Groups in o-min^l structures: bi-interpretability

Th^m [PPS1]: $(G;\cdot)$ group def^ble in an o-min^l struct^re M

which is $(G;\cdot)$-def^bly simple (no $(G;\cdot)$-def^ble proper non-trivial normal subgroup)

Then $(G;\cdot)$ is M-def^bly iso^ic to a semialg^c subgroup of $GL_n(R)$

where R is an M-def^ble RCF.

("M-def^ble" := def^ble in the structure M with parameters from M (i.e. $L_M(M)$-def^ble))

// [PPS2] considers definability in $(G;\cdot)$

Def^n: M, N structures

· An ~~interpretation~~ Δ of M in N, $\Delta : M \rightsquigarrow N$,
is a bij^n $\Delta : M \to X/E$ where X is an N-def^ble set $X \subseteq N^n$, $E \subseteq X^2$ is an N-def^ble equiv^ce rel^n

s.t. if $Z \subseteq M^n$ is M-def^ble
then $\Delta(Z)$ is N-def^ble (i.e. $\pi_E^{-1}(\Delta(Z)) \subseteq X^n$ is)

· Note: an IM $\sigma : M \to N$ is an interpretation; the comp^n of terps is a terp.

· An interp Δ of M in N
and an " Γ " N " M
form a bi-interpretation
if $\Gamma \circ \Delta$ is M-def^ble
and $\Delta \circ \Gamma$ is N-def^ble

Rem^k: Then $Z \subseteq M^n$ is M-def^ble iff $\Delta(Z)$ is N-def^ble

pf: $\Rightarrow$: by def^n of terp
$\Leftarrow$: $\Gamma(\Delta(Z))$ is M-def^ble hence Z is □

Th^m [PPS2]: If $(G;\cdot)$ is a $(G;\cdot)$-def^bly simple group def^ble in an o-min^l str^re

then either

(I) (Unstable case) (e.g. $PSL_2(\mathbb{R})$, $SO_3(\mathbb{R})$) $(G;\cdot)$ is bi-interpretable with an RCF K

Moreover, $(G;\cdot)$ is $(G;\cdot)$-def^bly iso^ic to the connected comp^nt of $\hat{G}(K)$ for some K-simple linear alg^c group $\hat{G}$ over K

OR (II) (stable case) (e.g. $PSL_2(\mathbb{C})$) ACF K ... to $\hat{G}(K)$ for some simple linear alg^c group $\hat{G}$ over K

Rem^k: Cherlin–Zilber conj^e: (II) holds for G a simple gr^p of fin RM.

Cor^y: The def^ble sets of $(G;\cdot)$ are precisely the semialg^c resp. Zariski-constructible sets.

Cor^y: If $\sigma : G_1 \to G_2$ is a group IM
then $\sigma = \theta \circ \tau$ for some semialg^c $\tau : G_1 \xrightarrow{\cong} G_3$
and some $\theta : G_3 \xrightarrow{\cong} G_2$ induced via the interpretation from a field IM

In particular, if G is an unstable semialg^c real Lie group, so $K = \mathbb{R}$ and so $\theta = id$,
~~then~~ σ is a Lie group AM.

~~[illegible]~~
"o-min' Borel-Tits"

Cor^{y}: Suppose G_i, K_i are as in Th^m PPS2 $(i=1,2)$

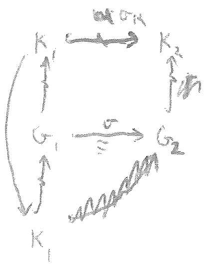
and $\sigma : G_1 \to G_2$ is a group IM.

Then $\sigma = \tau \circ \theta$ where θ is induced by a field IM $K_1 \xrightarrow{\cong} K_2$

and τ is a K_2-~~[illegible]~~ def^{ble} group IM. (so semialg^c / rational)

In particular σ is cont^s, and if $K_1 = \mathbb{R} = K_2$ then σ is a Lie group IM.

Pf: Let $\sigma_K : K_1 \to K_2$ make the diagram commute:



~~[illegible]~~

Now $\sigma_K(K_1)$ is K_2-def^{bly} IM^c to K_2

(since $K_1, K_2 \models RCF$ [OPP] or $K_1, K_2 \models ACF_0$)

say by $\tau_K^{-1} : \sigma_K(K_1) \to K_2$

Let $\theta_K := \tau_K^{-1} \sigma_K : K_1 \to K_2$ field IM

Fact: $K_1 \xrightarrow{\alpha} K_2$
If $K \models ACF$ and F is an inf^{te} field interpreted in K then F is K-def^{bly} IM^c to K

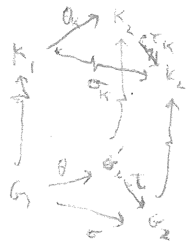
So $\sigma_K = \tau_K \circ \theta_K$

so $\sigma = \tau \circ \theta$

where $\theta : G_1 \xrightarrow{\cong} G_2'$ is induced by θ_K

and $\tau : G_2' \xrightarrow{\cong} G_2$ — — — — τ_K

(hence K_2-~~[illegible]~~ def^{ble})



PPS1 reduces PPS2 to case G semialg^c in an RCF R.

~~[illegible]~~

The case $R = \mathbb{R}$, G compact was Nesin-Pillay.

~~[illegible]~~ New ingredient:

~~Th^m~~ Th^m(2.12): Let G be a def^{ble} inf^{te} group in an o-min^l structure which is solvable and not nilpotent-by-finite.

Then a field K is interpretable in $(G;\cdot)$.

($Fin \hookrightarrow G \twoheadrightarrow Nil_p$)

Example: $PSL_2(\mathbb{R})$ $\quad \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \cong \mathbb{R}_+ \rtimes \mathbb{R}^\times$

G is R-isotropic if there is $T \leq G$ R-def^{ble} subg^p with T R-def^{bly} IM^c to $(R^\times)^n$ some $n > 0$.

Cases: · G R-anisotropic ⟶ use [NP]-style arguments (+[Buchman]) to get field

G R-isotropic ⟶ use th^y of alg^c g^{ps} and 2.12 to get field

SO_3^{00}

${}^*\mathbb{R} := (\mathbb{R};+,\cdot)^{U} \succ (\mathbb{R};+,\cdot)$ U non-princ^l ufilter on ω

$st : {}^*\mathbb{R} \dashrightarrow \mathbb{R}$

$\mathcal{O} := \mathrm{dom}(st)$

$\mathfrak{m} := \ker(st)$

$SO_3^{00} := SO_3^{00}({}^*\mathbb{R}) := \ker(st : SO_3({}^*\mathbb{R}) \to SO_3(\mathbb{R}))$

$= SO_3({}^*\mathbb{R}) \cap \begin{pmatrix} 1+\mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & 1+\mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & 1+\mathfrak{m} \end{pmatrix}$ = "rotations by infinitesimal angle about any (non-standard) axis"

Consider def^{ble} sets in $(SO_3^{00};*)$.

Sim G^{00} for $G \leq GL_n(\mathbb{R})$ a compact linear alg^c group

<u>Th^m</u> (B-Peterzil): If $G \leq GL_n(\mathbb{R})$ is a simple compact linear alg^c gr^p, then $(G^{00};*)$ is bi-interpretable with $({}^*\mathbb{R};+,\cdot;\mathcal{O}) \models RCVF$.

In particular, $X \subseteq G^{00}$ is $(G^{00};*)$-def^{ble} iff $({}^*\mathbb{R};+,\cdot;\mathcal{O})$-def^{ble}.

<u>Pf outline:</u>

I) Find $J \subseteq SO_3^{00}$ ordered interval def^{ble} in $(SO_3^{00};*)$

II) Trichotomy $\Rightarrow$ field K on J

III) Find copy of SO_3^{00} in G^{00}, $(G^{00};*)$-def^{ble}

IV) Adjoint rep^n in $K \rightsquigarrow$ bi-interp.

I) Let $S := SO_3(\mathbb{R})$ $S^{00} := SO_3^{00}({}^*\mathbb{R})$

Let $b \in S \setminus \{e\}$

$C_S(b) = \{\text{rotations with same axis}\} \cong SO_2({}^*\mathbb{R})$

$C_{S^{00}}(b) = C_S(b) \cap S^{00} \cong SO_2^{00}({}^*\mathbb{R}) \underset{\text{via exp}}{\cong} \mathfrak{m}$

$I := b^S * b^S \cap C_{S^{00}}(b)$

<u>Fact</u>: $I = [-b^2, b^2]$ in $C_{S^{00}}(b)$

$X := b^{S^{00}} * b^{S^{00}} \cap C_{S^{00}}(b) = (h, b^2]$ infinite interval

Then get $(S^{00};*)$-def^{ble} int^l $[e,p] \subseteq C_{S^{00}}(b)$

$(p := b^2 h^{-1},\ (e,p) = h^{-1}X \cap b^2 X^{-1})$

hence $J = (p^{-1}, p) \subseteq C_{S^{00}}(b)$ ordered group interval $(S^{00};*)$-def^{ble}

Trichotomy $\rightsquigarrow$ field on $K \subseteq J$

<u>Finding $SO_3^{00} \hookrightarrow G^{00}$:</u>

$\mathfrak{g}_0 := L(G)$

$\mathfrak{h}_0 \leq \mathfrak{g}$ Cartan (max^l ab^n subalg^a, L(max^l torus))

$\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ $\mathfrak{h} := \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$

Exists $X \in \mathfrak{g}$ and "root" $\alpha \in \mathfrak{h}^* \setminus \{0\}$ s.t. $[H,X] = \alpha(H)X$ $\forall H \in \mathfrak{h}$

Since G compact, α is purely imaginary i.e. $i\alpha \in \mathfrak{h}_0^*$

Let $U := X - \bar{X}$ $V := iX + i\bar{X}$ Then $U, V \in \mathfrak{g}_0$

and $H := [U,V] = 2i[X,\bar{X}] \in \mathfrak{h}_0$

$c := \alpha(H) \in i\mathbb{R} \setminus \{0\}$ so $[H,U] = cX + c\bar{X} \in \mathbb{R}V$

and $[V,H] = icX - ic\bar{X} \in \mathbb{R}V$

so $\mathfrak{s} := \langle U, V, H \rangle_{\mathbb{R}} \cong \mathfrak{so}_3$

Let $\mathfrak{s}' := \mathfrak{s} \oplus \mathfrak{h}_0$

then $[\mathfrak{s}', \mathfrak{s}'] = \mathfrak{s}$ and $C_{\mathfrak{g}_0}(C_{\mathfrak{g}_0}(\mathfrak{s}')) = \mathfrak{s}'$

Let $S, S' \leq G$ with $L(S) = \mathfrak{s}$, $L(S') = \mathfrak{s}'$

then $S'^{00} = C_{G^{00}}(C_{G^{00}}(S'^{00}))$ and $S^{00} = (S'^{00}, S'^{00})$,

so S^{00} is $(G^{00};*)$-def^{ble}

(and sim S is $(\mathbb{R};+,\cdot)$-def^{ble})

<u>Pf.</u> $C_{\mathfrak{g}_0}(\mathfrak{h}_0) = \mathfrak{h}_0$

so $C_{\mathfrak{g}_0}(\mathfrak{s}') = \mathfrak{h}_0 \cap C_{\mathfrak{g}_0}(\{U,V\}) = \ker \alpha|_{\mathfrak{h}_0}$

Now $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta$ $\quad \mathbb{C}\beta \cap \Delta = \{\beta, -\beta\}$, $\dim \mathfrak{g}_\beta = 1$

So $C_{\mathfrak{g}}(\ker \alpha|_{\mathfrak{h}_0}) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ (since $\ker \alpha|_{\mathfrak{h}_0} \subseteq \ker \beta \Leftrightarrow \beta \in \mathbb{C}\alpha$)

so $C_{\mathfrak{g}_0}(\ker \alpha|_{\mathfrak{h}_0}) = \mathfrak{s}'$ $\square$