Peterzil–Starchenko o-min^l trich^y Final chapter

<u>Recap</u>

M o-min^l $\aleph_1$-sat^d

$a \in M$ non-trivial

Then exists convex Λ-def^ble ordered gr^p $a \in (G_0, <, +, 0) \subseteq M$

~~convex~~

i.e. $(G_0, <) \subseteq (M, <)$ convex

and $\Gamma_+ = G_0^3 \cap X$ for some def^ble $X \subseteq M^3$.

~~Translating~~

Let $p \in G_0$, $p > 0$ Set $I := [-p, p]$. "group interval"

$M|_I$ = structure on I with $X \subseteq I^n$ 0-def^ble in $M|_I$ iff $X \subseteq M^n$ 0-def^ble in M

<u>Fact</u> stably embedded, i.e. " " def^ble in $M|_I$ " " def^ble " "

<u>Def^n</u> ~~[Loveys–Peterzil]~~

$J \subseteq I$ subint^l

def^ble $f : J \to I$ is <u>linear</u> if $f(a+x) - f(a) = f(b+x) - f(b)$

$\forall a, b, a+x, b+x \in J$

and a <u>partial endomorphism</u> (pe) if $0 \in J$ and $f(0) = 0$.

* $M|_I$ is <u>nonlinear</u> if ..

either (NL1) exists pe. which isn't 0-def^ble

or (NL2) exists $f : J \to I$ s.t. $f|_{J'}$ is not linear for any $J' \subseteq J$ inf^te subint^l

<u>Th^m</u> [Loveys–Peterzil '93]:

$M|_I$ is linear iff it is a reduct of ~~[...]~~ an interval $[-b, b]$ in an ordered VS

(i.e. $\sigma : V|_{[-b,b]} \to I$ order-IM s.t. $X \subseteq I^n$ def^ble $\Rightarrow \sigma^{-1}(X) \subseteq V^n$ def^ble)

<u>Th^m</u>: M $\aleph_1$-sat^d o-min^l, $a \in M$. Then exactly one holds:

(T1) a is trivial

(T2) the structure induced by M on some convex nb^d of a is an ordered vectorspace over a div^n ring

(T3) the structure induced by M on some open int^l around a is an o-min^l exp^n of an RCF.

So remains to show

<u>Prop^n</u> ~~[...]~~

Suppose $M|_I$ is nonlinear.

Then an RCF is def^ble on an open subinterval. (Hence, translating, on an open interval around a)

<u>Pf</u>: <u>Step I</u>: Find convex Λ-def^ble ordered groups $(H, <, *, 1)$

$(G, <, \oplus^m, 0^m)$

and def^ble cont^s faithful presentation $G \to \mathrm{Aut}(H)$

(i.e. $G \times H \to H$)

If (NL1) done last week.

So sps (NL2), $h : J \to I$ notlinear on any $J' \subseteq J$

Replacing h with $\pm h^{\pm 1}$ and shrinking J,

wMA $\forall a < b \in J$. $h(a+x) - h(a) \lesssim_0 h(b+x) - h(b)$

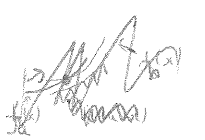
~~[...]~~ and $h(a) < h(b)$

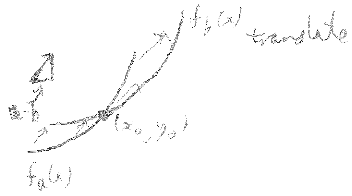
(" h' and h'' positive")

Let $(x_0, y_0) \in J \times I$ gen^c

Let $f_a(x) := h(a+x) + y_0 - h(a+x_0)$ for $a \in J$

so $f_a(x_0) = y_0$





Define $(a,b) R_+ (c,d) \iff f_a - f_b \lesssim^+_{x_0} f_c - f_d$

$(a,b) R_0 (c,d) \iff f_a \circ f_b^{-1} \lesssim^+_{y_0} f_c \circ f_d^{-1}$

~~[...]~~

Replacing J with a suitable convex subset, these are q-rel^n

Let $(G;\oplus,<)$ and $(H;*,<)$ be corresponding groups on

J^2/E_+ and J^2/E_0

where recall: $(a,b)E_+(c,d)$ iff $(c_0,d)R_+(a,b)R_+(c_+,d)\ \forall c_0<c<c_+$
sim E_0

and writing $a\ominus b$ for $(a,b)/E_+$; $(a\ominus b)\oplus(b\ominus c) := (a\ominus c)$

a/b for $(a,b)/E_0$; $a/b * b/c := a/c$

Given $a,b,c\in J$ $\exists! d\in J$ $a\ominus b = c\ominus d$

Define action of H on J by $(q/p)\cdot a = b \Leftrightarrow a/p = b/q$

Then $(r/q)\cdot((q/p)\cdot a) = b \Leftrightarrow \frac{(q/p)\cdot a}{q} = b/r \Leftrightarrow a/p = b/r \Leftrightarrow (r/p)\cdot a = b$

and since $*$ is comm^{ve} $a/p = b/q \Rightarrow a/b = a/p * p/b = b/q * p/b = p/q$, and sim conversely,

so $(q/p)\cdot a = b$ iff $b/a = q/p$. So $(\cdot)$ is well-defined & faithful.

It is cont^s.

~~Lemma: $(q/p\cdot)$ preserves E_+ (any $p,q\in J$)~~

~~i.e. $a\ominus b = c\ominus d \Rightarrow ((q/p)\cdot a)\ominus((q/p)\cdot b) = ((q/p)\cdot c)\ominus((q/p)\cdot d)$~~

Lemma: Let $p,q\in J$. Then $\alpha(x) := (q/p)\cdot x$ preserves E_+

i.e. If $a\ominus b = c\ominus d$, then $\alpha(a)\ominus\alpha(b) = \alpha(c)\ominus\alpha(d)$

Define $q/p\cdot(a\ominus b) := (q/p\cdot a)\ominus(q/p\cdot b)$.

So by def^n of $\oplus$, $(q/p\cdot)$ is an end^{ism} of G and $(p/q\cdot)$ is its inverse.

So have representation $H\to Aut(G)$

Lemma: The repr^n is faithful, $(q/p\cdot) = (s/r\cdot)$ iff $(q/p) = (s/r)$

Finally, fix arbitrary $a_G, a_H\in J$, and transport $(G;\oplus)$ and $(H;*)$ to J via $x\ominus a_G \mapsto x$

$x/a_H \mapsto x$

So G,H are convex A-def^{ble} ordered groups in M_I

Step II: Get ring

Write "xy" instead of "$x\oplus y$", and "gh" instead of "$g*h$" ~~Translating, WMA ...~~

For $g\in H$, $(g\cdot): G\to G$

Let $(g\cdot)(x) := g\cdot x$

Lemma: $\hat{R} := \{(g\cdot)-(h\cdot) \mid g,h\in H\}$ is a convex subring of $End(G) :=$ ring of def^{ble} endo^{ms} of G

ordered by $\eta>0$ iff $\eta(p)>0$ for any (some) $p>0$

Pf: $\hat{H} := \{(g\cdot)\mid g\in H\}\subseteq End(G)$ is convex, by cont^y of $(\cdot)$.

Claim: If $g,h,k\in H$ then $(g\cdot)+(h\cdot)+(k\cdot)\in\hat{H}$.

Pf: Cases: (i) $g\le k\le h$: Then $(g\cdot)\le(g\cdot)+(h\cdot)+(k\cdot)\le(h\cdot)$ so done by convexity.

(ii) $k\le g,h$: Now

since $(kg^{-1}\cdot)\le 1$

$(gk^{-1}h\cdot) = (g\cdot)+(gk^{-1}h\cdot)-(g\cdot) \ge (g\cdot)+(kg^{-1}\cdot)\circ((gk^{-1}h\cdot)-(g\cdot))$

$= (g\cdot)+(h\cdot)-(k\cdot)$

$\ge (g\cdot)+(h\cdot)-(g\cdot)$

$= (h\cdot)$ So done by convexity.

since $(g\cdot)\ge(k\cdot)$

(iii) $k\ge g,h$: sim.

PS-Trick cont^{d}

Hence $\hat{R}$ is closed under addition:

$$((g_1\cdot) - (h_1\cdot)) + ((g_2\cdot) - (h_2\cdot)) = (g_1\cdot) - ((h_1\cdot) + (h_2\cdot) - (g_2\cdot))$$

and hence under mult^n:

$$((g_1\cdot) - (h_1\cdot)) \circ ((g_2\cdot) - (h_2\cdot)) = \cancel{(g_1 g_2\cdot) + (h_1 h_2\cdot)}$$
$$((g_1 g_2\cdot) - (h_1 g_2\cdot)) + ((h_1 h_2\cdot) - (h_1 g_2\cdot))$$

$\square$ Lemma

Now let $p \in G$, $p > 0$

set $R := \{\eta(p) \mid \eta \in \hat{R}\}$ - convex subgroup of G iso^c as ordered gr^p to $\hat{R}$. Transport mult^n

so R is Λ-def^{ble} ordered integral domain in M

Step III: ~~Get~~ fraction field ~~[illegible]~~ on an int^l $(-\beta, \beta)$:

K := fraction field of R (ordering: $a, b > 0 \Rightarrow a/b > 0$)

Let $\beta \in R$, $\beta > 0$. Let $K := (-\beta, \beta) \subseteq R$

Then
$$x \longmapsto \frac{x}{|x| + \beta}$$
$$K \longrightarrow \hat{K}$$

is an order-preserving bij^n

so get def^{ble} ordered field on K, with order that of M

By Pillay, $K \models RCF$ $\square$ Prop^n

~~[illegible]~~

Examples:

a. $M := \langle \mathbb{R}; \leq, +, \cdot|_{[-1,1]^2} \rangle$: no RCF with universe $\mathbb{R}$ is def^{ble}, by [Pillay-Scowcroft-Steinhorn "between gr^{p}s & rings"]

but around every point is an RCF (by Trick^y)

Cor^y: If G is an n-dim^l group def^{ble} in a nonlinear o-min^l exp^n of an ordered group, let R be as above.

Then the group of def^{ble} auto^{isms} of G embeds in $GL_n(R)$

Pf: Otero-Peterzil-Pillay arg^{nts} go through locally $\square$

(e.g. skew Laurent rings, $\mathbb{C}((t))$ with $t\alpha = \alpha^\sigma t$, $\sigma \in Aut(\mathbb{C})$)

Cor^y: Let D be an ordered non-comm^{ve} div^n ring, as an ordered VS over itself.

Then D has no proper o-min^l exp^n.

Pf: Any such would be non-linear but then G := add^{ve} group in above cor^y $\Rightarrow$ $D^\times \leq GL_1(R)$

$\times$, D non-comm^{ve} $\square$

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