The Pila-Zannier proof of Manin-Mumford

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Contents

Notes for a talk in a Münster seminar, based largely on section 5.1 in Tom Scanlon's paper "O-minimality as an approach to the André-Oort conjecture".

1 Manin-Mumford

- $K \leq \mathbb{C}$ number field.
- G complex abelian variety (i.e. projective connected non-trivial algebraic group) over K. (e.g. $G = E^g$ for E an elliptic curve over K.)
- $g := \dim(G)$.
- Let $\exp : LG \to G$ be the exponential map of $G = G(\mathbb{C})$ as a complex Lie group.
- $LG = T_0(G)$ is a g-dimensional $\mathbb{C}$-vector space.
- exp is a surjective holomorphic homomorphism.
- $\Lambda := \ker(\exp) \cong \mathbb{Z}^{2g}$ is a full lattice in LG, i.e. $\Lambda = \langle \lambda_1, \ldots, \lambda_{2g} \rangle$ where $(\lambda_i)_i$ is an $\mathbb{R}$-basis of LG.
- So as a complex Lie group, $G \cong \mathbb{C}^g/\Lambda$ is a complex torus, and is diffeomorphic to $(\mathbb{R}/\mathbb{Z})^{2g}$.
- Torsion subgroup: $G[\infty] \cong (\mathbb{Q}/\mathbb{Z})^{2g}$.
- The non-trivial connected algebraic subgroups of G are precisely the abelian subvarieties.
- G has only countably many abelian subvarieties, and each is defined over $\mathbb{Q}^{\mathrm{alg}}$.
- $G[\infty]$ is Zariski dense in G.
- A **torsion coset** is a coset $H + \xi$ of an abelian subvariety $H \leq G$ by a torsion point $\xi \in G[\infty]$.

Theorem (Manin-Mumford Conjecture; Raynaud (1983))**.** *Let $X \subseteq G$ be an irreducible (complex) subvariety and suppose $X \cap G[\infty]$ is Zariski dense in X. Then X is a torsion coset.*

Corollary. *For $X \subseteq G$ Zariski closed,*

$$X \cap G[\infty] = \bigcup_{i=0}^{n} \xi_i + H_i[\infty]$$

where $\xi_i \in G[\infty]$ and $H_i \leq G$ are abelian subvarieties.

Proof.

$$\begin{aligned} X \cap G[\infty] &= (X \cap G[\infty])^{\mathrm{Zar}} \cap G[\infty] \\ &= \left(\bigcup_{i=0}^{n} \xi_i + H_i\right) \cap G[\infty] \\ &= \bigcup_{i=0}^{n} ((\xi_i + H_i) \cap G[\infty]) \\ &= \bigcup_{i=0}^{n} \xi_i + H_i[\infty] \end{aligned}$$

□

1.1 Special Locus

- Let $X \subseteq_{\mathrm{cl}} G$.
- The **special locus** (or Ueno locus, or Kawamata locus) of X in G is

 $$\mathrm{SpL}(X) := \bigcup \{\mathrm{SpL}(X, H) : H \leq G \text{ abelian subvariety}\},$$

 where

 $$\mathrm{SpL}(X, H) := \bigcup \{g + H : g + H \subseteq X,\ g \in G\} = \bigcap_{h \in H} (X - h).$$

 We will see that $\mathrm{SpL}(X)$ corresponds to the Pila-Wilkie 'algebraic part' of $\exp^{-1}(X)$.
- The **stabiliser** of X in G is

 $$\mathrm{Stab}(X) := \mathrm{Stab}_G(X) := \{g \in G : g + X = X\}.$$

Lemma. *Suppose X is irreducible. Then $\mathrm{SpL}(X) = X$ iff $\mathrm{Stab}(X)$ is infinite.*

Proof.

- $\mathrm{Stab}(X) = \bigcap_{x \in X}(X - x)$, so $\mathrm{Stab}(X) \subseteq_{\mathrm{cl}} G$, so $\mathrm{Stab}(X)$ is an algebraic subgroup.
- If $\mathrm{Stab}(X)$ is infinite, then the connected component of the identity $H := \mathrm{Stab}(X)^o \leq G$ is an abelian subvariety, so $X = \bigcup\{x + H : x \in X\} = \mathrm{SpL}(X)$.
- Conversely: If $X = \mathrm{SpL}(X)$, then $X = \mathrm{SpL}(X, H)$ for some H, since each $\mathrm{SpL}(X, H)$ is closed and G has only countably many abelian subvarieties and X is irreducible.

 So $H \leq \mathrm{Stab}(X)$.

□

Lemma. $\mathrm{SpL}(X) \subseteq_{\mathrm{cl}} X$.

Proof.

- We actually prove something stronger:

 Claim. *If $(X_a)_{a \in A}$ is a constructible family of subvarieties of G, then $(\mathrm{SpL}(X_a))_{a \in A}$ is a constructible family of subvarieties of G.*

 Where

 – a **constructible** subset of a variety is a boolean combination of Zariski closed subsets; (*Fact: constructible ⇔ definable in* $(\mathbb{C}; +, \cdot)$.)
 – $(X_a)_{a \in A}$ is a **constructible family** of subvarieties of G if
   * A is constructible,
   * $X \subseteq G \times A$ is constructible, and
   * $X_a := \{g \in G : (g, a) \in X\}$ is Zariski closed in G for all a.
- Since each $\mathrm{SpL}(X_a, H)$ is closed, it suffices to show that there are finitely many $H_1, \ldots, H_n$ such that for all $a \in A$, $\mathrm{SpL}(X_a) = \bigcup_i \mathrm{SpL}(X_a, H_i)$.
- Say an abelian subvariety $H \leq G$ **appears maximally** in $X \subseteq G$ if X contains some coset $\gamma + H$ which is maximal among the cosets of abelian subvarieties contained in X.
- Say H **appears maximally** in $(X_a)_a$ if it appears maximally in some X_a.
- Then it suffices to show:

 (∗) Only finitely many H appear maximally in $(X_a)_a$.
- We prove (∗) by induction on $d := \max_a \dim(X_a)$.
- We may assume that each X_a is irreducible, by:

 Fact. *There exists a constructible $(X'_{a'})_{a' \in A'}$ and a constructible map $\alpha : A' \to A$ such that $\{X'_{a'} : \alpha(a') = a\}$ are the irreducible components of X_a.*
- We may also assume that each X_a has finite stabiliser.

 Indeed, there is (fact) a uniform bound N on the size of the finite stabilisers $\mathrm{Stab}(X_a)$, and $A' := \{a \in A : |\mathrm{Stab}(X_a)| \leq N\}$ is constructible. So to perform the reduction, it suffices to see that (∗) holds for $(X_a)_{a \in A \setminus A'}$.

 UPDATE: the argument previously written here for that was nonsense. Thanks to Zoé Chatzidakis for pointing this out. One should proceed by induction by quotienting by the connected component of the stabiliser.
- Now suppose $H_0 \leq G$ appears maximally in $(X_a)_a$. Say H_0 appears maximally in X_a.
- Let $h \in H_0 \setminus \mathrm{Stab}(X_a)$.
- Then $X'_{a,h} := X_a \cap (h + X_a)$ is a proper subvariety of X_a and contains a coset of H_0.
- Since $X'_{a,h} \subseteq X_a$, actually H_0 appears maximally in $X'_{a,h}$.
- Now

 $$(X'_{a,g} : a \in A,\ g \in G,\ X'_{a,g} \neq X_a)$$

 is a constructible family of subvarieties of G each of dimension less than d. By the inductive hypothesis, only finitely many H appear maximally in it. So H_0 is one of these finitely many, as required.

□

2 Ax-Schanuel

2.1 Restricted exponentiation

- Recall $\Lambda = \ker \exp$ is freely generated by an $\mathbb{R}$-basis $\lambda_1, \ldots, \lambda_{2g}$ of $LG \cong \mathbb{C}^g$.
- We identify LG with $\mathbb{R}^{2g}$ with respect to this basis (instead of taking real and imaginary parts),

 $$0 \to \mathbb{Z}^{2g} \to \mathbb{R}^{2g} \to^{\exp} G \to 0.$$
- Let $\mathcal{F} := [0,1)^{2g} \subseteq LG$ ("fundamental domain"), so the restriction $\exp_| : \mathcal{F} \to G$ is a bijection.
- $\exp_|$ is definable in $\mathbb{R}_{\mathrm{an}}$.
- For $X \subseteq G$, $\exp_|$ yields a bijection $\exp_|^{-1}(X) \cap \mathbb{Q}^{2g} \to X \cap G[\infty]$.
- To apply Pila-Wilkie, we must determine $\exp_|^{-1}(X)^{\mathrm{alg}}$.
- The key tool for this is the Ax-Schanuel theorem (or its "Lindemann-Weierstrass case").

2.2 Ax-Schanuel

The original Ax-Schanuel theorem concerns usual complex exponentiation (i.e. the exponential map of the multiplicative group):

Fact (Ax '71)**.** *Suppose $f_i : \Delta \to \mathbb{C}$ are holomorphic functions on the unit disc, and $f'_1(0), \ldots, f'_n(0)$ are $\mathbb{Q}$-linearly independent. Then*

$$\mathrm{trd}(f_1(t), \ldots, f_n(t), e^{f_1(t)}, \ldots, e^{f_n(t)}/\mathbb{C}) \geq n + 1.$$

- Proved using differential algebra.
- Generalisations:
 – Brownawell-Kubota: for elliptic curves;
 – Kirby: for arbitrary (semi-)abelian varieties.
 – Ax '72: general analytic version for arbitrary complex algebraic groups. We will use this.

Fact (Ax '72)**.** *Let*

- *$\mathbb{H}$ be a complex algebraic group,*
- *$\Gamma \leq \mathbb{H}$ a connected analytic subgroup, and*
- *$e \in K \subseteq \Gamma$ an irreducible analytic subvariety.*

Then there is an analytic subgroup $\mathbb{H}' \leq \mathbb{H}$ containing K^{Zar} and Γ such that:

$$\dim K \leq \dim K^{\mathrm{Zar}} - (\dim \mathbb{H}' - \dim \Gamma).$$

Idea: If K is a component of $K^{\mathrm{Zar}} \cap \Gamma$, the "expected" dimension is

$$\dim K = \dim K^{\mathrm{Zar}} - (\dim \mathbb{H} - \dim \Gamma);$$

Ax's theorem says this can be exceeded only if the intersection is really happening in a smaller group.

2.3 Algebraic part

Lemma. $\exp_|^{-1}(X)^{\mathrm{alg}} = \exp_|^{-1}(\mathrm{SpL}(X))$.

Proof.

- We show $\exp^{-1}(X)^{\mathrm{alg}} = \exp^{-1}(\mathrm{SpL}(X))$.
- Let $x \in \exp^{-1}(X)^{\mathrm{alg}}$, and suppose $x \notin \exp^{-1}(\mathrm{SpL}(X))$.
- Replacing X with $X - \exp(x)$, we may assume $x = 0$.
- So $0 \in C' \subseteq \exp^{-1}(X)$ for a semialgebraic curve C'.
- So $0 \in C \subseteq \exp^{-1}(X)$ for an irreducible algebraic curve C.
- Replace G with the smallest abelian subvariety G' containing $\exp(C)$, and X with $X \cap G'$.

 We still have $0 \notin \mathrm{SpL}(X)$.
- Consider $\Gamma_{\exp} \leq LG \times G =: \mathbb{H}$.
- Let $K \ni (0,0)$ be an analytic irreducible component of $(C \times X) \cap \Gamma_{\exp} = \Gamma_{\exp|_C}$.
- Now $\langle \exp(C)^{\mathrm{Zar}} \rangle = G$ by assumption, so $\pi_2(\langle K^{\mathrm{Zar}} \rangle) = G$, so $\langle K^{\mathrm{Zar}} \rangle \supseteq \{0\} \times G$ (by consideration of the algebraic subgroups of $LG \times G$).

 So $\langle K^{\mathrm{Zar}} \rangle + \Gamma = \mathbb{H}$.
- Meanwhile $X \neq G$ since $0 \notin \mathrm{SpL}(X)$.
- By Ax, $\dim K \leq \dim K^{\mathrm{Zar}} - (\dim \mathbb{H} - \dim \Gamma)$, so:

 $$\begin{aligned} \dim(G) &= \dim(\mathbb{H}) - \dim(\Gamma) \\ &\leq \dim K^{\mathrm{Zar}} - \dim K \\ &\leq \dim(C \times X) - 1 \\ &\leq \dim(X) \\ &< \dim(G). \end{aligned}$$

 Contradiction.

□

Remark. Kawamata *On Bloch's Conjecture* (1980, Inventiones) gives an alternative proof of this (and also of the closedness of $\mathrm{SpL}(X)$).

3 Masser

- Let $\gamma \in G(\mathbb{Q}^{\mathrm{alg}})$.
- In co-ordinates according to our projective embedding (possibly permuting the co-ordinates),

 $$\gamma = [1 : \gamma_1 : \ldots . : \gamma_n] \in G \subseteq \mathbb{P}^n(\mathbb{C})$$

 with $\gamma_i \in \mathbb{Q}^{\mathrm{alg}}$.
- Then $K(\gamma) := K(\gamma_1, \ldots, \gamma_n)$.
- For $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/K)$,

 $$\sigma(\gamma) := [1 : \sigma(\gamma_1) : \ldots : \sigma(\gamma_n)] \in G.$$
- $\deg(K(\gamma)/K)$ is the size of the orbit of γ under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/K)$.

Theorem (Masser '84)**.** *Exist $\lambda = \lambda(g) > 0$ and $C > 0$ such that for $\xi \in G[\infty]$ of order k,*

$$\deg(K(\xi)/K) \geq Ck^{\lambda}.$$

4 Pila-Zannier

4.1 Reduction

Theorem (Manin-Mumford Conjecture)**.** *If $X \subseteq_{\mathrm{cl}} G$ is irreducible and $(X \cap G[\infty])^{\mathrm{Zar}} = X$ then X is a torsion coset.*

Suffices to show:

Lemma. *Suppose $X \subseteq_{\mathrm{cl}} G$ is irreducible and defined over $\mathbb{Q}^{\mathrm{alg}}$ and suppose* $\mathrm{Stab}_G(X)$ *is finite. Then $(X \setminus \mathrm{SpL}(X)) \cap G[\infty]$ is finite.*

Proof of Theorem from Lemma.

- Let H be the connected component of the stabiliser $H := \mathrm{Stab}_G(X)^o$.

 So $H \leq G$ is an abelian subvariety or the trivial subgroup.
- Then (fact) G/H is an abelian variety, and $X/H \subseteq G/H$ is an irreducible subvariety, and $\mathrm{Stab}_{G/H}(X/H)$ is finite.

 (Indeed, if $\mathrm{Stab}_{G/H}(X/H)$ contains an infinite algebraic subgroup S, then $S' = \pi_H^{-1}(S)$ stabilises $X = \pi_H^{-1}(X/H)$, but $\dim(S') = \dim(S) + \dim(H) > \dim(H)$, contradicting the choice of H).
- X is over $\mathbb{Q}^{\mathrm{alg}}$, because $G[\infty] \subseteq G(\mathbb{Q}^{\mathrm{alg}})$ and so X is $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{\mathrm{alg}})$-invariant.

 Also H is over $\mathbb{Q}^{\mathrm{alg}}$.

 So X/H is over $\mathbb{Q}^{\mathrm{alg}}$.
- So by the Lemma, $(X/H \setminus \mathrm{SpL}(X/H)) \cap (G/H)[\infty]$ is finite.
- But:

 $$\pi_H^{-1}(X/H \cap (G/H)[\infty]) \supseteq X \cap G[\infty],$$

 so

 $$\pi_H^{-1}((X/H \cap (G/H)[\infty])^{\mathrm{Zar}}) \supseteq (X \cap G[\infty])^{\mathrm{Zar}} = X,$$

 so

 $$(X/H \cap (G/H)[\infty])^{\mathrm{Zar}} = X/H.$$
- Now $\mathrm{Stab}_{G/H}(X/H)$ is finite, so $\mathrm{SpL}(X/H)$ is a proper closed subvariety of X/H.
- So the finite set $(X/H \setminus \mathrm{SpL}(X/H)) \cap (G/H)[\infty]$ is Zariski dense in X/H.
- So $X/H = \{\xi\}$ for some $\xi \in (G/H)[\infty]$.
- So $X = \pi^{-1}(\xi)$ is a coset of H, and it contains a torsion point since $X \cap G[\infty]$ is dense in X, so X is a torsion coset.

□

4.2 Concluding by Pila-Wilkie

Proof of Lemma.

- Let $X \subseteq_{\mathrm{cl}} G$ be irreducible and over $\mathbb{Q}^{\mathrm{alg}}$ with $\mathrm{Stab}_G(X)$ finite.
- WTS: $(X \setminus \mathrm{SpL}(X)) \cap G[\infty]$ is finite. Suppose not.
- Increasing K, we may assume X is over K.

 Then $\mathrm{SpL}(X)$ is also over K.
- So by Masser: For arbitrarily large k, there are Ck^{λ} points of order k in $X \setminus \mathrm{SpL}(X)$.
- If ξ has order k, then the height of

 $$\exp_|^{-1}(\xi) \in \exp_|^{-1}(X \setminus \mathrm{SpL}(X)) \cap \mathbb{Q}^{2g} = \exp_|^{-1}(X)^{\mathrm{tr}} \cap \mathbb{Q}^{2g}$$

 is at most k.
- But by Pila-Wilkie, exists $C' > 0$ such that

 $$N(\exp_|^{-1}(X)^{\mathrm{tr}}, k) \leq C' k^{\frac{\lambda}{2}}.$$
- For large enough k, these bounds contradict each other, i.e.

 $$Ck^{\lambda} > C' k^{\frac{\lambda}{2}}.$$

□