Distal Theories and the Type Decomposition Theorem

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Setting

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- Let $\mathcal{L}$ be some first-order language, T some complete $\mathcal{L}$-Theory and $\mathcal{U} \models T$ be a monster model.
- T_{co} in the language $\{<, E\}$ will denote a coloured DLO with infinitely many dense colours.

Distal Sequences

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- For every set A, tuple b and A-indiscernible sequence $J = J_0 + J_1$ such that $I \equiv_{EM} J$ and J_0, J_1 are without endpoints,
  if $J_0 + b + J_1$ is indiscernible, then it is A-indiscernible.

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  if $J_0 + b + J_1$ is indiscernible, then it is A-indiscernible.
- An NIP theory is called *distal* if every indiscernible sequence is distal in this theory.

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- Non-constant totally indiscernible sequences are not distal.
- A non-constant indiscernible sequence in T_{co} is distal if and only if it has constant colour.

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- For totally indiscernible sequences only the first case can happen, for distal sequences only the second case.

Compressible Types

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Definition

- Let $a \in M$. A type $tp(a/A)$ is *compressible* if for an $|M|^+$ saturated elementary extension (M', A') of (M, A), for any $\mathcal{L}$-formula $\phi(x, y)$, there is $\zeta(x, e) \in tp(a/A')$ such that $\zeta(x, e) \vdash tp_\phi(a/A)$.

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- This is equivalent to the following property: For any $\mathcal{L}$-formula $\phi(x, y)$ there is some $\zeta(x, t)$ such that for any finite $A_0 \subseteq A$, there is $e \in A$ such that $\models \zeta(a, e)$ and $\zeta(x, e) \vdash tp_\phi(a/A_0)$.

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Fact

A type $tp(a/A)$ is compressible iff $tp(a/A')$ is weakly orthogonal to every $q(y) \in S(A')$ finitely satisfiable in A, i.e. $tp(a/A') \cup q$ determines a complete type.

Sketch of distal implies compressible

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- We can find some set D of size $< \kappa$ between C and A' such that for any two realizations $I, I' \subseteq A'$ of $q^{(\omega)}|_D$ we have $tp_{\mathcal{L}}(aI/C) = tp_{\mathcal{L}}(aI'/C)$. Take such an I and some $J \models q^{(\omega)}|_{\mathcal{U}}$.

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- Now, one can show that $I + J$ is indiscernible over aC.
- Take any $b \models q$. Then $I + b + J$ is C-indiscernible and by distality, $I + b$ is aC-indiscernible.

Sketch of distal implies compressible (continued)

- So for any other $b' \models q$ we have $tp(ab/C) = tp(ab'/C)$. Since we can take C to be as big as we want, orthogonality of p and q follows.

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- It is also possible to show uniform compressibility, this requires some combinatorial tricks (i.e. the (p,k)-theorem).

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- Take some finite tuple a in A and some $\mathcal{L}$-formula $\phi(x, y_0 ... y_{2n})$. Let $\zeta(x, z)$ be given by compressibility for $tp(a/I)$. Assume $\models \phi(a, b_0 ... b_{2n})$ for all $b_0 < ... < b_{2n} \in I + J$.

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- Consider $I_0 \subset I$ finite. Then there is some tuple b in I with $\models \zeta(a, b)$ and $\zeta(x, b) \vdash tp_\phi(a/I_0)$. For a large enough I_0 we can also find $b_0 < ... < b_{2n}$ disjoint from b.

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- We have: $\models (\forall x)(\zeta(x, b) \to \phi(x, b_0 ... b_{2n}))$. Now use indiscernibility of $I + d + J$:

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- We have: $\models (\forall x)(\zeta(x,b) \rightarrow \phi(x, b_0 ... b_{2n}))$. Now use indiscernibility of $I + d + J$:
- For any $b'_0 < ... < b'_{n-1}$ in I and $b'_{n+1} < ... < b'_{2n}$ in J there is some b' in $I + J$ such that $\models (\forall x)(\zeta(x,b') \rightarrow \phi(x, b'_0 ... d ... b'_{2n}))$.

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- We have: $\models (\forall x)(\zeta(x,b) \rightarrow \phi(x, b_0 \ldots b_{2n}))$. Now use indiscernibility of $I + d + J$:
- For any $b'_0 < \ldots < b'_{n-1}$ in I and $b'_{n+1} < \ldots < b'_{2n}$ in J there is some b' in $I + J$ such that $\models (\forall x)(\zeta(x,b') \rightarrow \phi(x, b'_0 \ldots d \ldots b'_{2n}))$.
- We have $\models \zeta(a, b')$ since $I + J$ is indiscernible over a. So $\models \phi(a, b'_0 \ldots d \ldots b'_{2n})$ follows.

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Fact

- If T is NIP, then distal and co-distal are equivalent.
- co-distal implies compressible for complete types.

Generically stable partial Types

Definition

- A partial type π over $\mathcal{U}$ is *Ind-definable* over a set A if for every $\phi(x, y)$, the set $\{b \mid \phi(x, b) \in \pi\}$ is a union of A-definable sets.

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- π is *generically stable over* A if π is Ind-definable over A and the following holds:
  If $(a_k)_{k<\omega}$ is such that $a_k \models \pi|_{Aa_{<k}}$ and $\phi(x, b) \in \pi$, then for all but finitely many values of k, we have $\models \phi(a_k, b)$.

The Type Decomposition Theorem

Theorem (Simon, 2017)

Let T be NIP and $p(x) = tp(a/A)$ be any type. Then there is $\pi(x)$ generically stable over A with $\pi(x)|_A = p(x)$, such that the following holds:

1. If $(M, A) \prec (M', A')$ is $|M|^+$-saturated and
2. $q(y)$ is a global type finitely satisfiable in A,

then $tp_x(a/A') \cup (\pi \otimes q)|_{A'}(x, y)$ implies the complete type $(q \otimes p)|_A(y, x)$.

The Type Decomposition Theorem

- Let $A \subset \mathcal{U}$ be small. Let $p(x) = tp(a/A)$ and $q(y)$ a global A-finitely satisfiable type. Let $\bar{\alpha}_*$ be an A-indiscernible sequence, encoding the *q-stable part of a*.

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- We define the partial type $\pi_q(x) := p(x) \cup (\exists \bar{\alpha}'_*)(tp(x\bar{\alpha}'_*/A) = tp(a\bar{\alpha}_*/A) \wedge \bar{\alpha}'_*$ is indiscernible over $\mathcal{U})$.

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- π will be the conjunction of all π_q's.

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- So $a' \models \pi_q|_B$ iff the colour of a' is not contained in B.
- For any q finitely satisfiable in A that is not of a new colour, we have $\pi_q = p$.
- It follows that $tp_x(a/A') \cup (\pi \otimes q)|_{A'}(x, y)$ simply says that the colour of a realization of $tp(a/A')$ is not contained in $A' \cup \{b\}$ for some $b \models q$.

Shrinking of Indiscernibles

Fact

- Let $I = (a_i)_{i \in \mathcal{I}}$ be indiscernible, d any tuple and $\phi(y_0, ..., y_{n-1}, d)$ a formula. Then there is a finite convex equivalence relation $\sim_\phi$ on $\mathcal{I}$ such that given:

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- Let $T(I, \phi)$ denote the number of equivalence classes for the coarsest $\sim_\phi$ for I. This is bounded by some integer only depending on $\phi(y_0, \ldots, y_{n-1}, z)$.

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- Let $T(I, \phi)$ denote the number of equivalence classes for the coarsest $\sim_\phi$ for I. This is bounded by some integer only depending on $\phi(y_0, ..., y_{n-1}, z)$.
- If $I \subseteq J$ and A is some set of parameters we write $I \trianglelefteq_A J$ if $T(I, \phi) = T(J, \phi)$ for every $\mathcal{L}_A$-formula $\phi(y_0, ..., y_{n-1}, d)$.

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  - there is some Morley sequence I in q over Aa such that $I + \bar{\alpha}$ is A-indiscernible.
- Then there is some $s(\bar{x}) \in \Omega$ such that, if $\bar{\alpha}_* \models s$ and $\bar{\alpha}_* \trianglelefteq_A \bar{\alpha}'$ with $tp(\bar{\alpha}'/Aa) \in \Omega$, then $\bar{\alpha}_* \trianglelefteq_{Aa} \bar{\alpha}'$. (Follows by Shrinking of indiscernibles).