The maximal divisible type-definable subgroup $G^{\#} = [p^{\infty}]G(K) = \bigcap_n [p^n]G(K)$ of a semiabelian variety over a separably closed field K played a crucial role in Hrushovski's celebrated proof of function-field Mordell-Lang in positive characteristic, the crucial idea being that $G^{\#}$ is the positive characteristic analogue of the "Manin kernel" in characteristic zero (also denoted $G^{\#}$).

This paper treats in detail the behavior of the $\cdot^{\#}$ functor (in all characteristics) when applied to short exact sequences of semiabelian varieties, and finds that the question of when it preserves exactness is closely linked to descent: if $0 \to G_1 \to G_2 \to G_3 \to 0$ is a (separable) exact sequence of ordinary semiabelian varieties and G_1 and G_3 are defined over the absolute constants, then G_2 descends to the absolute constants iff $0 \to G_1^{\#} \to G_2^{\#} \to G_3^{\#} \to 0$ is also exact.

Using this equivalence, the authors find examples in all characteristics of semiabelian varieties $0 \to \mathbb{G}_m^n \to S \to A \to 0$ where the $\cdot^{\#}$ functor is not exact. In positive characteristic, they find that in this case $S^{\#}$ does not have ordinal-valued "relative Morley rank" (being the analogue of Morley rank for relatively definable subsets of a type-definable set). The existence of such S contradicts a claim in [Hru96]. The present paper includes in the introduction a sketch of how to avoid using that claim in the relevant part of the argument of [Hru96], so the validity of [Hru96] is unaffected.

The authors also take the opportunity to discuss some of the basic properties of relative Morley rank. This is a very natural notion, which appeared in a less general form in [Hru96], and is of interest in itself.

The results on exactness and descent are proven in arbitrary characteristic. This proceeds by exploiting the same correspondence between separably closed fields and differentially closed fields, and between $[p^{\infty}]G(K)$ and the Manin kernel, exploited in Hrushovski's work. The authors formalise this correspondence by developing the theory of "D-structures" in arbitrary characteristic, generalising the theory of Buium in characteristic zero. In particular, they find that taking the inverse limit of copies of G under $[p^n]$ as the positive characteristic analogue of the universal vectorial extension in characteristic zero, applying the characteristic zero definition of $G^{\#}$ in positive characteristic does indeed yield $[p^{\infty}]G(K)$.

References

[Hru96] Ehud Hrushovski. The Mordell-Lang conjecture for function fields. *J. Amer. Math. Soc.*, 9(3):667–690, 1996.