem which says that a function with constant derivative must be linear.

THEOREM 10.6. *For $\mathcal{F}$ as above, let $h(x)$ be a definable continuous, $\mathcal{F}$-differentiable function on I. If $d_{\mathcal{F}} h(x)$ is constant on I then $h \in \mathcal{F}$.*

Proof. We assume that all functions are definable over the empty set.

Let p be generic in I. By Theorem 10.5, $\tau_{\mathcal{F}} h(p)$ touches $h(x)$ at p, say, from above. By genericity, there are $p_1 < p < p_2$ such that for every $p' \in (p_1, p_2)$, if $f = \tau_{\mathcal{F}} h(p')$ then f touches h at p' from above and $f(x) \geqslant h(x)$ for all $x \in (p_1, p')$ and all $x \in (p', p_2)$. Take $p' \in (p_1, p)$. Then

$$\tau_{\mathcal{F}} h(p')(p') = h(p') \leqslant \tau_{\mathcal{F}} h(p)(p')$$

and

$$\tau_{\mathcal{F}} h(p')(p) \geqslant h(p) = \tau_{\mathcal{F}} h(p)(p).$$

By continuity, there is an $x_0 \in I$ such that $\tau_{\mathcal{F}} h(p')(x_0) = \tau_{\mathcal{F}} h(p)(x_0)$. By our assumptions, $d_{\mathcal{F}} h(p') = d_{\mathcal{F}} h(p) = a$ for some a. Hence there are v' and v such that $\tau_{\mathcal{F}} h(p')(x) = f_{a,v'}(x)$ and $\tau_{\mathcal{F}} h(p)(x) = f_{a,v}(x)$. It follows that $f_{a,v'}(x_0) = f_{a,v}(x_0)$. By the comment preceding the statement of the theorem, $v' = v$; hence $\tau_{\mathcal{F}} h(p') = \tau_{\mathcal{F}} h(p)$. We can similarly show that $\tau_{\mathcal{F}} h(p') = \tau_{\mathcal{F}} h(p)$ for all $p' \in (p, p_2)$. But then $h(x) = \tau_{\mathcal{F}} h(p)(x)$ for all $x \in (p, p_2)$.

Since p was an arbitrary generic point in I, we can partition I into finitely many intervals on each of which $h(x) = f_{a,v}(x)$ for some v. By the continuity of h, the functions from $\mathcal{F}$ must agree on the endpoints of these intervals; hence, as before, they all are equal to each other. Namely, $h(x) = f_{a,v}(x)$ for all $x \in I$.

Theorem 10.6 seems surprising since it depends on a particular parametrization of $\mathcal{F}$. However, note that if we change the parametrization then we might need first to restrict the domain of $\mathcal{F}$ in order to ensure that it is a nice family. After this has been done, the collection of $\mathcal{F}$-bounded functions could change and the theorem would not apply to the same functions.

Similarly to Theorem 10.6, one can formulate and prove the $\mathcal{F}$-version of the Mean Value Theorem and other calculus results. The notion of concavity with respect to $\mathcal{F}$ can also be defined by saying that a function $h(x)$ is *$\mathcal{F}$-concave up on I* if $d_{\mathcal{F}} h(p)$ is increasing in p. Some basic properties which are usually related to the second derivative can be proved using the above methods.

References

1. C. C. CHANG and J. H. KEISLER, *Model theory*, 3rd edn (North Holland, Amsterdam, 1990).
2. L. VAN DEN DRIES, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series 248 (Cambridge University Press, 1998).
3. E. HRUSHOVSKI, 'Strongly minimal expansions of algebraically closed fields', *Israel J. Math.* 79 (1992) 129–151.
4. E. HRUSHOVSKI, 'A new strongly minimal set', *Ann. Pure Appl. Logic* 62 (1993) 147–166.
5. E. HRUSHOVSKI and A. PILLAY, 'Weakly normal groups', *Logic colloquium '85* (eds The Paris Logic Group, North Holland, Amsterdam, 1987), pp. 233–244.
6. E. HRUSHOVSKI and A. PILLAY, 'Groups definable in local fields and pseudo-finite fields', *Israel J. Math.* 85 (1994) 203–262.
7. E. HRUSHOVSKI and B. ZIL'BER, 'Zariski geometries', *J. Amer. Math. Soc.* 9 (1996) 1–56.
8. J. KNIGHT, A. PILLAY and C. STEINHORN, 'Definable sets in ordered structures II', *Trans. Amer. Math. Soc.* 295 (1986) 593–605.
9. J. LOVEYS and Y. PETERZIL, 'Linear o-minimal structures', *Israel J. Math.* 81 (1993) 1–30.
10. A. MEKLER, M. RUBIN and C. STEINHORN, 'Dedekind completeness and the algebraic complexity of o-minimal structures', *Canadian J. Math.* 44 (1992) 843–855.

11. C. MILLER, 'A growth dichotomy for o-minimal expansions of ordered field', *Logic, from foundations to applications* (*Staffordshire 1993*) (eds W. Hodges, J. Hyland, C. Steinhorn and J. Truss, Oxford University Press, New York, 1996), pp. 385–399.

12. M. OTERO, Y. PETERZIL and A. PILLAY, 'Groups and rings definable in o-minimal expansions of real closed fields', *Bull. London Math. Soc.* 28 (1996) 7–14.

13. A. PILLAY, P. SCOWCROFT and C. STEINHORN, 'Between groups and rings', *Rocky Mountain J. Math.* 295 (1989) 871–885.

14. Y. PETERZIL, 'A structure theorem for semibounded sets in the reals', *J. Symbolic Logic* 57 (1992) 779–794.

15. Y. PETERZIL, 'Zil'ber's conjecture for some o-minimal structures over the reals', *Ann. Pure Appl. Logic* 61 (1993) 223–239.

16. Y. PETERZIL, 'Constructing a group-interval in o-minimal structures', *J. Pure Appl. Algebra* 94 (1994) 85–100.

17. Y. PETERZIL and S. STARCHENKO, 'Geometry, calculus and Zil'ber's conjecture', *Bull. Symbolic Logic* 2 (1996) 72–83.

18. A. PILLAY, 'On groups and fields definable in o-minimal structures', *J. Pure Appl. Algebra* 53 (1988) 239–255.

19. A. PILLAY and C. STEINHORN, 'Definable sets in ordered structures I', *Trans. Amer. Math Soc.* 295 (1986) 565–592.

20. A. PILLAY and C. STEINHORN, 'Discrete o-minimal structures', *Ann. Pure Appl. Logic* 34 (1987) 275–290.

21. A. PILLAY and C. STEINHORN, 'Definable sets in ordered structures III', *Trans. Amer. Math. Soc.* 309 (1988) 469–476.

22. A. J. WILKIE, 'Model completeness results for expansions of the ordered real field numbers by restricted Pfaffian functions and the exponential function', *J. Amer. Math. Soc.* 9 (1996) 1051–1094.

23. B. ZIL'BER, 'Structural properties of models of $\aleph_1$-categorical theories', *Logic methodology and philosophy of science VII* (eds R. Barcon Marcus *et al.*, North Holland, Amsterdam, 1986), pp. 115–128.

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